Quelques Familles de Lois Paramétriques pour les Processus de Branchement Univariés

Some Parametric Families of Single-Type Branching Distributions

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RÉSUMÉ. Dans cet article, nous présentons quelques familles paramétriques de lois de probabilité univariées associées à des mécanismes particuliers de branchement de dynamique de populations en temps continu. La pertinence et la simplicité de l'interprétation des paramètres de telles familles seront d'un grand intérêt pour de nombreux domaines d'application et ce spécifiquement pour concernant les problèmes d'inférence statistique. Ces familles sont mieux adaptées à modéliser des systèmes dynamiques de populations où l'hypothèse de Poisson est généralement, mais on le montre ici à tort, admise. Les calculs et les caractéristiques importantes de ces lois de probabilité sont directement dérivées de leurs fonctions génératrices. Ces dernières satisfont à des équations linéaires aux dérivée partielles linéaires homogènes faciles à appréhender. De plus, ces équations permettent d'établir une formule de récurrence générale pour les moments factoriels.

ABSTRACT. In this paper, we present some parametric families of probability distributions associated to particular single type homogeneous branching processes in continuous time. Their simplicity and the relevance of the interpretation of parameters for many domains of applications are of valuable interest for statistical inference. These families are particularly well adapted to handle branching dynamical systems of populations where Poisson assumption is generally but mistakenly assumed. Calculations and pertinent properties concerning these probability distributions are derived from their generating functions satisfying specific linear partial differential equations. As a by product, these equations allow the statement of a general recurrence formula for factorial moments.

MOTS-CLÉS. Processus de branchement en temps continu, Equations aux dérivées partielles, fonction génératrice des moments, moments factoriels, ditributions alourdies en zero.

KEYWORDS. Continuous-time Branching Processes, Partial Differential Equations, probability generating function, factorial moments, zero inflated distributions.

1. Introduction

Initially, the idea sustaining the definition of a branching process (BP) is the description of simple random mechanisms allowing each individual of a population to breed into several clones or to die. More precisely, the population starts with an initial random number of individuals, then at any time and independently from the others, a living individual after an exponentially distributed lifetime gives rise to a progeny of random, possibly null (meaning its death), size. Additionally to independence, the random behaviors of individuals are identically distributed (i.i.d.). These processes are proven to be Markovian and therefore inherit many important properties. Standard references on Markov branching processes, among many others, are Harris (2022), Sevast'yanov (1951), Kendall (1966), Krishna, Athreya and Keiding (1977).

The general theory of branching processes considers appropriate stochastic models to describe dynamical systems of population dynamics whose members (cells, particles, etc.) breed, evolve, interact or die. Indeed, branching processes have been successfully applied in many diverse domains (e.g Taylor and Karlin (2014)) highlighting each time specific issues related to areas in molecular biology, cell biology, developmental biology, epidemiology, demography, particle physics, evolution, ecology, medicine, and others, see Jagers et al (1975), Mode, Kimmel and Axelrode (2003), Jacob (2010), Slavtchova-Bojkova, Trayanov and Dimitrov (2017). However, considering the issue of statistical inference within all this variety of dynamics unveil a flagrant lack of existing consistent families of probability distributions modeling the time varying population size $X(t)$ of these very dynamics (see Taylor and Karlin (2014), Gonzales et al (2008), Becker (1977) for some statistical issues).

Much of the early work on the continuous-time branching processes was initiated by the Russian school in the middle of the XXth century (Sevast'yanov (1951), Kendall (1966), Ghikman and Skororohod (1975)). Later, many prime theoretical results regarding the stochastic behavior of branching processes were established by an ever greater international community of probabilists (see Pakes (1999), Chen (2002-1), Chen (2002-2), Jagers (1975)). First, it was proven that, depending on the mean value of the offspring size m, discrete time and then continuous time branching processes has a dichotomous behavior in the sense that the population becomes extinct or explodes almost surely (a.s) :

$$\lim_{t\longrightarrow\infty} X(t) = 0 \;\; or \;\; \lim_{t\longrightarrow\infty} X(t) = +\infty.$$

BP are divided into sub-critical ($m < 1$), critical ($m = 1$) and supercritical ($m > 1$) types. An important characteristic for BP is the extinction probability function $p_0(t) = P(X(t) = 0)$, so that $P(X(\infty) = 0) = p_0(\infty)$ and $P(X(\infty) = \infty) = 1 - p_0(\infty)$. Actually, we prove in this paper that $p_0(t)$ determines the probability distribution of the whole process. On the other hand, general BP can explode at finite times, letting $\tau_\infty = sup\{t > 0 : X_t < \infty\}$, ie the explosion time of the branching process, we call X regular if $P(\tau_\infty < \infty) = 0$. Conditions for regularity are also well known (see Ghikman and Skororohod (1975), Harris (2002)) and were assumed to hold in the present article.

This paper is mainly devoted to provide some sensible parametric distribution families of BP so that one can unroll the usual efficient maximum likelihood estimation procedure. For that purpose, we followed an unusual approach based on the study of the partial differential equation (PDE) related to the probability generating function (pgf) of BP. Hence, we obtained a general recurrence formula for factorial moments when they exist. It also enlighten the bona fide relation between the extinction probability function and the pgf.

Chapter 2 introduces continuous time branching processes. Chapter 3 uses the Markovian property to derive the main differential and partial differential equations issued from Kolmogorov's equations. In Chapter 4 we investigate some particular branching dynamics.

2. Continuous Time Branching Processes

2.1. *Some Notations and Reminders*

First, let us fix some notations used in this paper and remind some well known facts related to branching processes.

For enough smooth and well defined functions f, g, the n^{th} derivative of f is denoted $f^{(n)}(x)$ (or f', f'' if $n = 1, 2$), while the composition of f by g is written $f \circ g(x) = f(g(x))$ and $f^{\circ n}$ for the n-fold. In the case of several variables, e.g. $f(x, y)$, we use the classical notations for the partial derivatives e.g. $\partial^n_{x^n} \partial^m_{y^m} f(x, y)$.

