Limits of Landauer's principle and of the second law of thermodynamics

Limites du principe de Landauer et du second principe de la thermodynamique

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ABSTRACT. Since 1857, the second law of thermodynamics has faced the challenge of Maxwell's imagined demon. The widely accepted response to this challenge suggests that the demon requires a bit of information to work, and according to Landauer's principle, the erasure of this bit must offset the entropy reduction achieved by the demon. Recent experiments involving two-state physical systems subject to thermal fluctuations at the nanoscale have aimed to either prove Landauer's principle or to demonstrate Szilard engines or Maxwell's demons in practice. We wrote the equations and developed a numerical model to simulate the evolution of these systems. The results highlight the distinction between thermodynamic entropy and information entropy. They demonstrate that Landauer's principle has a limited range of applicability and that, using a two-state memory, it is possible to eliminate a small amount of entropy without expending energy—challenging the second law of thermodynamics at the nanoscale.

RÉSUMÉ. Depuis 1857, le second principe de la thermodynamique est confronté au défi d'un démon, imaginé par Maxwell, qui serait capable de diminuer l'entropie d'un gaz à l'aide d'une information sur son état. La réponse généralement admise à ce défi suggère que l'effacement de cette information compenserait, selon le principe de Landauer, la réduction d'entropie obtenue par le démon. Des expériences récentes portant sur des systèmes physiques à deux états, soumis à des fluctuations thermiques à l'échelle nanoscopique, ont cherché soit à prouver le principe de Landauer, soit à réaliser une machine de Szilard ou un démon de Maxwell. Nous avons écrit les équations et développé un modèle numérique permettant d'observer et de comprendre l'évolution de ces systèmes. Les résultats montrent que l'entropie thermodynamique et l'entropie d'information ne sont pas équivalentes. Ils démontrent également que le principe de Landauer a un domaine d'application limité et que, grâce à une mémoire à deux états, il est possible d'éliminer une faible quantité d'entropie sans dépenser d'énergie, ce qui constitue une violation locale du second principe de la thermodynamique à l'échelle nanoscopique.

KEYWORDS. entropy, information, Landauer's principle, Maxwell's demon, second law of thermodynamics, Szilard engine.
MOTS-CLÉS. entropie, information, principe de Landauer, démon de Maxwell, second principe de la thermodynamique, machine de Szilard.

1. Introduction

Ludwig Boltzmann formulated the statistical version of the second law of thermodynamics, which he expressed in 1877 with the equation $H = k_B \log W$, where H is the entropy of a system, W is its statistical weight, and $k_B = 1.38 \times 10^{-23}$ J/K. A few years earlier, James Clerk Maxwell had developed the kinetic theory of gases and envisioned a demon capable of violating the second law [1]. In 1929, Leo Szilard [2] proposed a thought experiment involving an engine in which the entropy of a single molecule is reduced without expending energy, after detecting the molecule's position. According to John von Neumann [3], it is the knowledge of the molecule's position that enables an operator to achieve this reduction in entropy—a hypothesis that preserves the validity of the second law by establishing a link between information and entropy. In 1962, Rolf Landauer [4], building on Boltzmann's formula, asserted that erasing a bit of information irreversibly generates an entropy of at least $k_B \log 2$ — a principle that has since gained wide acceptance.

Transient violations of the second law have long been observed in small systems, and it has been shown that they are consistent with fluctuation theorems [5] which apply to systems driven in a time-reversed manner. These apparent violations vanish when the results are averaged over many repeated identical experiments.

Since the 2010s, numerous experiments have been conducted at very low energy levels —on the order of $k_B T$ at temperature T — to support Landauer's principle, implement a Szilard engine or a Maxwell's demon, or explore the thermodynamics of two-state physical systems, which can be regarded as one-bit memories. These systems fall into two categories: bistable memories [6-11] characterized by an energy landscape with two minima, and tilt memories [12-13] , which operate as all-or-nothing systems. In both cases, an actuator is used to alter the system's potential energy. The stability of these memories is limited by thermodynamic fluctuations or quantum tunneling, which induce random transitions between states.

In a first article [14], we proposed an exploratory approach to several experiments, along with a brief overview of the historical connection between information and entropy. In a second article [15], we formulated the equations governing the quasistatic and out-of-equilibrium evolution of these systems, assuming the transition frequency is known as a function of the potential difference between the two states. We also suggested explaining apparent violations of the second law through a hypothesis on the temporality of entropy formulated by Landau and Lifshitz.

In the present article, we analyse recent experiments through two differents methods. First by computing the out-of-equilibrium evolution of bistable memories using the Langevin equation. Second, by directly derivating the equations for the quasistatic evolution of both bistable and tilt memories, starting from Gibbs' formula for statistical entropy. Our analysis shows that thermodynamic entropy and information entropy do not always align and therefore cannot be regarded as equivalent. Landauer's limit emerges in out-of-equilibrium processes but can be circumvented by slowing them to quasistatic regimes. We examined the arguments Landauer used to support his principle and identified a shortcoming in his reasoning. Finally, we analysed the behavior of these memory systems and the Szilard engines and Maxwell's demons they can be used to implement, leading us to reveal a limitation of the second law at the nanoscale.

In the following we use k_B as the unit of entropy and $k_B T$ as the unit of energy, unless otherwise indicated.

2. Reset-to-zero operation of a bistable memory

In his 1961 paper [4], Landauer primarily referred to a bistable memory defined by a symmetric energy landscape, with two potential wells separated by a barrier that ensures stability (Fig. 1, phase 1, and Fig. 7 at t = 0 in Appendix A1). This type of memory has been realized, for example, by Bérut et al. [6] and Jun et al. [8], who used a colloidal particle in a liquid, subjected to a potential landscape created by an optical tweezer in the first study and by an electric field in the second. The particle's movement is influenced by four factors: Brownian motion, viscosity, gradient and temporal variation of the potential.

