

# Analytical solutions in short-time limits for computationally efficient droplets evaporation models

## Solutions analytiques dans la limite des temps courts pour des modèles numériquement efficaces d'évaporation de gouttelettes

Edoh Tossou<sup>1</sup>, Kwassi Anani<sup>2</sup> and Roger Prud'homme<sup>3</sup>

<sup>1</sup> Laboratoire d'Analyse, de Modélisation Mathématique et Applications, Département de Mathématiques, Facultés des Sciences, Université de Lomé, Togo, [tossoud35@gmail.com](mailto:tossoud35@gmail.com)

<sup>2</sup> Laboratoire d'Analyse, de Modélisation Mathématique et Applications, Département de Mathématiques, Facultés des Sciences, Université de Lomé, Togo, [kanani@univ-lome.tg](mailto:kanani@univ-lome.tg)

<sup>3</sup> Jean Le Rond d'Alembert Institute, UMR 7190 CNRS, Université de Sorbonne, [roger.prudhomme@sorbonne-universite.fr](mailto:roger.prudhomme@sorbonne-universite.fr)

**ABSTRACT.** This paper presents a semi-analytical model of the spherically symmetric droplet transient heating and evaporation in a subcritical environment. In the numerical procedure of the model, the droplet radius is assumed to be fixed during a short time step, but varies from each time step to the next. This variation is obtained through an approximate analytical solution of the heat diffusion equation inside the stagnant droplet, by considering the volume-average or core temperature at the beginning of each time step. Depending on the temperature evolution of the gas phase assumed in quasi-steady state at the immediate vicinity of the droplet, explicit solutions are obtained in the Laplace domain for the droplet internal and surface temperatures. Next, analytical approximations in short time limits, corresponding to the asymptotic expansions of the Laplace domain solutions are derived. In particular, the reduction in droplet radius during the evaporation process is approximated by an analytical expression that is in close agreement with the numerical results. All equations are then applied in their dimensionless form to model the transient heating and evaporation of pure fuel droplets of various sizes. The new model provides a consistent description of both the initial heating phase and the entire evaporation phase of the droplet. Furthermore, the results demonstrate significantly greater computational efficiency than evaporation models using successive time steps, particularly when the series solution of the heat diffusion equation is employed within the droplet.

**RÉSUMÉ.** Cet article présente un modèle semi-analytique décrivant le chauffage transitoire et l'évaporation d'une gouttelette en symétrie sphérique dans un environnement sous-critique. Dans la procédure numérique du modèle, le rayon de la gouttelette est supposé constant sur un court pas de temps, mais peut varier d'un pas de temps à l'autre. Cette variation est obtenue à partir d'une solution analytique approximative de l'équation de diffusion thermique à l'intérieur d'une gouttelette au repos, en considérant la température moyenne de la goutte au début de chaque pas de temps. En fonction de l'évolution de la température de la phase gazeuse, supposée en état quasi-stationnaire au voisinage immédiat de la gouttelette, des solutions explicites sont obtenues dans le domaine de Laplace pour les températures interne et de surface de la gouttelette. Ensuite, des approximations analytiques correspondant aux développements asymptotiques des solutions dans le domaine de Laplace, sont dérivées dans la limite des temps courts. En particulier, la réduction du rayon de la gouttelette au cours du processus d'évaporation est approximée par une expression analytique en bon accord avec les résultats numériques. Toutes les équations sont ensuite appliquées sous leur forme adimensionnelle afin de modéliser l'échauffement transitoire et l'évaporation de gouttelettes de combustible pur de différentes tailles. Le nouveau modèle fournit une description cohérente tant de la phase d'échauffement initiale que de l'ensemble de la phase d'évaporation de la gouttelette. De plus, les résultats montrent une efficacité de calcul nettement supérieure à celle des modèles d'évaporation utilisant des pas de temps successifs, en particulier lorsque la solution en série de l'équation de diffusion thermique est appliquée à l'intérieur de la gouttelette.

**KEYWORDS.** spherical droplet, transient heating, evaporation, time-step analysis, asymptotic expansions, analytical approximations, CFD spray modelling.

**MOTS-CLÉS.** Gouttelette sphérique, chauffage transitoire, évaporation, analyse par pas de temps, développements asymptotiques, approximations analytiques, modélisation CFD des pulvérisations.

## List of symbols

Symbols	Signification	Symbols	Signification
$B_M$	Spalding number	$R_s$	Droplet radius
$\dot{M}$	Instantaneous evaporation rate	$\dot{R}_s$	Derivative of the droplet radius with respect to time
$\alpha_l$	Thermal diffusivity of the liquid fuel	$c_{pg}$	Specific heat capacity at constant pressure of the gas
$h$	The convection heat transfer coefficient	$Y_{fs}$	Vapor mass fraction on the droplet surface
$\overline{T_0^0}$	Initial temperature of the droplet	$P_g$	Ambient gas pressure
$\overline{T_g^0}$	Initial gas temperature	$P_{fs}$	Pressure of saturated fuel vapor near the surface of the droplet
$\rho_l$	Density of the liquid	$T_s$	Surface temperature of the droplet
$c_l$	Specific heat capacity of liquid phase fuel	$Q_s$	Temperature gradient at the droplet surface
$c_g$	Specific heat capacity of gas phase fuel	$T_g^1$	Estimate value of the gas-phase temperature
$k_g$	The thermal conductivity of gas phase	$M_g$	Molar masses of air of gas
$k_l$	The thermal conductivity of liquid fuel	$P_g$	Ambient gas pressure
$\ell$	Latent or specific heat of vaporization	$a, b$	Constant related to the fuel physical properties
$M_f$	Molar masses of air of fuel	$K$	Ratio of gas and liquid conductivity

## 1. Introduction

Transient heating and evaporation of liquid fuel droplet problems arise in fluid dynamics engineering, especially in combustion processes. These multiphase flow problems are classically described by parabolic partial differential equations, each one associated with a set of initial and moving boundary conditions. The evaluation of the processes governing the evaporation of an isolated spherical droplet at rest in a quiescent environment is of fundamental importance in several applications involving sprays. Classically, the equations of the gas and liquid phases of a single evaporating droplet are studied under specific assumptions. The solutions are then coupled at the droplet surface using the local evaporation equilibrium hypothesis. Complex analytical models for the gas phase have been developed by several authors. But, the study of the liquid phase during evaporation has remained under simplifying assumptions. Over the past few decades, various methods have been used by different authors to treat the droplet transient heating problem. Numerical methods (see for example [MIN 96]) such as the finite difference method (FDM), the finite element method (FEM) and the boundary element method (BEM) have been well established and successfully applied to droplets transient heating and evaporation problems [ELW 11]; [MIT 11]. Likewise, a number of approximate analytical models formulated through physical considerations, such as the power law, the

polynomial approximations and the heat balance integral method, have been used as in [SNE 13] and [ZHO 17]. But, analytical solutions to the time-step models of droplet transient heating problem are limited due to the moving boundary condition at the droplet surface.

However, the exact series solution of the symmetrical problem of heating or cooling of a spherical solid body, has been addressed in several research and text books as in [LUI 68] and [BRE 17]. In [LUI 68] for example, the method of separation of variables in one hand, and that of the Laplace integral transform in the other hand, have been used and exact series solutions are obtained for the symmetrical heating/cooling problem of a spherical solid body with prescribed expressions of the ambient temperature. In another physical context, that is in radial mass diffusion problems inside finite-volume spheres, the case of constant gas temperature for small times was treated far ago ([MIN 79]). Since 2004, Sazhin et al. proposed an analytical solution, namely, a convergent series in time and radius for the droplet transient heating problem with a time dependent gas temperature and a constant convection heat transfer coefficient during the time step. This solution has been used for example in [RYB 16] and [RYB 18] for the implementation of mono and multi-component droplets vaporization models. But, to the best of our knowledge, the present analytical approximations in short time limits for a vaporizing droplet surface, center and volume-average temperatures have not yet been proposed, though they are simple enough to minimize computational cost and error in various CFD calculations for droplets and sprays.