Recall that the probability generating function (pgf) of an integer valued r.v. Y with probability density (pdf) $P(Y=j)=p_j,\ j\geqslant 0$, is defined by

$$G(z)=E(z^Y)=\sum_{j=0}^{\infty}p_jz^j$$

where $z\in D(0,1)$, the complex open unit disc. G has nice properties. For example G is analytic in D and as such is completely determined by its values on the real segment $[0,1]$. It also inherits most of classical Fourier transforms properties. For example, Y has finite moment of order k if and only if $\lim_{z\uparrow 1}G^{(k)}(z)=G^{(k)}(1^-)<\infty$. If this the case, the factorial moments $m^{[k]}=E(Y(Y-1)...(Y-k+1))$ are then given by :

$$m^{[k]}=G^{(k)}(1^-),\quad k\geq 0.$$

For example $E(Y)=G'(1),\ Var(Y)=G''_Y(1)+G'(1)(1-G'(1))$ if k=2. In duality, the derivatives of the pgf at $z=0$, yield the pdf

$$p_j=\frac{G^{(j)}(0)}{j!},\ j\geq 0.$$

2.2. *Branching Processes*

Continuous time Markov branching process $X=\{X_t : t\geqslant 0\}$ are the analogous of discrete time Galton-Watson process $(X_n)_{n\geqslant 0}$, counting the number of individuals or particles of a population undergoing a branching dynamics. We briefly recall some of the definitions, tools and results related to these latter processes before dealing with the continuous time context.

2.2.1. Discrete Time Branching Processes

Discrete time Markov branching processes $(X_n)_{n\geqslant 0}$ are defined by an initial integer valued r.v. X_0, and by the following recursive rule. At time $n\geqslant 1$, if $X_n=0$ then $X_{n+1}=0$ a.s., whereas for $X_n>0$, we have $X_{n+1}=\sum_{j=1}^{X_n}Y_{n,j}$, where the $Y_{n,j}$ are i.i.d. integer valued random variables with probability distribution $(P(Y=j)=p_j, j\geqslant 0)$. This construction-definition implies that $(X_n)_{n\geqslant 0}$ is is a homogeneous Markov process.

The random variable $Y_{n,j}$ represents the size of the offspring issued from the individual j living at time n. The event $Y_{n,j}=0$ means the death of individual j at time n. The event $Y_{n,j}=1$ can be interpreted without loss of comprehension, either as the survival or the *death instantly followed by regeneration* of the individual j.

The identical and total independent branching behaviors of particles entail that if $X(0)$ has pgf $g_0(.)$ and if the $Y_{n,j}$ have a common pgf G, then the iterations of G yield the pgf of X_n :

$$g_n(z)=g_0\circ G\circ\cdots\circ G(z)..=g_0\left(G^{\circ n}(z)\right).$$

Actually, the particular case $g_0(z)=z$ ie $X(0)=1$ a.s, yielding $g_n(z)=G^{\circ n}(z)$ is actually sufficient to characterize all subsequent probability issues of the process $(X_n)_{n\geqslant 0}$.

2.2.2. Continuous Time Branching Processes

Homogeneous continuous time branching processes also assume that particles are independent, have the same offspring size distribution save that particles have continuous independent proper lifetimes that are exponentially distributed with the same parameter λ. The process $X(t),\ t \geq 0$ counting the number of living particles at time t is also Markovian. Similarly to the discrete time case, the independence, homogeneity and Markov properties entail that the probability distribution of the whole process is entirely determined by the one-dimensional distributions of the $X(t)$ conditionally to the event $X(0) = 1$ or equivalently by the probability generating function (pgf) $g(t,z) = E(z^{X(t)}|X(0) = 1)$. Similarly, the pgf reveals to be the best tool for the theoretical investigation of continuous time branching dynamics, see Athreya and Ney (2012).

For sake of a consistent interpretation throughout the paper, we assume that at a death instant τ, the particle cannot be restored (reborn) instantly so that there is a genuine jump for the population size. The other representation allowing instantaneous restoration of individuals is also possible but involves a distinct expression for the infinitesimal generator of the Markov process.

Being Markovian and homogeneous, branching processes are entirely characterized by their transition probabilities $P(t) = (P_{i,j}(t))$ where $P_{i,j}(t) = P(X(t) = j \mid X(0) = i),\ i,j \in N,\ \ t \geq 0$. One may refer to Athreya and Ney (2012), Harris (2002), Kendall(1966) and for the general theory of continuous time jump Markov processes Ghikman and Skorohod (1975) and Ethier and Kurtz(1986) : They satisfy the semi group property $P(t+s) = P(t)P(s)$. Under regularity conditions, P has an infinitesimal generator $Q = (Q_{i,j},\ i,j \in N)$ (*i.e.* a derivative) defined by

$$Q_{i,j} = \lim_{t \downarrow 0} \frac{P_{i,j}(t) - \delta_{\{i=j\}}}{t}.$$

For branching processes and $0 \leq i,j < \infty$, the generator is simply written :

$$Q_{i,j} = \begin{cases} 0 & \textit{if } j < i-1 \\ i\lambda\mu & \textit{if } j = i-1 \\ -i\lambda & \textit{if } j = i \\ i\lambda(1-\mu)p_k & \textit{if } j = i+k, \end{cases}$$

where $0 \leq \mu \leq 1$ and $\lambda \geq 0$. The plain cases $\mu = 0,\ \ \mu = 1$ and $\lambda = 0$ corresponding to almost surely monotonic or constant processes are elementary and are not considered thereafter.