We have thoroughly analysed the work of Jun et al., which yielded highly accurate results. The study began with a numerical simulation based on the Langevin equation to model the out-of-equilibrium evolution. The potential landscape $U(x,t)$ applied to the particle is deterministic, but Brownian motion introduces considerable stochasticity to the system. We calculated the mean values of the variables over a large number of experiments, including the particle's energy $U_s(t)$, the entropy $S(t)$, and the probability $P(t)$ of state 1. The work W done by the actuator and the heat Q transferred from the thermostat were derived from the changes in U_s and S, with both quantities defined up to a constant. The simulation results align closely with the experimental data, demonstrating high accuracy.

We have developed a second approach by applying Gibbs' formula for statistical entropy to derive the equations governing any quasistatic evolution of the system, allowing us to directly obtain the mean values of the variables. Both methods are described in detail in Appendix.

They yield the same temporal evolution of the variables in quasistatic processes with high accuracy and produce identical results to those observed in the experiments of Jun et al.[8] for both quasistatic and out-of-equilibrium processes. However, we disagree with their interpretation of the results, which will be discussed later.

In their experiment, the range of the potential is very large (from 13 to −40). The corresponding calculations are reported in Appendices A1 and A2. In contrast, we propose applying the equations to a simpler protocol where the potential varies between 0 and 13 and consists of three phases instead of four. This approach yields results that are much easier to observe and interpret. Both methods were applied to this protocol (Fig. 1). The reset-to-zero occurs over three phases of equal duration. With a total duration of 1200 s, we obtained the same result as Jun et al.'s experiment, which had a duration of 940 s. The initial potential landscape is a double parabolic well. First, the barrier is symmetrically lowered to create a single well with a flat bottom. This flat well then evolves into a parabolic well with a horizontal segment on the right. Finally, the horizontal segment is transformed into a parabolic well, returning the system to the initial double-well landscape.
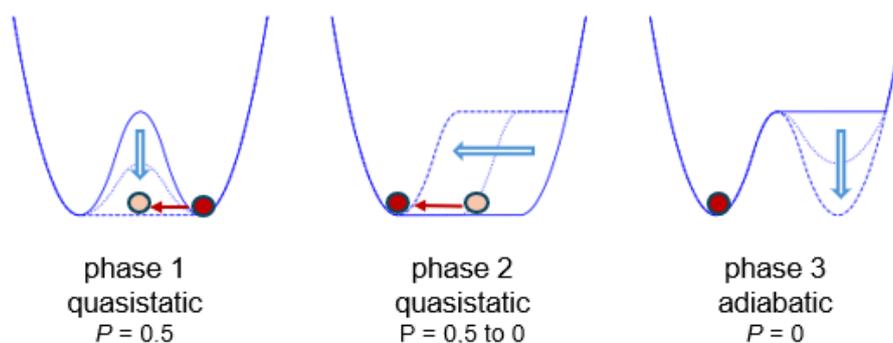

phase 1	phase 2	phase 3
quasistatic	quasistatic	adiabatic
$P = 0.5$	$P = 0,5$ to 0	$P = 0$

Fig. 1. *Reset-to-zero operation of a bistable memory*

At the start of phase 1, the memory state is randomly set to either 0 or 1 (indicating the particle is in the left or right well). For this example, we represent the initial state as 1. During phase 1, the barrier is lowered to create a flat-bottomed landscape, allowing the particle to move freely in a single well. In phase 2, the right side of the flat well is moved to the left, pushing the particle and confining it to a newly formed left well. Finally, in phase 3, the right well is restored while the particle remains in the left well.

Fig. 2 shows the evolution of the key variables during the operation. The final work provided by the actuator is $W = 0.71$, matching the result from Jun et al.'s experiment and closely approaching Landauer's limit of $\log 2 = 0.693$. The values of the probability P of state 1, heat Q, and work W are presented for both the simulation (subscript *sim*) and the quasistatic process (subscript *qs*). For the first two phases, the curves align perfectly. In the quasistatic limit, the actuator receives work $W = - 1.130$ during the barrier lowering, but provides $W = 1.823$ to confine the particle in the left well, resulting in a total work $W = \log 2$.

In the simulation, if the third phase lasts the same duration as the first two, almost no heat or work is exchanged. The process is adiabatic, and the work done during the first two phases is not recovered. It has been dissipated, which appears to support Landauer's principle. However, if we allow enough time for the third phase, it can also become quasistatic. As calculated in the Appendix A2, this duration is approximately 1 year.

In this case, the work W is recovered, the entire process becomes reversible, and the reset-to-zero is not dissipative, which contradicts Landauer's principle.
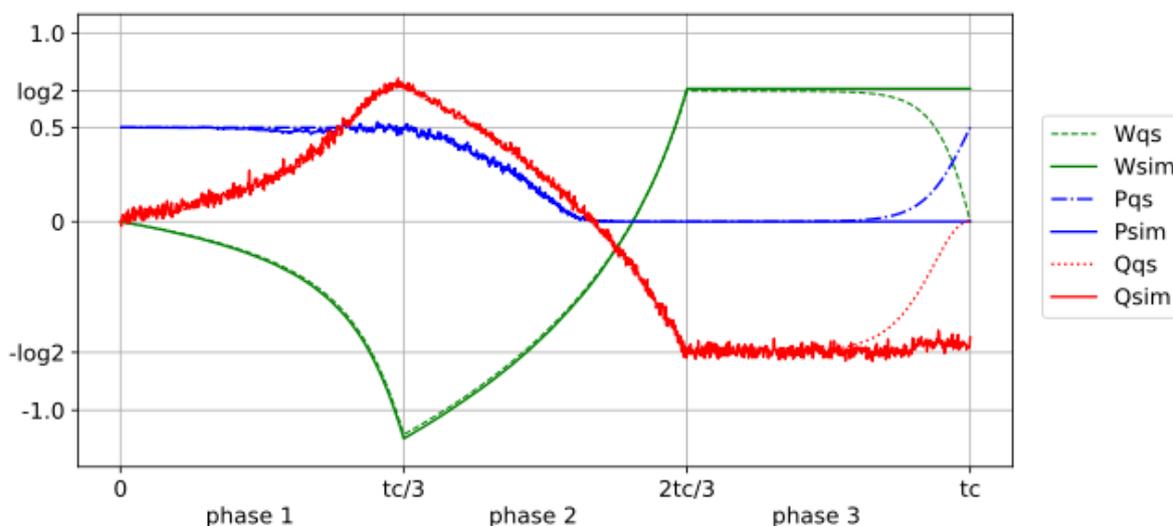

Fig. 2. *Evolution of the variables during reset-to-zero operation*

(*sim* = simulation of the experience for tc = 1200 s, *qs* = quasistatic limit)
During phase 1, the probability P remains constant at P = 0.5. The memory receives heat Q ≈ 0.71 from the thermostat and transfers to the actuator work W ≈ - 1.13. In phase 2, the work W increases to log2, while the heat Q decreases to − log2. During phase 3, the simulation shows no change, indicating an adiabatic process. However, if the process is quasistatic, which requires an extremely long time (the time scale of the figure does not apply in this case), the work W is fully recovered, making the entire operation reversible.