In this paper, analytical approximations are proposed for the time-step modeling of spherical droplet evaporation, when assuming a time-varying temperature of the gas phase at the immediate vicinity of the droplet. In the next section, the droplet evaporation process is described in the framework of the time-step analysis, and the Laplace transforms of the droplet surface, center and volume-average (core) temperatures are obtained for the heating problem in Section 3. In Section 4, analytical approximations are obtained for the temperature fields involved, which correspond to the asymptotic expansions of the previous Laplace-domain solutions in short time limits. In Section 5, the dimensionless analogue of the analytical approximations are also deduced, since they can furnish a single numerical procedure for the treatment of vaporizing droplets, regardless of their initial radii. In Section 6, the new solutions are applied to the transient heating and evaporation of typical diesel fuel droplets. They provide a consistent description of the initial heating and the entire evaporation stage. In addition, the results demonstrate significant numerical efficiency compared to those of previous conventional and non-conventional time step models. Finally, Section 7 outlines the conclusions.

## 2. Quasi-steady evaporation

This section describes the spherically symmetric evaporation of a single mono-component droplet suspended in a hot gas environment. Radiation effects are ignored and no relative motion is assumed between the droplet and the gas. The droplet radius  $R_s$  is assumed to remain constant for a short time step  $\Delta t$ , ( $t \in [0, \Delta t]$ ), but eventually decreases at the end of each time step. The droplet energy equation considered during  $\Delta t$ , is stated as:

$$\frac{\partial(RT_l)}{\partial t} - \alpha_l \frac{\partial^2(RT_l)}{\partial R^2} = 0, \quad (1)$$

subject to the following initial and boundary conditions:

$$T_l(R, t = 0) = \overline{T_0} \quad (2)$$

$$\frac{\partial T_l}{\partial R}(R = 0, t) = 0 \quad (3)$$

$$\frac{\partial T_l}{\partial R}(R = R_s, t) = Q_s(t) = \frac{h}{k_l} (T_g(t) - T_s(t)), \quad (4)$$

where  $T_l = T_l(R, t)$  is the temperature to be determined at time  $t$  and distance  $R$  from the center of the fixed droplet. Note that the equation (1), which is the spherically symmetric heat conduction equation, can be recast as:

$$\frac{\partial(RT_l)}{\partial t} - \alpha_l \left( \frac{\partial^2 T_l}{\partial R^2} + \frac{2}{R} \frac{\partial T_l}{\partial R} \right) = 0.$$

(5) As an example of time steps used in liquid fuel combustion modeling, Sazhin et al. in [SAZ 10] considered  $\Delta t = 0,003 \text{ ms} = 3 \times 10^{-6} \text{ s}$  for studying the transient heating of diesel fuel droplets with initial radius  $R_s = 10 \mu\text{m}$ . The authors reported that further reduction in the time steps has practically no effect on the results. In the present model, the volume-average or core temperature  $\bar{T}_0$  is considered inside the droplet at the beginning of each time step. The zero temperature gradient at the droplet center assures the spherical symmetry of the system during the evaporation process. The droplet is surrounded at its immediate vicinity by a thin layer of gas mixture at slow time-evolving temperature  $T_g(t)$ , whose values can be significant with time. The thermal diffusivity of the liquid fuel is denoted by  $\alpha_l = k_l / \rho_l c_l$ , where the specific heat capacity  $c_l$ , the thermal conductivity  $k_l$  and the density  $\rho_l$  of the liquid are treated as constant as well as the corresponding thermophysical properties  $c_g$ ,  $k_g$  and  $\rho_g$  of the gas phase at the immediate vicinity of the droplet ([SIR 10]). Therefore, it is assumed that the convection heat transfer coefficient  $h = k_g / R_s$  is constant during each time step  $\Delta t$  and only changes from one time step to the next. The time-varying temperature  $T_s(t)$  is related to the temperature gradient  $Q_s(t)$  at the droplet surface but,  $Q_s(t)$  and  $T_s(t)$  are not known beforehand, and must be determined as part of the solution of the equations (1)-(4). In order to approach this moving boundary problem, an iterative procedure is required since the actual radius  $R_s = R_s(t)$ , is a time function related to the evolution of the droplet temperature  $T_l(R(t), t)$ , which is the solution to Equation (1) for the whole lifetime of the droplet. Indeed, as mentioned among others in [SNE 13] and [ZHO 17], the droplet surface and core temperature values are sufficient, instead of the whole interior temperature field, to allow estimations of the related heat and mass quantities released at each time step in the Computational Fluid Dynamics (CFD) calculations for sprays.

At the surface, if effects of thermal swelling are ignored during the vaporization, the droplet actual radius  $R_s(t)$  is expected to decrease continuously with time  $t$ . The energy balance condition (4) at the surface can be rewritten as:

$$k_l \frac{\partial T_l}{\partial R}(R = R_s, t) = h(T_g^1(t) - T_s(t)) + \rho_l \ell \dot{R}_s(t), \quad (6)$$

where the dot symbol denotes differentiation with respect to time  $t$  and  $\ell$  is the specific heat of evaporation of the liquid. The convection heat transfer coefficient  $h = h(t)$  is recalled as  $h = k_g / R_s(t)$ , where  $k_g$  is the thermal conductivity of the ambient gas. As mentioned above, this coefficient  $h$  can be treated as a constant during each time step, since the physical properties are assumed to be constant with time in liquid and gas phases. The so-called effective temperature  $T_g$  corresponds to the temperature of the gas phase at a new time step. As in [SAZ 04], it can be written as:

$$T_g = T_g^1 + \frac{\rho_l \ell \dot{R}_s}{h}, \quad (7)$$

where  $\dot{R}_s$  is the derivative of the droplet radius with respect to time and  $T_g^1$  is the estimated value of the gas-phase temperature, both taken from the previous time step. The fuel vapor diffusion from the droplet surface controls the decrease of the moving boundary  $R = R_s(t)$ , and when stationary droplets are involved, the derivative  $\dot{R}_s$  satisfies the following equation ([SAZ 10]):

$$\dot{R}_s = -\frac{k_g \ln(1+B_M)}{\rho_l c_{pg} R_s}, \quad (8)$$

where  $c_{pg}$  is the gas specific heat capacity at constant pressure assumed to be equal to  $c_g$ . Equation (8) is solved by iteration in the framework of the time-step modeling. The Spalding mass transfer number  $B_M$  is calculated as  $B_M = Y_{fs} / (1 - Y_{fs})$ , where  $Y_{fs}$  is the vapor mass fraction on the droplet surface given by the relation:

$$Y_{fs} = \left[ 1 + \left( \frac{P_g}{P_{fs}} - 1 \right) \frac{M_g}{M_f} \right]^{-1}. \quad (9)$$

Here,  $P_g$  is the ambient gas pressure and  $P_{fs}$  the pressure of saturated fuel vapor near the surface of the droplet, while  $M_g$  and  $M_f$  are molar masses of air and fuel. Finally, the pressure  $P_{fs}$  is obtained from the Clausius-Clapeyron equation ([MIT 11]) as:

$$P_{fs} = \exp\left(a - \frac{b}{T_s - 43}\right), \quad (10)$$

where  $T_s(t) = T_l(R_s(t), t)$  is the surface temperature of the droplet and the coefficients  $a$  and  $b$  are some constant related to the fuel physical properties.