The probability distribution of a regular branching process is therefore characterized by the pair (λ, π) where $\lambda > 0$ and $\pi = (q_j)_{0 \leqslant j \neq 1}$, a probability distribution on $\mathbb{N} - \{1\}$ with $q_0 = \mu$ and

$$Q_{i,j} = i\lambda q_{j-i+1}, \quad j \neq i.$$

2.2.3. Kolmogorov's Equations

The infinite dimension matrix function of transition probabilities $(P(t), t \geqslant 0)$, satisfying the initial condition $P_{i,j}(0) = \delta_{\{i=j\}}, i,j \in N$, is the unique solution of the Backward Kolmogorov Equation (BKE) :

$$\frac{dP_{i,j}(t)}{dt} = \sum_{k\geqslant 0} Q_{i,k}P_{k,j}(t) \quad [1]$$

Formally, Equation (1) can be written as the linear infinite dimensional differential equation $P'(t) = QP(t)$ satisfying the initial condition $P(0) = I_d$. As for finite dimensional differential equations, its solution is written $P(t) = \exp(tQ) = \sum\limits_{j\geqslant 0} t^j Q^j/j!$.

$P(t)$ commutes with Q, so that in the case of homogeneous Markov processes, $P(t)$ also obeys the linear Forward Kolmogorov Equation (FKE) : $\frac{dP_{i,j}(t)}{dt} = \sum\limits_{k\geqslant 0} P_{i,k}(t)Q_{k,j}$, formally written $P'(t) = P(t)Q$.

Being infinite dimensional, both differential equations are difficult to handle in general. So instead of dealing with a such infinite transition probability matrix, we resort to the more convenient generating functions.

3. Fundamental Differential Equations Driving Branching Processes

The branching systems are based on the assumption that the sub-processes resulting from ancestors (*i.e.* branches) are independent and add to each other. Therefore if $G(t, z)$ denote the pgf of $X(t)$ when $X(0) = 1$, we get $g(t, z) = g_0(G(t, z))$ if X_0 has pgf $g_0(z)$. Accordingly, one has to consider only the case $g_0(z) = z$ and investigates the properties of the time dependent pgf :

$$G(t, z) = \sum_{j=0}^{\infty} P_{1,j}(t)z^j, \quad z \in D(0, 1). \quad [2]$$

satisfying the initial condition $\quad G(0, z) = z$.

Since the semi group property of $P(s + t) = P(t)P(s)$, can be translated into the property

$$G(t + s, z) = G(t, G(s, z)),$$

then taking account of the initial condition $G(0, z) = z$, we get

$$\begin{aligned}
\partial_t G(t, z) &= \lim_{s\longrightarrow 0} \frac{G(t + s, z) - G(t, z)}{s} \\
&= \lim_{s\longrightarrow 0} \frac{G(t, G(s, z)) - G(t, z)}{s} \\
&= \lim_{s\longrightarrow 0} \frac{G(s, z) - G(0, z)}{s} \frac{G(t, G(s, z)) - G(t, z)}{G(s, z) - z} \\
&= \partial_t G(0, z)\partial_z G(t, z),
\end{aligned}$$

that is

$$\partial_t G(t, z) = \partial_t G(0, z)\partial_z G(t, z) \quad [3]$$

Next, from Equations 1 and 2, we deduce that

$$\partial_t G(t,z) = \sum_{j\geq 0} Q_{1,j} G^j(t,z). \quad [4]$$

In particular we get $\partial_t G(0,z) = \sum_{j\geq 0} Q_{1,j} z^j$. For its importance and for convenience, we denote this last expression

$$V(z) = \sum_{j\geqslant 0} Q_{1,j} z^j. \quad [5]$$

Eventually, Equation 3 can be written as the following fundamental partial differential equation (PDE) for G :

$$\partial_t G(t,z) = V(z)\partial_z G(t,z), \quad [6]$$

with initial condition $G(0,z) = z$.

3.1. *Properties of* V

For interpretation purpose, V can be rewritten in a unique canonical form as follows :

$$V(z) = \lambda(G_Y(z) - z) \quad [7]$$

where $\lambda = -Q_{1,1}$ and $G_Y(z)$ is the pgf of the instantaneous offspring variable Y with values in $\mathbb{N} - \{1\}$ and pdf

$$P(Y = j) = -\frac{Q_{1,j}}{Q_{1,1}} = q_j, \; 0 \leqslant j \neq 1. \quad [8]$$

It is convenient to think Y as the product $U(W+2)$ of two independent r.v. where U is Bernoulli with $P(U = 0) = q_0$ and W has pdf

$$P(W = j) = \frac{q_{j+2}}{1 - q_0}, \; j \geqslant 0.$$

For instance W may be binomial, Poisson or geometric distributed.

The function V has the following properties on real interval $[0, 1]$:

— V is analytic and satisfies $V(0) = \lambda q_0 > 0$ and $V(1) = 0$.

— $E(Y) \geqslant 2(1 - q_0)$, since

$$E(Y) = \sum_{j\geqslant 2} j q_j \geqslant 2\sum_{j\geqslant 2} q_j = 2(1 - q_0).$$

— The derivatives $V^{(k)}(z) = \lambda \sum_{m=0}^{\infty} \frac{(k+m)!}{m!} q_{k+m} z^m, \;\; k \geqslant 2$, satisfy in particular

$$V^{(k)}(0) = \lambda k! q_k, \qquad V^{(k)}(1) = \lambda \sum_{m=0}^{\infty} \frac{(k+m)!}{m!} q_{k+m}. \quad [9]$$

— $V^{(k)}$ is non negative for $k \geqslant 2$, V and all of its derivatives are convex functions. V is strictly convex if $q_0 < 1$ and therefore has at most one zero z_V^* on the open real interval $]0, 1[$.

More precisely, we have the following factorization result :

LEMMA 1– $V(z) = \lambda(1-z)\left(q_0 - \sum_{j\geqslant 2} q_j(z+\cdots+z^{j-1})\right)$.

Moreover if $0 < q_0 < 1$*,* V *has a (unique) zero within* $]0,1[$ *if and only if* $E(Y) > 1$*. This condition is satisfied in particular for* $q_0 < 1/2$*.*

Proof.