3. Difference between information and thermodynamic entropy

Claude Shannon defined the information entropy of a one-bit memory by $H = - (P \log P + (1 - P) \log (1 - P))$, where P is the probability of either state.

In our experiment, we can compare it with the statistical entropy defined by Gibbs' formula:

$S = - \sum_i P_i \log P_i$, where P_i is t he probability of the particle to be at abscissa x_i at equilibrium (see Appendix A2).

During the reset-to-zero operation, we observe that H and S are equal at both the beginning and the end of the process but differ considerably in between (Fig. 3). In phase 1, the probability of state 1 remains at P = 0.5, resulting in H = log2. However, as the barrier lowers, the phase space of the memory increases, causing S to increase to 1.403 in the quasistatic process.

This result challenges the commonly accepted equivalence between thermodynamic and Shannon entropies. Moreover, in the case of bistable memory, the choice of the boundary x between states 0 and 1 is conventional. For a symmetric double well, it is typically placed along the axis of symmetry, though this is not a strict requirement. Changing the position of this boundary alters the Shannon entropy, while the thermodynamic entropy remains unaffected.
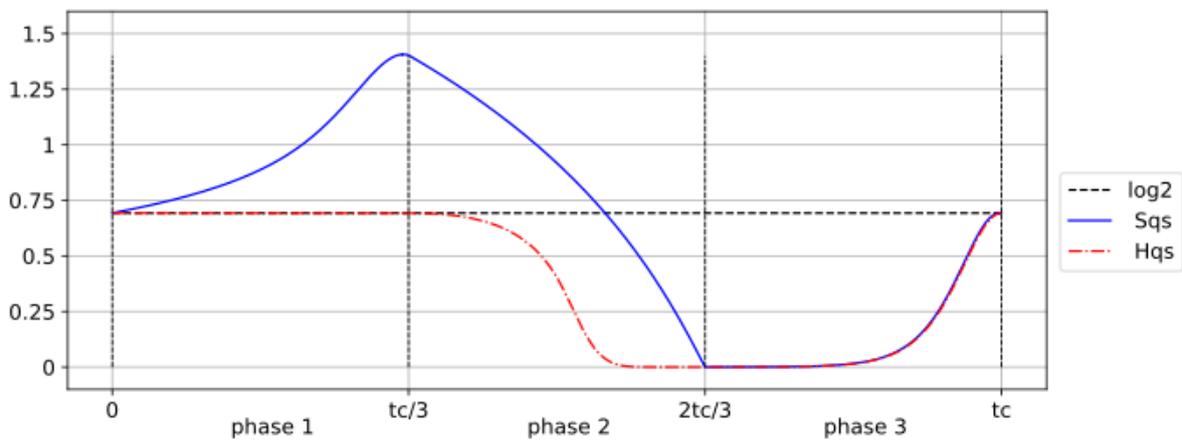
Fig. 3. *Statistical entropy S vs. information entropy H at the quasistatic limit*

At the start of phase 1, $S = H = \log2$. During phase 1, the state of the memory remains random ($H = \log2$), while the statistical entropy increases to $S = 1.403$ as the phase space expands. In phase 2, as the memory is pushed into state 0, both entropies decrease to 0. In phase 3 they revert back to their initial value while remaining equal.

If these two entropies are not equivalent, we must reconsider the arguments that Landauer used to establish his principle. He stated [4] that for a bit, a « well-defined initial state corresponds, by the usual statistical mechanical definition of entropy, $S = k_B \log W$, to zero entropy ». He further explained, « The degrees of freedom associated with the information can, through thermal relaxation, go to any one of 2^N possible states (for N bits in the assembly), and therefore the entropy can increase by $k_B N \log2$ as the initial information becomes thermalized ». He writes that the reset-to-one operation of N thermalized bits results in an entropy decrease of $k_B N \log2$, and concludes that « The entropy of a closed system cannot decrease; hence, this entropy must appear somewhere else in the form of a heating effect ». The most widely stated form of his principle, which is equivalent to the original, asserts that erasing a bit causes an energy dissipation of at least $k_B T \log2$.

Landauer applied the Boltzmann formula $S = k_B \log W$ to a memory in a 'well-defined state,' meaning 0 or 1, which gives S = 0. However, he noted that after thermal relaxation, the entropy can increase to $S = k_B \log2$, acknowledging that this state is not at equilibrium. The limitation of Landauer's approach was applying the Boltzmann formula to a 'well-defined' state of bistable memory, which is inherently out of equilibrium. Yet the Boltzmann formula does not apply unless a system is at equilibrium. If the system is macroscopic, it is subject to fluctuations whose time average is zero. For a nanoscopic system, as in the present case, the formula gives the average entropy over a very large number of experiments, or, equivalently, over a very long time with respect to the relaxation time of the system which is here about one year (see Appendix A2).

Finally, the experiment by Jun et al. does not prove Landauer's principle. As in our simulation, if the duration of the final phase is long enough for the process to become quasistatic, the entire process becomes reversible, and the work W = log2 expended during the previous phases can be recovered. Therefore, the dissipation, which is often cited as evidence for Landauer's principle, only occurs when the last phase is adiabatic and can be avoided if the phase is long enough to reach quasistatic conditions.