### 3. Laplace domain solutions

If  $f(t)$  is a function defined in  $t \geq 0$ , then its unilateral Laplace integral transform (LIT) is given in the complex  $p$ -plane by (see [ANA 21]):

$$F(p) = \mathcal{A}(f(t)) = \int_0^{\infty} f(t) e^{-pt} dt,$$

provided that  $f(t)$  be of exponential order, that is, there are constants  $C$  and  $\sigma$  so that  $|f(t)| < C e^{\sigma t}$ , when  $t$  is sufficiently large. The inversion, from the Laplace domain  $p$  to the time domain  $t$  is given by the complex integral:

$$f(t) = \mathcal{A}^{-1}(F(p)) = \frac{1}{2i\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} F(p) e^{pt} dp,$$

where  $\gamma > \sigma$  is chosen so that  $F(p)$  converges absolutely on the real part of the  $p$  line  $\text{Re}(p) = \gamma$  and  $F(p)$  is analytic at the right of this line.

To find Laplace domain solutions for the liquid-phase temperatures, the unilateral Laplace transform is applied to the problem (1)-(4). The Laplace transform of the gas phase temperature  $T_g(t)$  is denoted by  $\mathcal{A}T_g(p)$ . The surface temperature  $T_s(t)$  and the gradient  $Q_s(t)$  are respectively transformed to  $\mathcal{A}T_s(p)$  and  $\mathcal{A}Q_s(p)$ . The temperature distribution  $T_l(R,t)$  is transformed into  $\mathcal{A}T_l(R,p)$ . In the Laplace domain, the time-domain equations (1)-(4) are transformed into:

$$pR\mathcal{A}T_l(R,p) - \frac{d^2(R\mathcal{A}T_l(R,p))}{dR^2} = R\bar{T}_0, \quad (11)$$

$$\begin{cases} \left. \frac{d\mathcal{A}T_l(R,p)}{dR} \right|_{R=0,p} = 0, \\ \left. \frac{d\mathcal{A}T_l(R,p)}{dR} \right|_{R=R_s,p} = \mathcal{A}Q_s(p) = \frac{h}{k_l}(\mathcal{A}T_g(p) - \mathcal{A}T_s(p)). \end{cases} \quad (12)$$

The initial condition in the Laplace domain is included in the governing equation (11), due to the Laplace transform of the derivative theorem (see [DEB 14]). Taking into account the conditions (12), the solution of the differential equation (11) for the transform  $\mathcal{A}T_l(R,p)$  can be sought under the form of:

$$\mathcal{A}T_l(R,p) - \frac{\bar{T}_0}{p} = C(p) \frac{\sin\left(R\sqrt{\frac{p}{\alpha_l}}\right)}{R} \quad (13)$$

where  $C(p)$  is to be determined. This form of the solution is used in [LUI 68] (see Formula (6.5.21)) for the case of a constant gas temperature. However, it remains valid here as long as the initial temperature of the droplet is uniform, even in the context of a time-varying gas-phase temperature  $T_g(t)$ . The condition at the droplet surface leads to:

$$\left. \frac{d\mathcal{A}T_l(R,p)}{dR} \right|_{R=R_s,p} = \mathcal{A}Q_s(p) = \frac{K}{R_s}(\mathcal{A}T_g(p) - \mathcal{A}T_s(p)), \quad (14)$$

where the dimensionless parameter  $K = k_g/k_l$  is the ratio of gas and liquid conductivities. The coefficient  $C(p)$  is determined from Equation (14) as:

$$C(p) = \frac{KR_s(\mathcal{A}T_g(p) - \bar{T}_0/p)}{\left[ (k-1)\sinh\left(R_s\sqrt{\frac{p}{\alpha_l}}\right) + R_s\sqrt{\frac{p}{\alpha_l}} \cosh\left(R_s\sqrt{\frac{p}{\alpha_l}}\right) \right]}. \quad (15)$$

Therefore, the Laplace domain solution of the original problem (1)-(4) is determined as:

$$\mathcal{A}T_l(R,p) = C(p) \frac{\sinh\left(R\sqrt{\frac{p}{\alpha_l}}\right)}{R} + \frac{\bar{T}_0}{p}. \quad (16)$$

This Laplace domain solution can be equally recovered by applying a combined method of the Fourier sine and the Laplace integral transforms to the original problem (1)-(4), as performed in

[ANA 21]. By substituting  $R=R_s$ , the droplet surface temperature is obtained from relation (16) as:

$$\mathcal{A}T_s(p) = C(p) \frac{\sinh\left(R_s \sqrt{\frac{p}{\alpha_l}}\right)}{R_s} + \frac{\bar{T}_0}{p}. \quad (17)$$

Likewise, by tending  $R$  to 0 in relation (16), we obtain the droplet center temperature as:

$$\mathcal{A}T_c(p) = C(p) \sqrt{\frac{p}{\alpha_l}} + \frac{\bar{T}_0}{p}. \quad (18)$$

The temperature gradient at the droplet surface is expressed by differentiating relation (16) at  $R=R_s$ .

This leads to:

$$\mathcal{A}Q_s(p) = C(p) \frac{\left[ R_s \sqrt{\frac{p}{\alpha_l}} \cosh\left(R_s \sqrt{\frac{p}{\alpha_l}}\right) - \sinh\left(R_s \sqrt{\frac{p}{\alpha_l}}\right) \right]}{R_s^2}, \quad (19)$$

where  $C(p)$  is given by the equation (15). In order to obtain the instantaneous volume-average temperature  $T_0(t)$  of the droplet, we recall that:

$$T_0(t) = \frac{3}{R_s^3} \int_0^{R_s} R^2 T_l(R,t) dR, \quad (20)$$

where the temperature  $T_l$  of a sphere of center  $O$  and radius  $R_s$  is a function only of radius  $R$  and time  $t$ , i.e.,  $T_l \equiv T_l(R,t)$  as in symmetrical problems. Relation (20) is transformed in the Laplace domain into:

$$\mathcal{A}T_0(p) = \frac{3}{R_s^3} \int_0^{R_s} R^2 \mathcal{A}T_l(R,p) dR. \quad (21)$$

Using equations (16) and (21), the instantaneous volume-average temperature  $T_0(t)$  of the droplet is calculated in the Laplace domain as:

$$\mathcal{A}T_0(p) = 3\alpha_l \frac{\mathcal{A}Q_s(p)}{R_s p} + \frac{\bar{T}_0}{p}, \quad (22)$$

where  $\mathcal{A}Q_s(p)$  is given by the equation (19).

In the case of a steady gas-phase temperature, when  $T_g(t)$  is assumed to be constant with time ( $T_g(t) = \bar{T}_g$ ), one has  $\mathcal{A}T_g(p) = \bar{T}_g/p$  and formula (16) reduces to:

$$\mathcal{A}T_l(R,p) = \frac{KR_s(\bar{T}_g - \bar{T}_0) \sinh\left(R \sqrt{\frac{p}{\alpha_l}}\right)}{Rp \left[ (K-1) \sinh\left(R_s \sqrt{\frac{p}{\alpha_l}}\right) + R_s \sqrt{\frac{p}{\alpha_l}} \cosh\left(R_s \sqrt{\frac{p}{\alpha_l}}\right) \right]} + \frac{\bar{T}_0}{p}. \quad (23)$$

As a comparison, this solution (23) is identical to the one obtained by [LUI 68] (see Equation (6.5.24)), for the symmetrical problem of heating or cooling of a spherical solid body. Inverse Laplace transforms can be accomplished numerically, regardless of the complexity of the involved solutions (see among others [DEH 82] and [COH 07]). Nevertheless, the application of formulae (17), (18) and (22) to the time-step vaporization modeling, will need to perform the numerical inverse Laplace transform of these equations over each time step. The computational cost of such a treatment may increase considerably, especially if too small values of time steps  $\Delta t$ , ( $t \in [0, \Delta t]$ ) are considered. Exact series solutions can be sought from the Laplace domain to the time domain by means of convolution and Heaviside's expansion theorems. But, as involved in [SAZ 04], the description by a series solution of droplets transient heating and evaporation in term of the convection heat transfer coefficient might not be adequate at the very initial stage of droplet heating, when the boundary layer around the droplet does not have time to be established. This limitation pushes us to seek for a more improved approximate analytical model, which is more computationally efficient and includes the whole heating stage as well as the evaporation stage of spherical droplets.