Let us write z as $z = \sum_{j\neq 1} q_j z$ and factorize $z^j - z$ as usual so that we get

$$V(z) = \lambda(1-z)\left(q_0 - \sum_{j\geqslant 2} q_j(z+\cdots+z^{j-1})\right).$$

Next, let us notice that $\theta(z) = \sum_{j\geqslant 2} q_j(z+\cdots+z^{j-1})$ is strictly increasing on $[0,1]$, satisfies $\theta(0) = 0$ and that

$$\theta(1) = \sum_{j\geqslant 2} q_j(j-1) = \sum_{j\geqslant 2} jq_j - \sum_{j\geqslant 2} q_j = E(Y) - (1-q_0).$$

Therefore, for V to have a zero within $[0,1[$, it is necessary and sufficient that

$$q_0 - \theta(1) = 1 - E(Y) < 0.$$

The last assertion results from the inequality $E(Y) \geqslant 2(1-q_0)$. $\square$

3.2. *Solving the PDE*

Usually to solve the linear PDE 6, we have to exhibit a first integral, that is, a function $G(t,z)$ that keeps constant on graphs $(t, I(t))$, ie the paths of the solution of the ordinary differential equation (ODE) :

$$\frac{dI(t)}{dt} = -V(I(t)), \qquad I(0) = y, \tag{10}$$

denoted hereafter $I(t,y)$.

Indeed, if G is a first integral, then $v(t,y) \overset{def}{=} G(t, I(t,y)) = G(0,y) = v(0,y)$ satisfies

$$\begin{aligned} \partial_t v(t,y) &= \partial_t G(t, I(t,y)) + \partial_z G(t, I(t,y))\partial_t I(t,y) \\ &= \partial_t G(t, I(t,y)) - \partial_z G(t, I(t,y)) V(I(t,y)) = 0. \end{aligned}$$

Therefore, setting $I(t,y) = z$ yields $\partial_t G(t,z) = V(z)\partial_z G(t,z)$ and this corresponds exactly to Equation (6).

Next, let us notice that the more natural first integral one may suggest consists in the determination of the origin coordinate y of solutions I passing through a point z at time t since this coordinates remains constant along each trajectory graph. So, this actually amounts to seek the reciprocal of the vector function :

$$\begin{pmatrix} t \\ y \end{pmatrix} \overset{F}{\to} \begin{pmatrix} t \\ I(t,y) \end{pmatrix} = \begin{pmatrix} t \\ z \end{pmatrix}, \; that\ is\ \begin{pmatrix} t \\ z \end{pmatrix} \overset{F^{-1}}{\to} \begin{pmatrix} t \\ G(t,z) \end{pmatrix} = \begin{pmatrix} t \\ y \end{pmatrix}$$

Moreover, Equation (10), rewritten as

$$\frac{dI}{V(I)} = -dt$$

has, on any interval not containing zeroes of V, primitives H that satisfy

$$H(I) - H(I_0) = \int_{I_0}^{I} \frac{du}{V(u)} = -t.$$

So, for any fixed t, $I(t,y)$ with $I_0 = I(0,y) = y$, is implicitly defined by :

$$H(I(t,y)) = H(y) - t$$

Equivalently, the use of the equalities $G(t,z) = y$ and $I(t,y) = z$, proves that the pgf $G(t,z)$ satisfies the implicit equation :

$$H(G(t,z)) = H(z) + t. \qquad [11]$$

Let us detail now the two cases related to the existence of a zeroes of V. Since V has at most one zero z_V^* in the interval [0,1[, local inverses H^{-1} of H exist in both intervals $[0, z_V^*[$ and $]z_V^*, 1[$. Consequently, if these inverses are known by explicit expressions, we readily obtain

$$G(t,z) = H^{-1}(H(z) + t).$$

Otherwise, one can apply the implicit function theorem to get a Taylor series expansion at any order of G, or better to get, as we will see, the probability density of $X(t)$ by using the equality :

$$\partial_{z^k}^k H(G(t,z)) = \partial_{z^k}^k H(z). \qquad [12]$$

Without loss of generality, one can assume that $H(0) = 0$, so that, setting $z = 0$ in 11, yields the following formula for the extinction probability $P_{1,0}(t) = G(t,0)$:

$$H(P_{1,0}(t)) = t. \qquad [13]$$

COROLLARY 1– *The primitive H of $1/V$ on the interval $[0, z_v^*[$ satisfying H(0)=0 is the increasing inverse function of the extinction probability function $P_{1,0}(t)$.*

COROLLARY 2– *$G(t,z)$ is entirely characterized by the extinction probability function, that is :*

$$G(t,z) = P_{1,0}(t + H(z)) = P_{1,0}(t + P_{1,0}^{-1}(z)).$$

3.3. *Recovering Transition Probabilities*

In order to characterize the pdf $P_{1,n}(t) = P(X(t) = n/X(0) = 1),\ n \geqslant 0$ for any fixed t, let us recall that $P_{1,n}(t) = \partial_{z^n}^n G(t,0)/n!$ and then resort to Faà di Bruno's formula related to the $n-th$ derivatives of a function composition. Using the Bell polynomial formulation, the latter reads :

$$(f\circ g)^{(n)}(x)=\sum_{k=1}^{n}f^{(k)}(g(x))\mathbf{B}_{n,k}\left(g^{(1)}(x),g^{(2)}(x),\ \dots\ ,\ g^{(n-k+1)}(x)\right) \qquad [14]$$

where

$$\mathbf{B}_{n,k}(u_1,\dots,u_{n-k+1})=\sum_{j\in\sigma_{n,k}}\frac{n!}{j_1!\cdots j_{n-k+1}!}\left(\frac{u_1}{1!}\right)^{j_1}\cdots\left(\frac{u_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}}, \qquad [15]$$

The sum over $j\in\sigma_{n,k}$, includes nonnegative integers $j=(j_1,j_2,\cdots,j_{n-k+1})$ satisfying the constraints :

$$\begin{cases} j_1+j_2+\cdots+j_{n-k+1}=k, \\ j_1+2j_2+3j_3+\cdots+(n-k+1)j_{n-k+1}=n. \end{cases} \qquad [16]$$

We notice that when derivatives of the functions f and $f\circ g$ are known, Fa di Bruno Equation (14) generates a recursive relation for the derivatives of g.