4. Shift memory for an energy-neutral reset-to-zero

Using the same equipment employed to create the bistable memory — a colloidal particle in a fluid — we can design a different type of memory in which the information entropy is controlled and stable, without the need for a potential barrier. This concept is based on the zero-energy protocol introduced by Gammaitoni [16], leading to a single potential well that we propose to call *shift memory*.
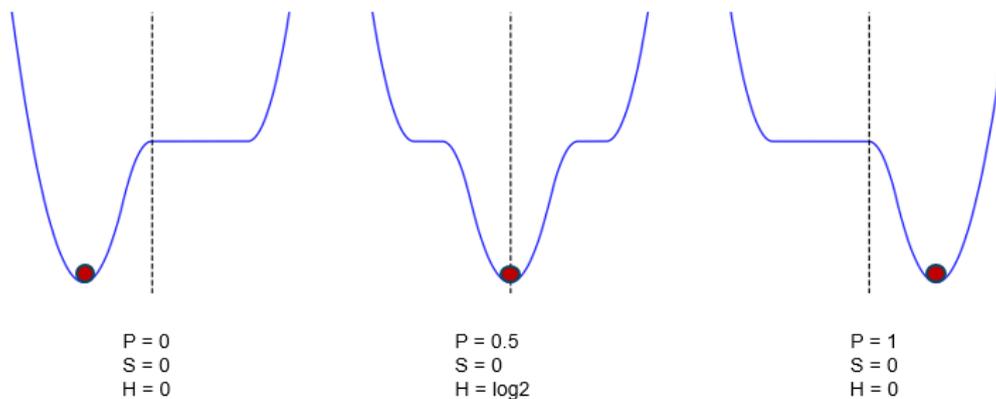
Fig. 4. *NOT operation with shift memory*

The potential well is gradually shifted from left to right. The physical parameters of the memory (energy U_s, entropy S) remain constant throughout. No work or heat is exchanged, but the probability of state 1 changes from 0 to 1, and the information entropy H fluctuates from 0 to log2 and back to 0. The operation is energy-neutral.

The single well can be shifted from one side to the other to invert the bit between states (NOT operation, Fig**.** 4). When performed quasistatically, this operation is energy-neutral. In this process, the thermodynamic entropy remains constant, while the information entropy fluctuates from 0 to $\log 2$ and back to 0.

For the shift memory, the reset-to-zero operation is also energy-neutral. If the initial state is 0, no action is needed. If the initial state is 1, it is inverted with zero energy. The initial position of the particle must be known to apply the protocol, so one possible counterargument is that the information about the position needs to be stored somewhere and erased after the operation. However, the information is already contained within the memory itself, and there is no need to copy and store it elsewhere. Therefore, this type of memory provides a direct challenge to Landauer's principle.

5. Tilt memory and Szilard engine

A tilt memory is a physical system with two states, separated by a controlled potential difference $U(t)$. We refer to two sets of experiments on this type of memory. The experiments by Koski et al. [12] aimed at realizing a Szilard engine, involved a single-electron box. An electric potential difference applied between two metal islands influences the probability of an electron being present (state 1) or absent (state 0) in an intermediate region that forms the box. The equations describing the quasi-static evolution of this system are reported in Appendix A3.

The experiments of Ribezzi et al. [13] designed to create a Maxwell's demon, used a 'hairpin' DNA fragment, where the two branches are linked by hydrogen bonds at rest (state 1) and can be separated by pulling on their ends (state 0). An actuator adjusts the electric potential in the first case and the traction force on the molecule in the second case to apply a potential difference $U(t)$ between the two states.

In this case, unlike bistable memory, there is no degree of freedom to define the boundaries between states 0 and 1 of the one-bit memory. There is a logical correspondence between the physical and informational states, and we will observe that thermodynamic and Shannon entropy remains equal in this scenario.

Koski et al. [12] and Ribezzi et al. [13] used tilt memories to construct a Szilard engine, as illustrated in Fig. 5. The energy landscape consists of two horizontal segments, with a potential of 0 for the left segment and a controlled potential U for the right segment.
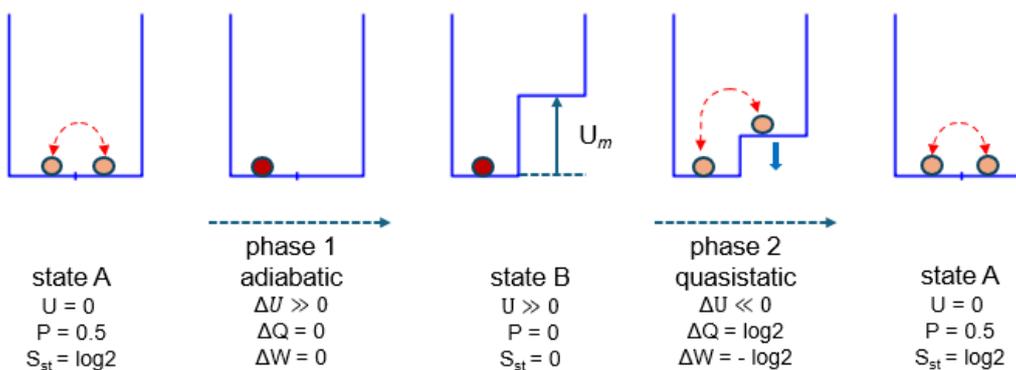

Fig. 5. *A Szilard engine*

In state A, the memory is randomly in state 0 or 1. When state 0 is detected, U is suddenly increased to a high value U_m (phase 1), stabilizing the bit at 0. The statistical entropy S_{st} decreases from log2 to 0. In phase 2, U is gradually reduced back to 0, returning to the initial state. During this phase, heat Q = log2 is converted into work [1].

Initially, $U = 0$, and the memory bit is randomly hopping between 0 and 1, with $P = 0.5$. The first phase begins when the memory is in state 0, after waiting as long as necessary. The potential U is then abruptly increased to a value of $U_m \gg 0$, ensuring that state 0 becomes stable and $P \simeq 0$. This process is adiabatic ($\Delta Q = 0$) and energy-neutral ($\Delta U_s = \Delta W = 0$). However, the system's statistical entropy is now $S_{st} = 0$, having decreased by $\Delta S_{st} = -\log2$.