#### 4. Analytical solutions in short time limits

In this section, analytical approximations are derived from the previous Laplace domain solutions, for the most important temperatures involved in the time-step analysis. The formulae are valid for any small values of the time step  $\Delta t$ , as practiced in CFD spray modeling. The limiting case of short time duration ( $t$  tending to 0) corresponds to a very large Laplace domain variable ( $p$  tending to  $\infty$ ). Therefore, asymptotic expansions of at least second order can be derived from the above Laplace domain solutions. As the transform  $(\mathcal{A}_{T_g}(p) - \bar{T}_0/p)$  tends to zero when  $p$  tends to  $\infty$ , equations (17), (18) and (22) lead respectively to:

$$\mathcal{A}_{T_s}(p) = K \left( \mathcal{A}_{T_g}(p) - \bar{T}_0/p \right) \left[ \frac{1}{R_s} \left( \frac{\alpha_l}{p} \right)^{1/2} + \frac{(1-K)\alpha_l}{R_s^2 p} + \frac{(1-K)^2}{R_s^3} \left( \frac{\alpha_l}{p} \right)^{3/2} \right] + \frac{\bar{T}_0}{p} + O\left(\frac{1}{p^2}\right); \quad (24)$$

$$\mathcal{A}_{T_c}(p) = 2K \left( \mathcal{A}_{T_g}(p) - \bar{T}_0/p \right) e^{-R_s \sqrt{\frac{p}{\alpha_l}}} \left[ 1 + \frac{(1-K)\alpha_l}{R_s} \left( \frac{\alpha_l}{p} \right)^{1/2} + \frac{(1-K)^2}{R_s^2} \frac{\alpha_l}{p} \right] + \frac{\bar{T}_0}{p} + O\left(\frac{e^{-\sqrt{p}}}{p^{3/2}}\right), \quad (25)$$

and

$$\mathcal{A}_{T_0}(p) = 3K \left( \mathcal{A}_{T_g}(p) - \bar{T}_0/p \right) \left[ \frac{K}{R_s^2} \left( \frac{\alpha_l}{p} \right) - \frac{K^2}{R_s^3} \left( \frac{\alpha_l}{p} \right)^{3/2} + \frac{K^2(K-1)}{R_s^4} \left( \frac{\alpha_l}{p} \right)^2 \right] + \frac{\bar{T}_0}{p} + O\left(\frac{1}{p^{5/2}}\right), \quad (26)$$

where the big  $O(\ )$  is the asymptotic notation.

Due to the convolution theorem of the Laplace transform ([DEB 14]), and to the converse to Watson's Lemma ([WON 89]), the results can be inverted in the time domain by using inverse tables (as in [POU 18]) or Maple software. Thus, analytical approximations of the droplet surface, center and volume-average temperatures at the earliest time of the process  $t \in [0, \Delta t]$  are found as follows:

$$T_s(t) = K \int_0^t (T_g(t-\eta) - \bar{T}_0) \left[ \frac{\sqrt{\alpha_l}}{R_s \sqrt{\pi \eta}} + \frac{\alpha_l(1-K)}{R_s^2} + \frac{2\alpha_l^{3/2}(1-K)^2}{R_s^3} \sqrt{\frac{\eta}{\pi}} \right] d\eta + \bar{T}_0 + O(t'); \quad (27)$$

$$T_c(t) = 2K \int_0^t (T_g(t-\eta) - \bar{T}_0) \left[ \frac{R_s}{2\sqrt{\alpha_l \pi}} \frac{e^{-\frac{\alpha_l}{4R_s^2 \eta}}}{\eta^{3/2}} \right] d\eta + \bar{T}_0 + O\left(e^{-\frac{1}{t'}}\right), \quad (28)$$

and

$$T_0(t) = 3K \int_0^t (T_g(t-\eta) - \bar{T}_0) \left[ \frac{K\alpha_l}{R_s^2} - \frac{2K^2\alpha_l^{3/2}}{R_s^3} \sqrt{\frac{\eta}{\pi}} + \frac{K^2(K-1)\alpha_l^2}{R_s^4} \eta \right] d\eta + \bar{T}_0 + O(t^{3/2}); \quad (29)$$

where  $t^n = f(T_g, T_0, K, \alpha_l, R_s) t^n$  with  $f(T_g, T_0, K, \alpha_l, R_s)$  being the corresponding coefficient of  $t^n$ , and  $n$  the order retained for the asymptotic expansion. Indeed, the decreasing of the droplet radius  $R_s$  toward 0 with time, needs to be taken into account in the application of these formulae in the time-step analysis, as practiced in CFD spray modeling. Therefore,  $R_s$  and all the parameters that intervene in the expansions (24)-(26) need to be appropriately included in the calculation of errors.

In the case of a constant temperature of the gas phase during the time step  $\Delta t$ , then  $T_g(t) = \bar{T}_g$  and the expressions (27)-(29) reduce to the following expressions:

$$T_s(t) = \bar{T}_0 + 2K(\bar{T}_g - \bar{T}_0) \frac{\sqrt{\alpha_l t}}{R_s \sqrt{\pi}} + O(t') \approx \bar{T}_0 + 2K(\bar{T}_g - \bar{T}_0) \frac{\sqrt{\alpha_l t}}{R_s \sqrt{\pi}}; \quad (30)$$

$$\begin{aligned} T_c(t) &= \bar{T}_0 + 2K(\bar{T}_g - \bar{T}_0) \left( 1 - \operatorname{erf}\left(\frac{R_s}{2\sqrt{\alpha_l t}}\right) \right) + O\left(e^{-\frac{1}{t'}}\right) \\ &= \bar{T}_0 + 4K(\bar{T}_g - \bar{T}_0) \frac{\sqrt{\alpha_l t}}{R_s \sqrt{\pi}} e^{-\frac{R_s^2}{4\alpha_l t}} + O\left(e^{-\frac{1}{t'}}\right) \\ &\approx \bar{T}_0 + 4K(\bar{T}_g - \bar{T}_0) \frac{\sqrt{\alpha_l t}}{R_s \sqrt{\pi}} e^{-\frac{R_s^2}{4\alpha_l t}} \end{aligned} \quad (31)$$

and

$$T_0(t) = \bar{T}_0 + 3(\bar{T}_g - \bar{T}_0) \frac{K^2 \alpha_l}{R_s^2} t + O(t^{3/2}) \approx \bar{T}_0 + 3(\bar{T}_g - \bar{T}_0) \frac{K^2 \alpha_l}{R_s^2} t, \quad (32)$$

where erf is the error function defined as:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz.$$