Indeed, $g^{(n)}$ only appears for $k=1$, and in that case, the sum in (15) reduces to a single index $(j_1=0,\dots,j_{n-1}=0,j_n=1)$, that is $\mathbf{B}_{n,1}(u_1,u_2,\dots,u_n)=u_n$.

Consequently, on the set $\{x: f^{(1)}(g(x))\neq 0\}$, we get the recurrence relation :

$$g^{(n)}(x)=\frac{(f\circ g)^{(n)}(x)-\sum\limits_{k=2}^{n}f^{(k)}(g(x))\mathbf{B}_{n,k}\left(g^{(1)}(x),\dots,g^{(n-k+1)}(x)\right)}{f^{(1)}(g(x))} \qquad [17]$$

In this context, if we fix t, put $f(z)=H(z)$ and $g(z)=G(t,z)=G_t(z)$ we notice that

$$\left(H\circ G_t\right)(z)=H(z)+t\ \Rightarrow\ \left(H\circ G_t\right)^{(n)}(z)=H^{(n)}(z).$$

So, the differentiation with respect to z yields

$$G_t^{(n)}(z)=\frac{H^{(n)}(z)-\sum\limits_{k=2}^{n}H^{(k)}\left(G_t(z)\right)\mathbf{B}_{n,k}\left(G_t^{(1)}(z),\dots,G_t^{(n-k+1)}(z)\right)}{H^{(1)}\left(G_t(z)\right)}. \qquad [18]$$

Consequently, if the inverse of H is not explicitly known, one can derive the Taylor expansion $G(t,z)=\sum\limits_{n=0}^{\infty}G_t^{(n)}(0)z^n/n!$ via the recursive relation :

$$G_t^{(n)}(0)=V\left(G_t(0)\right)\left[H^{(n)}(0)-\sum_{k=2}^{n}H^{(k)}\left(G_t(0)\right)\mathbf{B}_{n,k}\left(G_t^{(1)}(0),\dots,G_t^{(n-k+1)}(0)\right)\right].$$

H the primitive of $1/V$ with $H(0)=0$, also obeys the Fa di Bruno formula :

$$H^{(k)}(x)=\sum_{l=1}^{(k-1)}\frac{(-1)^l l!}{V(x)^{l+1}}B_{k-1,l}\left(V^{(1)}(x),\ \dots,\ V^{(k-l)}(x)\right),\quad k\geqslant 2.$$

This finally implies that the $P_{1,n}(t), n \geqslant 0$ obey the recursive relation :

$$P_{1,n}(t) = \frac{V\left(P_{1,0}(t)\right)}{n!}\left[H^{(n)}(0) - \sum_{k=2}^{n} H^{(k)}\left(P_{1,0}(t)\right)\mathbf{B}_{n,k}\left(1!P_{1,1}(t), \ldots, (n-k+1)!P_{1,n-k+1}(t)\right)\right]. \qquad [19]$$

Formula 19 shows how the extinction probability $P_{1,0}(t)$ determines completely the distribution probability of the branching process. The extinction probability itself, is completely determined by the coefficients $(Q_{1,j})_{j \geqslant 0}$ as follows.

PROPOSITION 1– *The extinction probability $P_{1,0}(t)$ satisfies the equation $H(P_{1,0}(t)) = t$ and has the series expansion*

$$P_{1,0}(t) = \sum_{n=1}^{\infty} \frac{P_{1,0}^{(n)}(0)}{n!} t^n,$$

with $P_{1,0}^{(1)}(0) = Q_{1,0}$. For $n \geqslant 2$, it obeys the recurrence formula :

$$P_{1,0}^{(n)}(0) = -Q_{1,0} \sum_{k=2}^{n} H^{(k)}(0) \boldsymbol{B}_{n,k}\left(P_{1,0}^{(1)}(0),\ \ldots,\ P_{1,0}^{(n-k+1)}(0)\right),$$

with

$$H^{(k)}(0) = \sum_{l=1}^{(k-1)} \frac{(-1)^l l!}{Q_{1,0}^{l+1}} B_{k-1,l}\left(1!Q_{1,1},\ \ldots,\ (k-l)!Q_{1,k-l}\right).$$

Proof.

Since $H(P_{1,0}(t)) = t$ and $P_{1,0}(0) = 0$, differentiation with respect to t, yields

$$P_{1,0}^{(1)}(t) = 1/H^{(1)}(P_{1,0}(t)) = V(P_{1,0}(t))$$

while for $n \geqslant 2$ we get $(H \circ P_{1,0})^{(n)}(t) = 0$. The recursive Fa di Bruno formula similar to equation (18), is written in this case :

$$H^{(1)}(P_{1,0}(t))P_{1,0}^{(n)}(t) = -\sum_{k=2}^{n} H^{(k)}(P_{1,0}(t))\mathbf{B}_{n,k}\left(P_{1,0}^{(1)}(t), \ldots, P_{1,0}^{(n-k+1)}(t)\right).$$

Putting $t = 0$ and substituting 0 for x in $H^{(k)}(x)$ ends the proof. $\square$

Using Lemma (2), we finally get the following result related to the pgf :

COROLLARY 3–

$$G(t,z) = \sum_{n=1}^{\infty} \frac{P_{1,0}^{(n)}(0)}{n!}\left(H(z) + t\right)^n.$$

3.4. *Moment Equations*

The n^{th}-factorial moments of $X(t)$ are defined by

$$m_X^{[n]}(t) = E\left(X(t)(X(t)-1)\cdots(X(t)-n+1)\right),\ n \geqslant 1.$$

We already know that, whether finite or infinite, the following formula holds true :

$$m_X^{[n]}(t) = \lim_{z\uparrow 1} \partial^n_{z^n} G(t,z) = \partial^n_{z^n} G(t,1).$$

Accordingly, recalling that Y is the instantaneous offspring variable defined by Equation 8, we have the following closed form formulas for the factorial moments.