In phase 2, the potential U is slowly reduced to 0, and the entropy S_{st} returns to 0. The energy balance for this phase is $\Delta U_s = 0$ and $\Delta S = \Delta Q = \log2$ [2], from which we obtain $\Delta W = -\log2$, meaning that a quantity of heat $Q = \log2$ was extracted from the thermostat, converted into work, and transferred to the actuator. These realizations of the Szilard engine demonstrate that it is possible to convert a quantity of heat $Q = \log2$ into an equivalent amount of work, thus reducing the entropy of an isolated system, which contradicts the second law of thermodynamics.

The authors of these studies offered an interpretation to explain the entropy decrease without violating the second law. According to this interpretation, the information that the memory is in state 0, which triggers the cycle, is stored in an external memory. A quantity of entropy $\Delta S = \log2$ is generated when this information is erased, in line with Landauer's principle. However, this argument no longer holds after the refutation of the principle.

Moreover, the information used to trigger the cycle resides within the memory itself; it is the bit corresponding to state 0. Therefore, there is no need to assume its storage and erasure outside of the memory.

6. Tilt memory and Maxwell's demon

Ribezzi et al. [13] realized a Maxwell's demon, the operation of which is illustrated in Fig. 6. Initially, the energy level of the memory is $U_i > 0$ with probability $P_i < 0.5$ for state 1. The memory is typically

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[1] The log2 value is an upper bound, which can be approached under the best experimental conditions.

[2] The equality $\Delta S = \Delta Q$ results from the choice of units (see at the end of the introduction).

in state 0, which is the most probable state. As soon as a transition to state 1 occurs, U is suddenly driven adiabatically to a value $U_m \ll 0$ to stabilize state 1, with the probability $P_m \simeq 1$.

Then, during phase 2, it is returned quasistatically to its initial value U_i. Experiment and theory (see Appendix A3, equation [13] show that the result of this operation is the transformation of heat into work $\Delta W = \log(1 + \exp(U_i))$, with a theoretically unlimited value. But the higher this ΔW value is, the longer the average waiting time of phase 1, as it is proportional to $\exp(U_i)$.
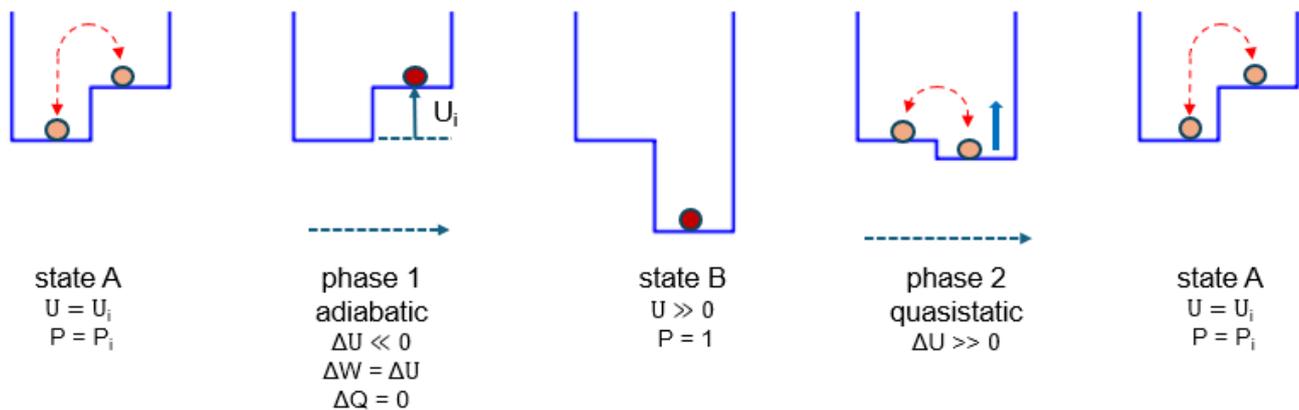

Fig. 6. *A Maxwell's demon*

In state A, the system can transition between state 0 (the most probable) and state 1. When state 1 is detected, the potential U is abruptly decreased to a very low value (phase 1 from A to B). This stabilizes the memory in state 1. During phase 2, the potential U is then gradually increased back to its initial value U_i.

According to Ribezzi et al., this reduction in entropy is offset by the erasure of the information required to operate the process. This information would have been stored as an increasing number of bits during the waiting time of the first phase and then erased, dissipating energy in accordance with Landauer's principle. However, as with the Szilard engine described earlier, this information resides within the memory itself, eliminating the need for external memory to store and erase it.

However, the local entropy reduction achieved within the system, as shown for the Szilard engine above, is far from being practically exploitable. In reality, the memory is part of an experimental setup that may include an optical tweezer with its laser beam, a cooling device, and other auxiliary equipment. In experiments of this type conducted to date, this setup generates much more entropy than it eliminates. Nonetheless, it remains true that the second law can be locally violated at energy levels on the order of $k_B T$ (here $k_B T = 4{,}14 \ 10^{-21}$ J at 300 K).

7. Statistical and thermodynamic entropies

These two experiments – the Szilard engine and the Maxwell's demon – present exceptions to the second law. They also highlight a distinction between thermodynamic and statistical entropies S_{th} and S_{st}, which arises from their definitions. For example, in state A of Fig. 6, the energy landscape remains unchanged, so the statistical entropy S_{st} stays constant. However, each transition of the memory between state 0 and state 1 involves an exchange of heat and thermodynamic entropy, $\Delta Q = \Delta S_{th} = \pm U_i$, between the memory and the thermostat. As a result, the entropies S_{th} and S_{st} of the memory exhibit transient deviations.

It may be noted that originally the statistical entropy introduced by Boltzmann concerned macroscopic systems involving a very large number of atoms or molecules, and at equilibrium. In the present case we consider a 2-state system which in general cannot be considered at equilibrium since it is subject to

stochastic transitions between the two states. To return to the notions of equilibrium and statistics it is necessary to carry out a large number of experiments and to consider the average of their results.

8. Conclusion

Since the works of Maxwell and Szilard the question of the absolute character of the second law of thermodynamics has long been beyond the scope of experiment at the nanoscale. Several recent experiments on Szilard machines and Maxwell demons have yielded sufficiently accurate results to allow us to establish local, minimal but reproducible violations of the second law, even if the experimental devices which are used produce more entropy than they can suppress.