The time-domain expansions (30)-(32) are directly derivable from formulae (24)-(26) after replacing  $\Delta T_g(p)$  by  $\overline{T_g}/p$ . As expected, the analytical approximations (30)-(32) of droplet surface, center and volume-average temperatures reduce all together to  $\overline{T_0}$  at  $t = 0$ . It is also remarkable that there exists a good match between the dimensions of the parameters involved in both sides of these approximations, since  $K$  is a dimensionless parameter. The big  $o(\ )$  notation can be replaced by the small asymptotic notation  $o(\ )$  in equations (28) and (31), which are valid for arbitrary order  $n \geq 1$  of the truncated expansions. The two latter truncation errors can be refined at any convenient value by increasing the order of the related time domain expansions. In the case of constant gas-phase temperature  $T_g(t) = \overline{T_0}$  during the time step  $\Delta t$ , the next order truncated expansions are calculated respectively as ( $T_c(t)$  is unchanged):

$$T_s(t) \approx \overline{T_0} + 2K(\overline{T_g} - \overline{T_0}) \frac{\sqrt{\alpha_l t}}{R_s \sqrt{\pi}} + K(1-K)(\overline{T_g} - \overline{T_0}) \frac{\alpha_l t}{R_s^2}; \quad (33)$$

$$T_c(t) \approx \overline{T_0} + 4K(\overline{T_g} - \overline{T_0}) \frac{\sqrt{\alpha_l t}}{R_s \sqrt{\pi}} e^{-\frac{R_s^2}{4\alpha_l t}}, \quad (34)$$

and

$$T_0(t) \approx \overline{T_0} + 3(\overline{T_g} - \overline{T_0}) \frac{K^2 \alpha_l}{R_s^2} t + 4K^3(\overline{T_g} - \overline{T_0}) \frac{(\alpha_l t)^{3/2}}{R_s^3 \sqrt{\pi}}. \quad (35)$$

Now, considering the first order expansion, it can be written that  $f(T_g, T_0, K, \alpha_l, R_s) = (\overline{T_g} - \overline{T_0}) K(1-K) \alpha_l / R_s^2$  in the evaluation of the surface temperature  $T_s(t)$ . This implies an absolute error less than  $\delta t' = (\overline{T_g} - \overline{T_0}) K(1-K) \alpha_l \Delta t / R_s^2$  in the evaluation of the surface temperature by formula (30). This absolute error is less than  $\delta t' = 3(\overline{T_g} - \overline{T_0}) K^2 \alpha_l \Delta t / R_s^2$  in the core temperature  $T_0(t)$  evaluation by the equation (32). The order of magnitude of the time step  $\Delta t$  is typically  $10^{-6}$  s for internal combustion engines as mentioned in Section 2. Therefore, considering the order of magnitude of the parameters involved in the expression of  $\delta t'$ , the approximate solutions (30)-(32) will be retained here for further numerical simulations. Indeed, they are sufficiently robust and can be assumed to be insensitive to larger expansion orders, even for very small values of the decreasing radius as  $R_s = 10^{-6}$   $\mu m$ .

This point of view can be verified in the numerical applications, by comparing the curves obtained when using the system (30)-(32) to those obtained from the greater order system (33)-(35). All the temperature curves obtained from the two different systems are identical and all solutions are virtually on top of each other.

## 5. Dimensionless formulas

Dimensionless forms of formulae (30)-(32) will give greater insight into the solutions. In order to allow further comparisons, we consider as in Mitchell et al. (2011)  $\Delta T = \ell / c_l$ , where  $\ell$  is recalled as the specific heat of evaporation of the liquid and we write:

$$\left\{ \begin{array}{l} r_s = \frac{R_s}{R_s^0}, \tau = \frac{t}{(R_s^0)^2 / \alpha_l}, \bar{\theta}_g = \frac{(\bar{T}_g - \bar{T}_0^0) - (\bar{T}_0 - \bar{T}_0^0)}{\Delta T} = \frac{(\bar{T}_g - \bar{T}_0^0)}{\Delta T}, \\ \theta_s = \frac{(T_s - \bar{T}_0^0)}{\Delta T}, \theta_c = \frac{(T_c - \bar{T}_0^0)}{\Delta T}, \theta_0 = \frac{(T_0 - \bar{T}_0^0)}{\Delta T}, \bar{\theta}_0 = \frac{(\bar{T}_0 - \bar{T}_0^0)}{\Delta T}, \end{array} \right. \quad (36)$$

where  $r_s$  is the dimensionless radius of the evaporating droplet,  $R_s^0$  is the initial radius of the droplet,  $\bar{T}_0^0$  and  $\bar{T}_0$  are respectively the initial temperature of the droplet and its core temperature at the beginning of a new time step  $\Delta t$ . The dimensionless temperatures  $\theta_s, \theta_c, \theta_0, \bar{\theta}_g$  and  $\bar{\theta}_0$  correspond respectively to the dimensional temperatures  $T_s, T_c, T_0, \bar{T}_g$  and  $\bar{T}_0$ . Then, the dimensionless analogue of the analytical approximate solutions (30)-(32) are written as:

$$\theta_s(t) = \bar{\theta}_0 + 2K \frac{\bar{\theta}_g}{r_s} \sqrt{\frac{\tau}{\pi}} + O(\tau') \approx \bar{\theta}_0 + 2K \frac{\bar{\theta}_g}{r_s} \sqrt{\frac{\tau}{\pi}} \quad (37)$$

$$\theta_c(t) = \bar{\theta}_0 + 4K \frac{\bar{\theta}_g}{r_s} \sqrt{\frac{\tau}{\pi}} e^{-\frac{r_s^2}{4\tau}} + O\left(e^{-\frac{1}{\tau'}}\right) \approx \bar{\theta}_0 + 4K \frac{\bar{\theta}_g}{r_s} \sqrt{\frac{\tau}{\pi}} e^{-\frac{r_s^2}{4\tau}} \quad (38)$$

and

$$\theta_0(\tau) = \bar{\theta}_0 + 3\bar{\theta}_g K^2 \frac{\tau}{r_s^2} + O(\tau^{3/2}) \approx \bar{\theta}_0 + 3\bar{\theta}_g K^2 \frac{\tau}{r_s^2}. \quad (39)$$

As for approximations (33)-(35), it can be written at the first expansion order that  $\tau' = K(1-K)\tau\bar{\theta}_g/r_s^2$  in the evaluation of  $\theta_s$ , and  $\tau' = 3K^2\tau\bar{\theta}_g/r_s^2$  in that of  $\theta_0$ . The absolute error in the estimations (37) and (39) are respectively less than  $\delta\tau' = K(1-K)\bar{\theta}_g\Delta\tau/r_s^2$  and  $\delta\tau' = 3K^2\bar{\theta}_g\Delta\tau/r_s^2$ , where  $\Delta\tau' = \alpha_l\Delta t / (R_s^0)^2$  can be considered as the non-dimensional time step corresponding to the time step  $\Delta t$  and the droplet initial radius  $R_s^0$ . However, attention needs to be paid to the criteria of convergence in the application of equations (37)-(39) to the time step analysis in droplets and sprays vaporization models. Indeed, due to the magnitude of the droplet initial radius  $R_s^0$ , the value of the non-dimensional time-step,  $\Delta\tau' = \alpha_l\Delta t / (R_s^0)^2$ , used in the application of equations (37)-(39), can be too large even if the dimensional time-step  $\Delta t$  is fixed small. For certain droplets, it is clear that  $\Delta\tau' = \alpha_l\Delta t / (R_s^0)^2 \gg \Delta t$  needs to be refined if  $R_s^0$  is too small. As an example, for typical diesel fuel droplets at initial radius  $R_s^0 = 5 \times 10^{-6} m$  and thermal diffusivity  $\alpha_l = 2.381 \times 10^{-8} m^2/s$ , calculations give  $\Delta\tau = 952.4\Delta t \gg \Delta t$ . Since  $\Delta\tau$  depends on  $\Delta t$ , its convenient value can be obtained by using a sufficiently small intermediate value  $\Delta t_i$  corresponding to a dummy dimensional time step. This issue will be briefly discussed in the application of the results (see comments on figure 3 in Section 6).