PROPOSITION 2– *If $m_X^{[n]}(t) < \infty$ for some integer n, then the following recursive formula holds true even for $E(Y) = 1$:*

$$m_X^{[1]}(t) = \exp^{\lambda(E(Y)-1)t} \qquad [20]$$

and, for $2 \leqslant n \leqslant N$,

$$m_X^{[n]}(t) = \lambda \sum_{k=2}^{n} \begin{pmatrix} n \\ k \end{pmatrix} m_Y^{[k]} \int_0^t \exp^{n\lambda(E(Y)-1)(t-s)} m_X^{[n-k+1]}(s)ds. \qquad [21]$$

Proof. According to Equation (6) and using the Leibniz rule for the derivatives of a product, we have (with some abuse of notation) :

$$\begin{aligned}\partial_t \partial^n_{z^n} G(t,z) &= \partial^n_{z^n}\partial_t G(t,z) = \partial^n_{z^n}\left(V(z)\partial_z G(t,z)\right)\\ &= \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} V^{(k)}(z) G_t^{(n-k+1)}(z).\end{aligned}$$

So, for $n = 1$, the previous equation gives :

$$\partial_t \partial_z G(t,z) = V(z)\partial^2_{z^2} G(t,z) + V^{(1)}(z)\partial_z G(t,z).$$

Next, put $z = 1$ and observe that $V(1) = 0,\ V^{(1)}(1) = \lambda(E(Y)-1)$ and that $V^{(k)}(1) = \lambda m_Y^{[k]}$. Then, equating $m_X^{[k]}(t) = \partial^k_{z^k} G(t,1)$ yields :

$$\partial_t m_X^{[1]}(t) = \lambda(E(Y)-1)m_X^{[1]}(t)$$
$$\partial_t m_X^{[n]}(t) = \lambda n(E(Y)-1)m_X^{[n]}(t) + \lambda \sum_{k=2}^{n} \begin{pmatrix} n \\ k \end{pmatrix} m_Y^{[k]} m_X^{[n-k+1]}(t)$$

This is a recursive linear system of homogeneous (for $n = 1$) and inhomogeneous (for $n > 1$) ODE for $m_X^{[n]}(t)$ when the $m_X^{[k]}(t)$, $1 \leqslant k \leqslant n-1$ are previously determined. These ODE solutions are classical and respectively given by Equations (20) and (21) by taking its account of the initial conditions $m_X^{[n]}(0) = \delta_{\{n=1\}}$.

Note that these formulas remain valid even if $E(Y) = 1$.

Since $E(X(t)) = m^{[1]}(t)$ and $Var(X(t)) = m^{[2]}(t) + m^{[1]}(t) - \left(m^{[1]}(t)\right)^2$, in particular we get the following formulas :

COROLLARY 4– *For* $E(Y)-1 \neq 0$,

$$\begin{aligned} E(X(t)) &= \exp^{\lambda(E(Y)-1)t} && [22] \\ Var(X(t)) &= \left(\frac{m_Y^{[2]}}{E(Y)-1}-1\right)\exp^{\lambda(E(Y)-1)t}\left(\exp^{\lambda(E(Y)-1)t}-1\right) && [23] \end{aligned}$$

while, for $E(Y)=1$ *:*

$$E(X(t))=1, \quad Var(X(t))=\lambda m_Y^{[2]} t. \qquad [24]$$

4. Examples of Branching Processes

We now investigate some examples of parameterized distribution of one type branching processes.

4.1. *Quadratic Branching*

It is the simplest one-type homogeneous branching process. It requires at most 2 parameters (λ, q) and corresponds to the following infinitesimal transition rules :

$$Q_{1,j} = \begin{cases} \lambda q & if j=0 \\ -\lambda & if j=1 \\ \lambda(1-q) & if j=2 \end{cases}$$

So, we have in this case

$$V(z)=\lambda(q-z+(1-q)z^2)$$

Here the instantaneous branching variable Y is dichotomous with pdf

$$P(Y=0)=q, P(Y=2)=1-q; \; and\ pgf\ G_Y(z)=q+(1-q)z^2.$$

This is the reason why it is named quadratic.

Regarding the factorial moments we have

$$E(Y)=m_Y^{[1]}=m_Y^{[2]}=2(1-q), \;\; whereas\ \ m_Y^{[k]}=0, k>2.$$

The condition $E(Y)-1>0$ corresponds to the explosive case $q<1/2$.

4.1.1. Searching Primitives of $1/V$

In order to integrate the rational function $1/V$, we have to distinguish two possible cases depending on the single or multiple nature of the zeroes of V. Clearly, V has the factorization

$$V(z)=\lambda(1-q)(1-z)\left(\frac{q}{1-q}-z\right).$$

The zeroes $z_1=1$ and $z_V=q/(1-q)$ coincide iff $q=1/2$.

Case A : $q = 1/2$ ***i.e.*** $z_V = 1$**.**
The primitive H of $1/V$, is defined for any $I \in [0, 1[$ by

$$H(I) = \int_{.}^{I} \frac{dz}{V(z)} = \frac{2}{\lambda} \int_{.}^{I} \frac{dz}{(1-z)^2} = \frac{2}{\lambda(1-I)} + C.$$

Taking $C = 0$, we retain the formula with $H(0) = 0$:

$$H(I) = \frac{2I}{\lambda(1-I)}.$$

Since $\lim_{z \uparrow 1} H(z) = +\infty$, H has an inverse on $[0, 1[$, that is

$$H^{-1}(v) = \frac{\lambda v}{2 + \lambda v}, \quad v \geqslant 0.$$

Case B : $q \neq 1/2$.
The partial fraction expansion of $1/V$ is given by

$$\frac{1}{V} = \frac{1}{\lambda(2q-1)} \left(\frac{1}{1-z} - \frac{1}{z_V - z} \right).$$

Any primitive of $1/V$, defined on any adequate sub interval of [0,1[is written :

$$H(I) = \int_{.}^{I} \frac{dz}{V(z)} = \frac{1}{\lambda(2q-1)} \log \left(\frac{|z_V - I|}{|1-I|} \right) + Cte.$$

We have to examine the two cases regarding the location of the root z_V with the interval $[0, 1[$.