We have also shown that the principle of equivalence between thermodynamic entropy and Shannon's information entropy must be abandoned. Shannon [17] defined his entropy $H = -(P_0 \log P_0 + P_1 \log P_1)$ as the only mathematical function satisfying three well-defined constraints, which he encountered while solving cryptographic problems during the Second World War [18]. Starting from Hartley's work [19], Shannon derived his formula for H to represent the minimum amount of information required to encode a given message. Its value depends on the frequency statistics within a specific context. Therefore, it is relative to that context, unlike the Gibbs formula, which applies to a well-defined physical system. While the two concepts are analogous, they apply to different domains.

Shannon's formula is useful as a measure for information storage and transmission. It has been reported [20] that von Neumann suggested the name "entropy" to Shannon for his statistical measure of information, which contributed to the widely accepted idea that there is an equivalence between thermodynamic and Shannon entropy [21]. The experiments studied above show that this is not the case. Moreover, they differ by nature. Thermodynamic entropy is an intrinsic property of a physical system, while information entropy generally rests on a code and depends on the statistics of that code in a given context [22].

Data Availability

The scripts and supporting documentation for this study are available in the following GitHub repository: https://github.com/argou/limits_of_landauer_and_second_law

Acknowledgments

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Appendix A

A1 Simulation of bistable memory by finite difference equations

In Jun et al.'s experiment [8], a particle was placed in water within a virtual potential U(x,t), which was updated every $\Delta t = 0.01\ s$.

Between two updates, the particle experienced a force $F = -\frac{\partial U(x,t)}{\partial x}$ and underwent stochastic Brownian motion. The particle's movement is governed by the Langevin equation, excluding the inertial term which can be neglected [3] :

$$\frac{\Delta x}{\Delta t} = -\frac{1}{\gamma}\frac{\partial U}{\partial x} + \sqrt{\frac{2\,D}{\Delta t}}\,w\ ,$$

where w is a white noise with mean $< w(t) > = 0$ and variance $< w(t)^2 > = 1$.

The friction coefficient is $\gamma = 6\,\pi\,\eta\,r$, with $\eta = 0.89\ 10^{-3}$ N.s/m^2 for water and the radius of the particle is r = 0.1 μm. Using the Einstein relation $D = \frac{k_B\,T}{\gamma}$, we obtain $D = 2.5$ μm^2/s.

Thus, the displacement of the particle due to the potential gradient and the Brownian motion during Δt is

$$\Delta x = \left(-D\frac{\partial U}{\partial x} + \sqrt{\frac{2\,D}{\Delta t}}\,w\right)\Delta t. \qquad [1]$$

During this displacement Δx a heat transfer (positive or negative) occurs from the thermostat to the particle:

$$\Delta Q = U(x + \Delta x, t) - U(x, t). \qquad [2]$$

The potential landscape is then updated from $U(x + \Delta x, t)$ to $U(x + \Delta x, t + \Delta t)$.

The total variation of the particle's potential, considering Brownian motion, the potential gradient, and the action of the actuator, is given by the following:

$$\Delta U = U(x + \Delta x, t + \Delta t) - U(x, t). \qquad [3]$$

Thus, according to the law of energy conservation, the actuator has provided work:

$$\Delta W = \Delta U - \Delta Q. \qquad [4]$$

Using these finite difference equations, we can simulate the evolution of the memory by knowing the initial state x0 along with function U(x,t). Due to the high stochasticity of Brownian motion, the variables must be averaged over a large number of trials.

We applied this method to the experiment of Jun et al., which yielded highly accurate results. The energy landscape $U(x,t)$ is a quartic function [4] of x, as shown in Fig. 7 at the phase limits. The initial position of the particle is randomly chosen at the bottom of either the left or the right well. We allowed the system a period of 20 s to reach equilibrium.
We simulated $ns = 1440$ runs from $t_0 = 0$ to $t_{nt} = tc$ (cycle time tc = 940 s). Thus, $nt = tc/\Delta t =$

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[3] According to Volpe and Volpe [23], the inertial term is negligible when Δt/τ <10. We have here Δt = 10 ms, and τ is the relaxation time of quantity of motion $\tau = m/\gamma$, the ratio of mass to viscous friction coefficient. Here we have $\Delta t/\tau = 40\ 10^{-9}$.

[4] Refer to https://github.com/argou/limits_of_landauer_and_second_law, for the definition of the U(x,t) functions of the 4 phases of the process.

940 000.

For every simulation, we computed $x_{i,j}$, $Q_{i,j}$, $U_{i,j}$ for indices $i = 0 \text{ to } nt, j = 1 \text{ to } ns$.

From $x_{i,j}$, we obtained the mean probability of state 1 at time t_i:

$P_i = \frac{1}{ns} \sum_{j=1}^{ns}(0 \text{ if } x_{i,j} < 0; 1 \text{ if } x_{i,j} > 0)$,

and the mean vectors: $\quad Q_i = \frac{1}{ns} \sum_{j=1}^{ns} Q_{i,j} \quad$ and $\quad U_i = \frac{1}{ns} \sum_{j=1}^{ns} U_{i,j}$.

With our units, defined in introduction, the thermodynamic entropy is such that $\Delta S = \Delta Q$. Thus, if we take $S_0 = \log 2$ (for $P_0 = 0.5$), we obtain $S_i = log 2 + Q_i$
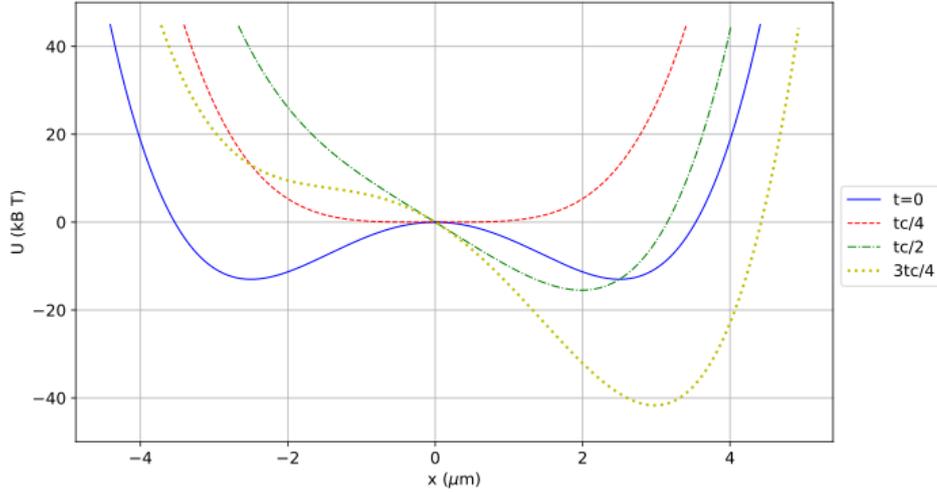