## 6. Application

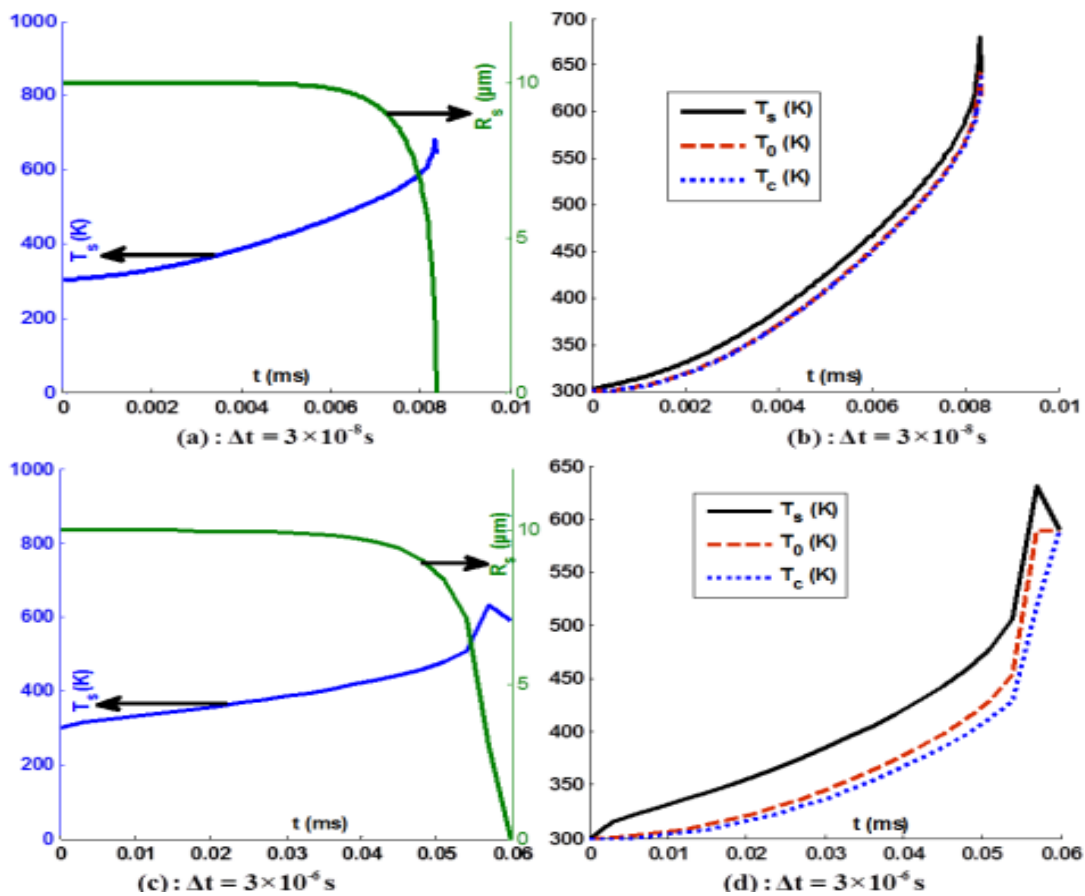
The theory developed in the precedent sections is applied to a specific problem of heating fuel droplets in diesel engines. The solutions (27) and (29) or their analogue (30) and (32) are expected to be used at the earliest time of the vaporization process and at each time step in the numerical code. The values of temperature predicted by those equations at the end of the previous time step are used as the initial values for the following time step, with updated values of input parameters (e.g. droplet radius  $R_s$ , mean temperature  $\overline{T_0}$  and gas-phase temperature  $T_g$ ). Based on the same input data as in [SAZ 10] and [MIT 11], the results of the present approximate analytical model are discussed and compared with those of classical, conventional and non-conventional time-step approaches reported in the above-mentioned references. The model input data, as reported in the above-mentioned references are summarized in Table 1. The superscript 0 indicates the initial value of the related parameter. The model formed by the system of equations (30) and (32) will be applied.

Parameters	Typical value	Units
$\overline{T_0}^0$	300	K
$\overline{T_g}^0$	1000	K
$\rho_l$	700	$kg\ m^{-3}$
$c_l$	2400	$J\ kg^{-1}\ K^{-1}$
$c_g$	1600	$J\ kg^{-1}\ K^{-1}$
$k_g$	0.03	$W\ m^{-1}\ K^{-1}$
$k_l$	0.04	$W\ m^{-1}\ K^{-1}$
$\ell$	$3 \times 10^5$	$J\ kg^{-1}$
$M_f$	0.180	$kg\ mol^{-1}$
$M_g$	0.029	$kg\ mol^{-1}$
$P_g$	$3 \times 10^6$	$Pa$
$a$	15.5274	–
$b$	5383.59	–

**Table 1.** Model input data, as used in [SAZ 10] and in [MIT 11]

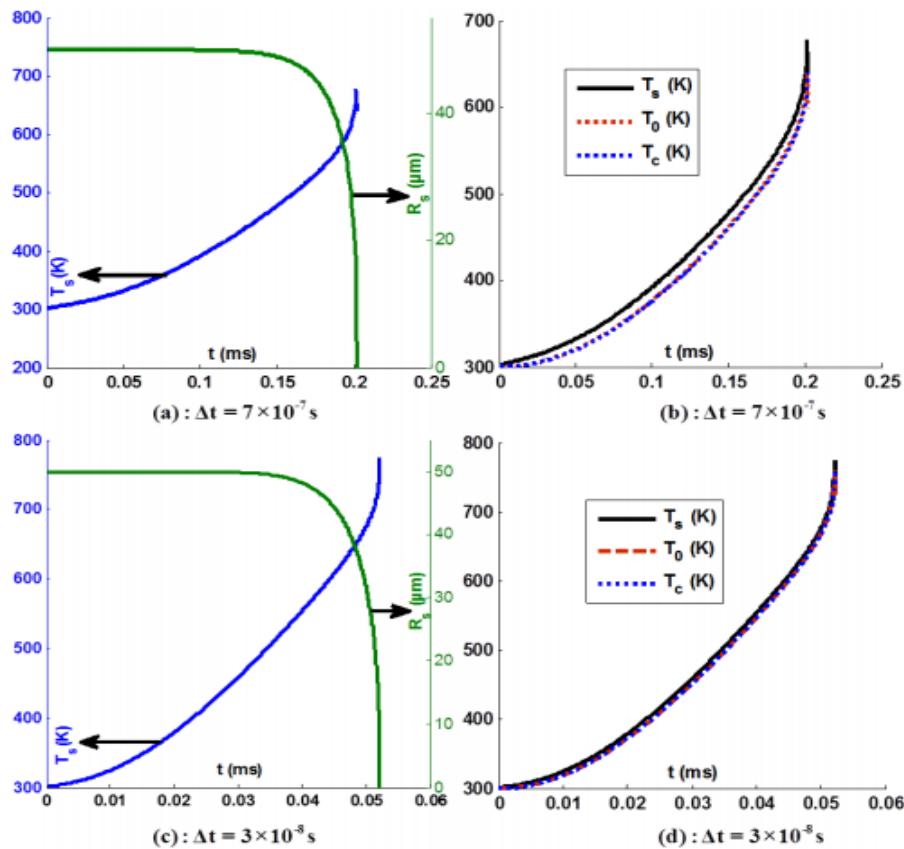
Figure 1 shows effects of the application of equations (30) and (32) to the transient heating of diesel fuel droplets at initial radius  $R_s^0 = 10\ \mu m$ , and for initial gas temperature  $T_g^0 = 1000\ K$ .