Case B1. The extinction case $z_V > 1$ *i.e.* $q > 1/2$.

H is well defined and is increasing on $[0, 1[$. Convening that $H(0) = 0$, we get

$$H(I) = \frac{1}{\lambda(2q-1)} \left[\log \left(\frac{z_V - I}{1-I} \right) - \log z_V \right]$$

Again, $\lim_{z \uparrow 1} H(z) = +\infty$, and H an inverse on $[0, 1[$:

$$H^{-1}(v) = \frac{1 - \exp^{\lambda(2q-1)v}}{\frac{1}{z_V} - \exp^{\lambda(2q-1)v}}, \quad v \geqslant 0.$$

Note that the exponential coefficient $\lambda(2q-1)$ is positive here.

Case B2. The explosive case $z_V < 1$ *i.e.* $q < 1/2$.

- On $D_1 = [0, z_V[$, H is increasing. So we can take for H and its inverse the same formulas as in B1, with $H(0) = 0$, save that $\lim_{z \uparrow z_V} H(z) = +\infty$ here.

- On $D_2 =]z_V, 1[$, H decreases and satisfies $\lim_{z \downarrow z_V} H(z) = +\infty$ whereas

$$\lim_{z \uparrow 1} H(z) = -\infty.$$

Moreover V attains it minimum at the inflection point $z* = 1/2(1-q) \in [z_V, 1[$, that also satisfies $z * - z_V = 1 - z*$. So one can choose the particular primitive such that $H(z*) = 0$, that is :

$$H(I) = \frac{1}{\lambda(2q-1)} \log\left(\frac{I - z_V}{1 - I}\right)$$

with inverse

$$H^{-1}(v) = \frac{z_V + \exp^{\lambda(2q-1)v}}{1 + \exp^{\lambda(2q-1)v}}, \quad v \in \mathbb{R}.$$

4.1.2. Pgf and Pdf of $X(t)$

Before stating the exact expression of pgf and pdf of $X(t)$, we give the following definition.

DEFINITION 1– *An integer valued random variable N is said to have a zero inflated geometric distribution with parameters $(\alpha, \beta),\ 0 \leqslant \alpha,\ \beta \leqslant 1$, if it is the mixture of a geometric distribution with a Dirac mass at 0, that is :*

$$\begin{aligned} P(N = 0) &= 1 - \alpha \\ P(N = k) &= \alpha\beta(1-\beta)^{k-1}, \quad k \geq 1 \end{aligned}$$

A zero inflated geometric distribution has the following characteristics :

$$G_N(z) = \frac{(1-\alpha) - ((1-\beta-\alpha)z}{1-(1-\beta)z},$$

$$E(N) = \frac{\alpha}{\beta} \textit{ and } Var(N) = \frac{\alpha(2-\beta-\alpha)}{\beta^2}.$$

PROPOSITION 3– *The pgf $G(t,z)$ of $X(t)$ has the following form*

$$G(t,z) = \frac{(1-\alpha_t) - (1-\beta_t-\alpha_t)z}{1-(1-\beta_t)z}$$

where :

1. $\alpha_t = \beta_t = \frac{2}{\lambda t + 2}$, *if* $q = 1/2$,
2. *for* $q \neq 1/2$ *and* $z_V = q/(1-q)$ *:*
$$\alpha_t = \frac{1-z_V}{1-z_V \exp^{\lambda(2q-1)t}}, \quad \beta_t = \frac{(1-z_V)\exp^{\lambda(2(q-1)t}}{1-z_V \exp^{\lambda(2(q-1)t}}.$$

Proof. One has only to compute $G(t,z) = H^{-1}(H(z)+t)$ in the different cases (A,B1,and B2) an notice that whatever are the B sub-cases, all formulas lead to the some expression. □

COROLLARY 5– *For quadratic branching, $X(t)$ obeys a zero inflated geometric distribution with parameters (α_t, β_t). In particular, we have :*

1. $P(X(t) = 0) = 1 - \alpha_t$
2. $E(X(t)) = 1,\ Var(X(t)) = \lambda t$ *for* $q = 1/2$,
3. *for* $q \neq 1/2$ *:*
$$E(X(t)) = \exp^{-\lambda(2q-1)t}, \quad Var(X(t)) = \frac{\exp^{-\lambda(2q-1)t}(\exp^{-\lambda(2q-1)t} - 1)}{1-2q}.$$

4.2. *Cubic Branching*

Here, the branching process is cubic and depends on 3 parameters, that is

$V(z) = \lambda(G_Y(z) - z)$, satisfies

$$G_Y(z) = q_0 + q_2 z^2 + q_3 z^3,$$

with the constraints : $q_0 + q_2 + q_3 = 1, \;\; 0 < q_0, q_3 < 1.$

In that case, we have

$$E(Y) = m_Y = 2q_2 + 3q_3, \;\; m_Y^{[2]} = G_Y''(1) = 2q_2 + 6q_3$$

and

$$Var(Y) = (3q_0 + q_2)(3q_3 + 2q_2) - 2q_2.$$

The explosion condition $E(Y) - 1 > 0$, equivalent to say that a zero of V is in $]0, 1[$, is thus written $q_2 + 2q_3 > q_0$.

The polynomial factorization of V gives

$$V(z) = \lambda(z-1)\left(q_3 z^2 + (1-q_0)z - q_0\right) = \lambda q_3 (z-1)(z-z_+)(z-z_-),$$

where

$$z_\pm = \frac{(q_0 - 1) \pm \sqrt{(1-q_0)^2 + 4q_0 q_3}}{2q_3}.$$

Clearly, under the parameter constraints we have $z_- < 0, z_+ > 0$, so using Lemma 1, one can see that V has a zero $z_V = z_+$ in $]0, 1[$ (resp. $z_+ = 1$) iff $q_2 + 2q_3 > q_0$ (resp. $q_2 + 2q_3 = q_0$).