Fig. 7. *Energy landscape at phase limits*

At the end of phase 4, the profile returns to its original shape (t = 0)

The experiment was designed to validate Landauer's principle. The final result, both from the experiment and the simulation, yielded work W = 0.71 $k_B T$, which is in agreement with Landauer's limit, log2. The evolution of the variables is shown in Fig. 8 and 9, where they are compared with the quasistatic values computed directly.

A2 Quasistatic equations of a bistable memory

The particle is confined between the abscissas x_{min} and x_{max}. The interval *[x_{min}, x_{max}]* is divided into N equal segments. For a quasistatic process, the probability of the particle being at abscissa x_i at time t follows the Boltzmann distribution at equilibrium.

$P(x_i, t) = \exp(-U(x_i, t)) / Z(t)$

or $P_i(t) = \frac{1}{Z(t)} \exp(-U_i(t))$ with $\quad Z(t) = \sum_{i=1}^{N} \exp(-U_i(t))$ [5]

In the following, all variables depend on time t. The potential energy of the system is $U_s = \sum_{i=1}^{N} P_i U_i$.

The statistical entropy is given by Gibbs' formula $S = -\sum_{i=1}^{N} P_i \log P_i$ [6]

thus, $\quad S = \frac{1}{Z} \sum_{i=1}^{N} U_i \exp(-U_i) + \frac{logZ}{Z} \sum_{i=1}^{N} \exp(-U_i)$

or $\quad S = U_s + \log Z$. [7]

The law of conservation of energy implies that $\Delta U_s = \Delta W + \Delta Q$. Based on our choice of units, Q and S vary identically as $\Delta Q = \Delta S$.

Therefore, $\Delta W = -\Delta(\log Z)$ [8]

Given the initial state and $U(x,t)$, we could directly compute the numerical values of all relevant physical variables for a quasistatic process, as illustrated in Fig. 8 and 9. These computed values align closely with the simulation results, except at the end of phase 4, which is not quasistatic in the experiment.

The agreement between these two independently derived methods — and their consistency with the final result of the experiment by Jun et al. — strongly supports the validity of both approaches.
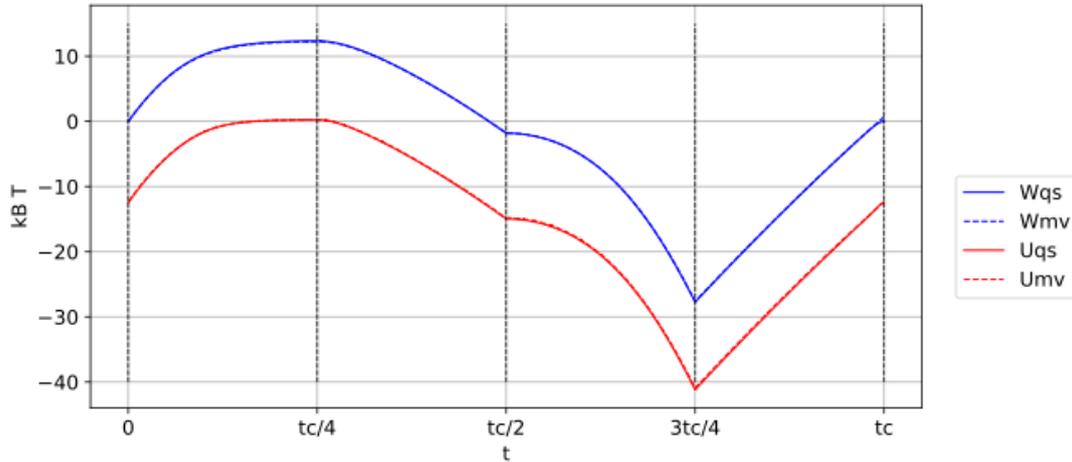

Fig. 8. *Evolution of work W and potential U of the particle*

The simulation values W and U fit perfectly with the quasistatic values (fluctuations of W and U are not visible at this scale).
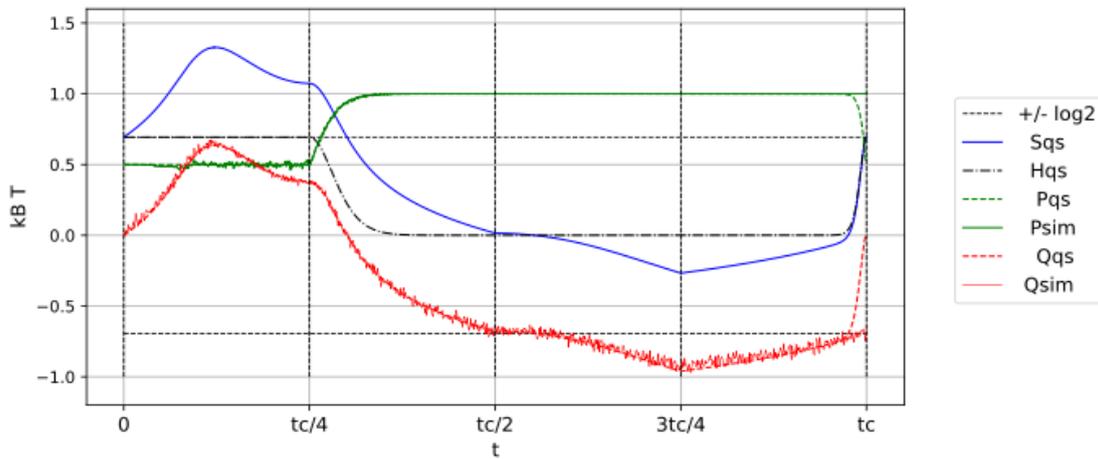

Fig. 9. *Evolution of P, Q, S and H*

The simulation values Psim and Qsim differ from quasistatic values Pqs and Qqs only by fluctuations, except at the end of phase 4, which is out of equilibrium for the simulation. The thermodynamic and information entropies S and H are in general different.