With the same time step  $\Delta t = 3 \times 10^{-6} s$  and droplet radius  $R_s^0 = 10 \mu m$ , the droplet lifetime  $t_{end} \approx 0.06 ms$  predicted on figure 1(c) is far reduced from that obtained on figure 5(b) in [SAZ 10], namely  $t_{end} \approx 0.06 ms$ . Thus, the determination of a convenient time step  $\Delta t$  in the present study turns out to be crucial for an efficient numerical procedure. After many attempts of time steps in a decreasing order beginning from  $\Delta t = 3 \times 10^{-6} s$ , we retained for this droplet, a convenient time-step to be equal to  $\Delta t = 3 \times 10^{-8} s$ . This value is proven to not leading to numerical instabilities and assures a better computational efficiency of the procedure. At this convenient value ( $\Delta t = 3 \times 10^{-8} s$ ), a sensible reduction down to  $0.0082 ms$ , i.e., a decrease about 86% can be noted in the droplet total lifetime as shown on figure 1(a). Moreover, the surface temperature curve on figure 1(a) exhibits a maximum value almost equal to 680 K according to this curve data obtained using Matlab. This value of the maximal surface temperature can be roughly considered in agreement with the values of 685.68 K obtained in [MIT 11] and that of 685.29 K obtained in [SAZ 10]. In fact, the maximum value of the surface temperature predicted by the present model is still lower by only 5 K than the referenced value about 685 K, since, at the beginning of each new time step, the core or volume average temperature is considered inside the droplet instead of the actual internal temperature field. On the contrary, this referenced value of the maximal surface temperature is largely under-predicted on figures 1(c) and 1(d), where the former time step  $\Delta t = 3 \times 10^{-6} s$  is used for the curves. Note also, when using the convenient time-step  $\Delta t = 3 \times 10^{-8} s$ , that the curve of the droplet center is almost merged with that of the core temperature, as it can be seen on figure 1(b).



**Figure 1.** Influence of  $\Delta t$  on computations of evaporating diesel fuel droplets for  $R_s^0 = 10 \mu m$  and  $\bar{T}_g^0 = 1000 K$ .

Figure 2 confirms the effects of the new approach, i.e., the application of the system of equations (30) and (32) to droplets evaporation model, provided that a convenient time step is used for the curves. Indeed, using again the above-mentioned time step  $\Delta t = 3 \times 10^{-8} s$  for a larger droplet at initial radius  $R_s^0 = 50 \mu m$ , the maximal value of the droplet surface temperature, as mentioned in the comments of figure 1 ( $T_s^{\max} \approx 680 K$ ), is largely over-predicted as shown on figures 2(c) and 2(d). It may be concluded that this time step ( $\Delta t = 3 \times 10^{-8} s$ ) leads to numerical instabilities. Therefore, a suitable time step  $\Delta t$  should increase with the droplet initial radius value. After a series of attempts, a convenient time step for the transient heating of diesel fuel droplets, at initial radius  $R_s^0 = 50 \mu m$  and in the related prescribed conditions, is found to be roughly equal to  $8 \times 10^{-7} s$ .

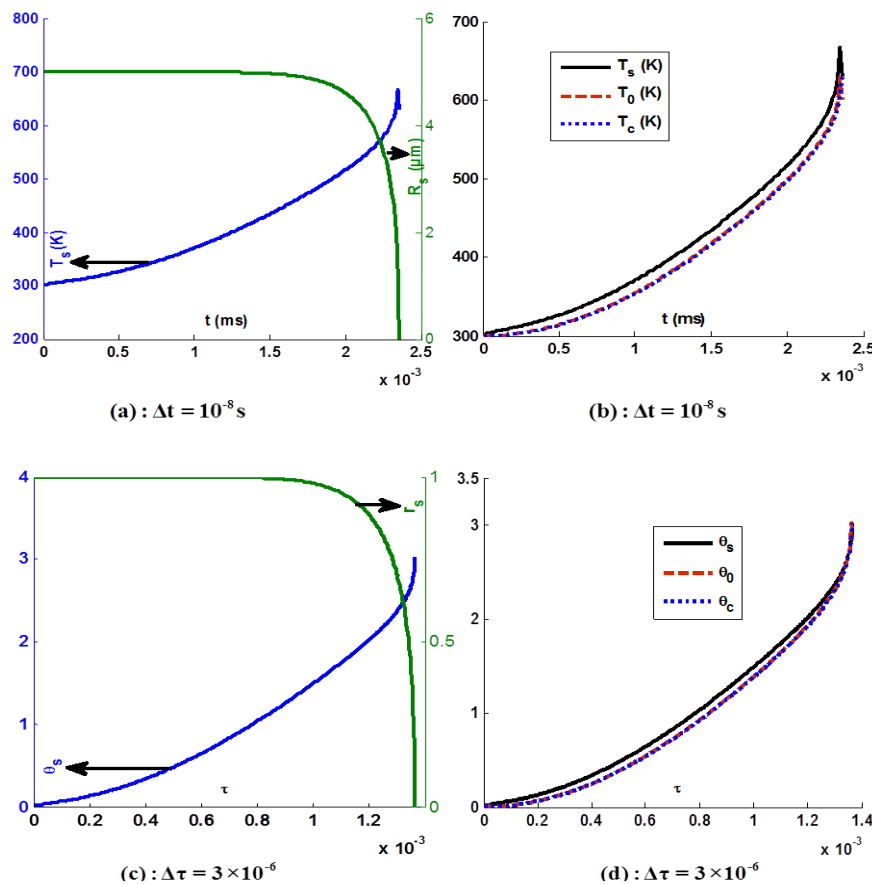


**Figure 2.** Influence of  $\Delta t$  on computations of evaporating diesel fuel droplets for  $R_s^0 = 50 \mu m$  and  $T_g^0 = 1000 K$

So, by considering  $\Delta t = 8 \times 10^{-7} s$ , the value of the droplet surface maximal temperature ( $T_s^{\max} \approx 680 K$ ) is relatively well predicted as shown on figures 2(a) and 2(b). Moreover, the droplet lifetime predicted by the present analytical model ( $t_{end} = 0.2 ms$ ) as shown by the radius curve on figure 2(a), remains far lower than those predicted by the conventional and nonconventional time step models ( $t_{end} \approx 62 ms$ ) as illustrated on figure 6(b) in [SAZ 10].

On figure 3, temperature and radius curves of initially small evaporating diesel fuel droplets ( $R_s^0 = 5 \mu m$ ) are presented (see figures 3(a) and 3(b)). Then, based on the formulae (37) and (39)

established in Section 5, the dimensionless analogue of figures 3(a) and 3(b) are also represented (confer respectively figures 3(c) and 3(d)). For such small droplets, which are often encountered in diesel-fueled engines, it is possible to refine the dimensional time step unto  $\Delta t = 10^{-8} s$  without causing numerical divergence or instabilities. Note that, for attempted values slightly above or under  $\Delta t = 10^{-8} s$ , the droplet surface maximal temperature ( $T_s^{\max} \approx 680 K$ ) as highlighted in the above comments of figure 2(a), was either largely under-predicted or over-predicted by the related temperature curves. Now, in order to perform valid graphical illustrations by keeping a regular time step for the dimensionless curves, it is sufficient to calculate the intermediate time step  $\Delta t_i$ , which will lead to a value of the dimensionless time step  $\Delta \tau = 3 \times 10^{-6}$ . The latter is proven to be convenient for non-dimensional representations of droplets temperatures and radius curves, since the dimensionless initial radius  $r_s^0 = 1$  can be considered large enough to avoid numerical divergence towards the end of the droplet lifetime. Thus, from the relation  $\Delta \tau = \alpha_l \Delta t_i / (R_s^0)^2 = 3 \times 10^{-6}$ , it can be deduced that  $\Delta t_i = (R_s^0)^2 \Delta \tau / \alpha_l \approx 3.15 \times 10^{-9} s$  for the involved values of radius  $R_s^0$  and liquid thermal diffusivity  $\alpha_l$  (see Table 1). Then, the non-dimensional surface temperature  $\theta_s$  exhibits a concave profile on figures 3(c) and 3(d), contrary to the convex profiles shown by the numerical, the conventional and the Sazhin et al. non-conventional solutions as represented on figures 2 and 3 from [MIT 11]. This concave profile suggests a more rapid heat-up and evaporation time, and corroborates the exponential reduction noted in the droplet lifetime. For comparisons, the convex profile shown by the surface temperature curve in former time-step models suggests rather a logarithmic or slow increase in the temperature.