Application to a simpler protocol

In Jun et al. [8], the potential could vary between 13 and $-40\,k_BT$ during the operation. These important variations make it difficult to visually follow the values of the main parameters.

We propose a simpler protocol that can be implemented using the same equipment, featuring a potential variation between 0 and 13 k_BT and a total duration of 1200 s, divided into three equal phases instead of 940 s divided into four phases. This protocol yields the same final results and is easier to interpret. It is shown in Fig. 1, with the evolution of the system variables illustrated in Fig. 2 and 3.

The energy landscape $U(x,t)$ is a continuously differentiable function composed of quadratic segments of the form $a\,x^2$, where $a = 10.8$ for the wells, $a = 0$ for the steps, and $a = \pm 54$ for the junctions, to ensure continuity of the derivatives. During each phase, the potential $U(x,t)$ evolves through linear interpolation between its values at the beginning $U(x,t_i)$ and at the end $U(x,t_f)$ of the phase.

The distance between the two minima is 4.8 μm, and the maximum potential of the barrier or the steps is $U = 13\ k_\mathrm{B}T$. The boundary between state 0 and state 1 is set along the axis of symmetry of the figure..

What should be the duration of the last phase for it to be quasi-static, such that the energy W = log2 can be recovered. This duration can be estimated using the Kramers formula law [15].

The hopping frequency given by Jun et al. [8] for this situation is $F = f0\ exp(\Delta U)$, where ΔU is the height of the barrier to overcome and $f0 = 2.0\ Hz$. The differential equation describing the probability of being in state 1 is

$$dP/dt = F01\,P - F10\,(1 - P), \qquad\qquad [9]$$

F_{ij} being the transition frequency frome state i to state j (see for example [24]). In our case $F_{01} = f_0 \exp(13)$.

The Kramers formula does not apply directly at the beginning of the process, in the absence of potential well in the right side. The Boltzmann distribution formula allows us to calculate $F_{10}(t)$ to solve the problem [5].

The numeric integration of equation [9] for several time scales of last phase gives the results shown in Fig. 10.
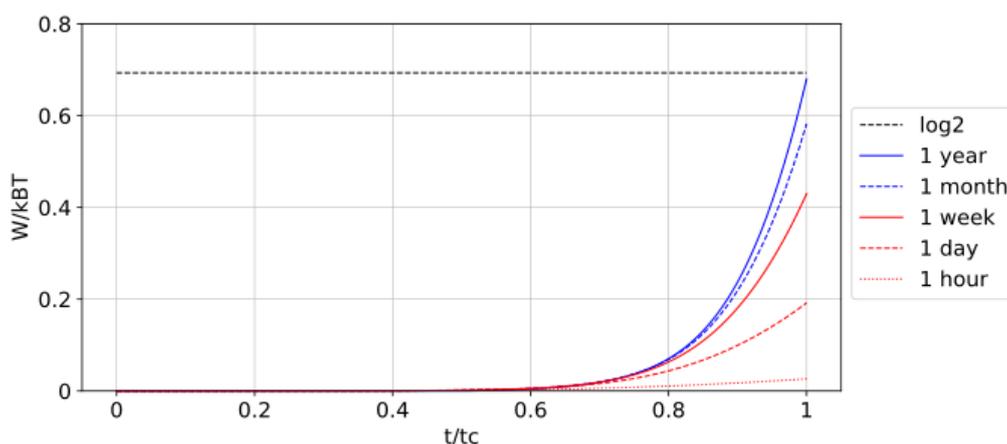

Fig. 10. *Energy W recovered by the actuator*
during the last phase as a function of its duration tc up to 1 year.

During the experiment, the result is W ≃ 0 (for t = 400 s). One needs approximately 1 year for the process to be quasistatic and reversible to recover W = log2.

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[5] The detailed calculation is available in the repository: https://github.com/argou/limits_of_landauer_and_second_law

A3 Quasistatic equations for tilt memory

The quasistatic equations for tilt memory, inferred from the previous equations of bistable memory with $N = 2$, are the following:

$$Z = 1 + \exp(-U) \tag{10}$$

$P = 1/(1 + \exp(U))$, equivalent to

$$U = \log\left(\frac{1-P}{P}\right) \quad \text{or} \quad \exp(U) = \frac{1-P}{P}, \tag{11}$$

$U_s = P\,U$, and

$S = -(P\log P + (1-P)\log(1-P))$.

Notably, we have in this case $\qquad Q = \Delta S = \Delta H$.

From equations [10] and [11], we obtain $\qquad Z = \frac{1}{1-P}$.

Then, from equation [8], we obtain $\qquad \Delta W = \Delta(\log(1-P))$. $\qquad$ [12]

Application to Maxwell's demon

In the Maxwell's demon setup represented in Fig. 6, the memory initially has an energy level of $U_i > 0$ with P_i as the probability of state 1. As soon as a transition to state 1 occurs, U is suddenly and adiabatically lowered to $U_m \ll 0$ to stabilize state 1 with $P_m \simeq 1$. The work required for this phase is

$$\Delta W1 = Pm\,Um - Ui = Pm\,Um - \log((1-P_i)/P_i).$$

Then, it is returned quasistatically to its initial value U_i. The work necessary for the second phase is $\Delta W_2 = \log(1-P_i) - \log(1-P_m)$.

Thus, $\Delta W = \Delta W_1 + \Delta W_2 = \log P_i + P_m\,U_m - \log(1-P_m)$.

As $P_m \simeq 1$, we finally obtain

$$\Delta W = \log P_i \text{ or } \Delta W = -\log(1 + \exp(U_i)). \tag{13}$$