**Figure 3.** Dimensional and dimensionless diagrams of evaporating diesel fuel droplets for  $R_s^0 = 5 \mu m$  and  $\overline{T}_g^0 = 1000 K$ .

The maximal dimensional temperature of the droplet surface at the end of the evaporation process, can be evaluated by using the data statistics of the surface temperature curve on figure 3(c). Noticing that  $\theta_s^{\max} \approx 3.03$ , will imply due to formulae (36), that  $T_s^{\max} \approx 680 K$  as shown on figures 1(a) and 1(b). As it can be seen on figure 3(c), the dimensionless radius curve needs not to be truncated during the numerical procedure contrary to the profiles shown on figures 4 and 5 from [MIT 11]. In brief, the use of dimensionless variables as performed above can be generalized to the time-step models of droplets evaporation, regardless of the initial radii values of the involved droplets. The present approximate analytical model leads to the prediction of temperature profiles and droplet lifetime far reduced, compared with those predicted by conventional and non-conventional former approaches. Comparisons with figures obtained in [SAZ 10], [MIT 11] and even with those derived from the more recent Discrete Component Model, the so-called DCM models (see [ELW 16] for example), show significant discrepancies. The differences noted in the spread of the curve profiles are significant and cannot be mitigated by the choice of the time step. Contrary to former models, the present model includes a consistent description of the very initial stage of droplet heating as well as the whole evaporation stage. Indeed, the heating without evaporation stage in the above-prescribed conditions, remains almost 50% of the total droplet lifetime as shown by radius curves on figures 1(a), 2(a) and 3(a). Although the new approach includes a complete heating stage without evaporation in evaluating fuel droplet vaporization, the related numerical procedure is still computationally efficient compared to the former models. Only a few elementary operations are performed during each time step. However, the use of dimensionless variables in droplets heating and evaporation models may be recommended by using the approximate analytical solutions (37)-(39), as performed above. As shown on figures 3(c) and 3(d), this system leads to consistent results with a fixed non-dimensional time step  $\Delta\tau = \alpha_l \Delta t_i / (R_s^0)^2 = 3 \times 10^{-6}$ . Droplets transient heating and vaporization are treated as an iterative time step procedure in almost all CFD codes for droplets and sprays.

## 7. Conclusion

In this study, explicit Laplace domain solutions are obtained for the spherically symmetric droplet evaporation problem. Then, analytical approximations in short time limits for the droplet surface, center and volume-averaged temperatures are derived from the asymptotic expansions of the Laplace domain solutions. The new analytical approximations are applied to the modeling of fuel droplet evaporation in diesel engine-like conditions. The model leads to the prediction of droplet lifetime far reduced from those of former approaches. The new approximate analytical solutions are sufficiently simple to be considered for implementation in spray CFD codes. They can also be used as intermediate results for the validation of new numerical methods in fuel droplets combustion models. The governing equations for heat and mass transfer have identical forms. Therefore, provided that the boundary and initial conditions are the same, the solutions obtained for the former are applicable to the latter. The present model can later be generalized to multi-component droplet heating and evaporation, based on the dimensionless analytical approximations retained for the heat transfer and their analogue to species diffusion equations. Droplet evaporation characteristics have a wide range of applications in engineering. Among others, one can cite the application to internal combustion engines, gas turbines and fire suppression.

## Bibliography

- [ANA 21] ANANI K., “Combined method of integral transforms for the spherically symmetric droplet heating problem”, *Advances in Mathematics: Scientific Journal*, vol. 10, p. 3141–3164, 2021.
- [BRE 17] BRENN G., *Analytical Solutions for Transport Processes*, Mathematical Engineering, Springer, Berlin, 2017.
- [COH 07] COHEN A. M., *Numerical Methods for Laplace Transform Inversion*, Springer Science & Business Media, New York, 2007.

- [DEH 82] DE HOOG F. R., KNIGHT J. H., STOKES A. N., “An improved method for numerical inversion of Laplace transforms”, *SIAM Journal on Scientific and Statistical Computing*, vol. 3, p. 357–366, 1982.
- [DEB 14] DEBNATH L., BHATTA D., *Integral Transforms and Their Applications*, CRC Press, Boca Raton, 2014.
- [ELW 11] ELWARDANY A. E., GUSEV I. G., CASTANET G., LEMOINE F., SAZHIN S. S., “Mono- and multicomponent droplet cooling/heating and evaporation: comparative analysis of numerical models”, *Atomization and Sprays*, vol. 21, n° 11, p. 907–931, 2011.
- [ELW 16] ELWARDANY A. E., SAZHIN S. S., IM H. G., “A new formulation of physical surrogates of FACE A gasoline fuel based on heating and evaporation characteristics”, *Fuel*, vol. 176, p. 56–62, 2016.
- [LUI 68] LUIKOV A. V., *Analytical Heat Diffusion Theory*, Academic Press, New York, 1968 (re-edited in 2012).
- [MIN 79] MINKOWYCZ W. J., *The Mathematics of Diffusion*, University Press, Oxford, 1979.
- [MIN 96] MINKOWYCZ W. J., *Advances in Numerical Heat Transfer*, CRC Press, Boca Raton, 1996.
- [MIT 11] MITCHELL S. L., VYNNYCKY M., GUSEV I. G., SAZHIN S. S., “An accurate numerical solution for the transient heating of an evaporating spherical droplet”, *Applied Mathematics and Computation*, vol. 217, n° 22, p. 9219–9233, 2011.
- [POU 18] POULARIKAS A. D., *Handbook of Formulas and Tables for Signal Processing*, CRC Press, Boca Raton, 2018.
- [RYB 16] RYBDYLOVA O., AL QUBEISSI M., BRAUN M., CRUA C., MANIN J., PICKETT L. M., DE SERCEY G., SAZHINA E. M., SAZHIN S. S., HEIKAL M., “A model for droplet heating and its implementation into ANSYS Fluent”, *International Communications in Heat and Mass Transfer*, vol. 76, p. 265–270, 2016.
- [RYB 18] RYBDYLOVA O., POULTON L., AL QUBEISSI M., ELWARDANY A. E., CRUA C., KHAN T., SAZHIN S. S., “A model for multi-component droplet heating and evaporation and its implementation into ANSYS Fluent”, *International Communications in Heat and Mass Transfer*, vol. 90, p. 29–33, 2018.
- [SAZ 04] SAZHIN S. S., KRUTITSKII P. A., ABDELGHAFAR W. A., MIKHALOVSKY S. V., MEIKLE S. T., HEIKAL M. R., “Transient heating of diesel fuel droplets”, *International Journal of Heat and Mass Transfer*, vol. 47, p. 3327–3340, 2004.
- [SAZ 10] SAZHIN S. S., KRUTITSKII P. A., GUSEV I. G., HEIKAL M. R., “Transient heating of an evaporating droplet”, *International Journal of Heat and Mass Transfer*, vol. 53, n° 13–14, p. 2826–2836, 2010.
- [SIR 10] SIRIGNANO W. A., *Fluid Dynamics and Transport of Droplets and Sprays*, 2nd ed., Cambridge University Press, Cambridge, 2010.
- [SNE 13] SNEGIREV A. Y., “Transient temperature gradient in a single-component vaporizing droplet”, *International Journal of Heat and Mass Transfer*, vol. 65, p. 80–94, 2013.
- [WON 89] WONG R., *Asymptotic Approximations of Integrals*, Academic Press, New York, 1989.
- [ZHO 17] ZHOU Z. F., LI W. Y., CHEN B., WANG G. X., “A 3rd-order polynomial temperature profile model for the heating and evaporation of moving droplets”, *Applied Thermal Engineering*, vol. 110, p. 162–170, 2017.