4.2.1. Moment Formulas

Proposition 2 readily yields the following recursive formulas.

PROPOSITION 4– *Depending on the mean value $m_Y = 2q_2 + 3q_3$, the two first moments are written :*

1. *For $m_Y = 1$, $E(X(t)) = 1$, $Var(X(t)) = \lambda(1 + 3q_3)t$,*
2. *for $m_Y \neq 1$: $E(X(t)) = \exp^{\lambda(m_Y - 1)t}$,*

$$Var(X(t)) = \frac{3q_3 + 1}{m_Y - 1} \exp^{\lambda(m_Y-1)t}\left(\exp^{\lambda(m_Y-1)t} - 1\right),$$

3. *whereas for $k \geq 3$, we obtain :*

$$m_X^{[k]}(t) = \lambda k(k-1) \int_0^t \exp^{k\lambda(m_Y-1)(t-s)} \left((q_2 + 3q_3) m_X^{[k-1]}(s) + (k-2) q_3 m_X^{[k-2]}(s)\right) ds$$

4.2.2. Probability Generating Functions

We now seek the form of the primitive H depending on the location of the zero z_+ of V.

Case A : $z_+ \neq 1$. The three zeroes are distinct, so taking $H(0) = 0$ yields

$$H(I) = \int_{\cdot}^{I} \frac{dz}{V(z)} = \frac{1}{\lambda q_3} \log\left(|I-1|^A |I-z_-|^B |I-z_+|^C\right),$$

where

$$A = -(B+C), \quad B = \frac{1}{(1-z_-)(z_+-z_-)}, \quad C = \frac{1}{(z_+-1)(z_+-z_-)},$$

that is,

$$H(I) = \frac{1}{\lambda q_3 (z_+-z_-)} \left(\frac{1}{(1-z_-)} \log\left(|1 + \frac{1-z_-}{I-1}| \right) - \frac{1}{(1-z_+)} \log\left(|1 + \frac{1-z_+}{I-1}| \right) \right).$$

Case B : $z_+ = 1$. So, 1 corresponds to a double root of V, and then

$$\frac{1}{V(z)} = \frac{1}{\lambda q_3} \left(\frac{A}{(z-z_-)} + \frac{B}{(z-1)} + \frac{C}{(z-1)^2} \right),$$

where

$$A = \frac{1}{(1-z_-)^2}, \quad B = -\frac{1}{(1-z_-)^2}, \quad C = \frac{1}{(1-z_-)}.$$

Therefore, taking $H(0) = 0$, yields

$$H(I) = \frac{1}{\lambda q_3} \left(\log(|I-z_-|^A |I-1|^B) - C(I-1)^{-1} \right).$$

Moreover, in the case B, we have : $1 - z_- = (3 + q_3^{-1})/2$, which ultimately leads to :

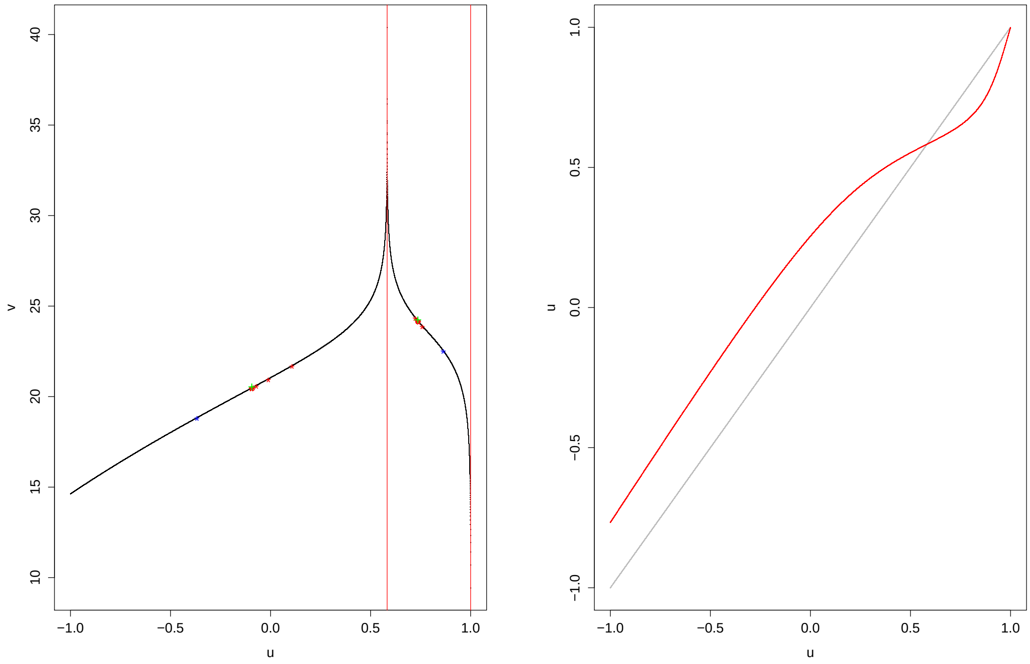
$$H(I) = \frac{1}{\lambda q_3 (1-z_-)^2} \left(\log(|1 + \frac{1-z_-}{I-1}|) + \frac{1-z_-}{I-1} \right)$$

Remark. For both cases, save for some specific parameter values, the inverse of H cannot be expressed by classical functions, however H being simple enough this allows fast computation procedures for inversion (see example in Figure 1). In the same vein, H being simple, one can use the recursive relations of Proposition 1 to approximate the extinction probability function $P_{1,0}(t)$ at any order and therefore get an approximation of the pgf $G(t,z) = P_{1,0}(t + H(z))$.

4.3. *Geometric Branching Distribution*

This is a family of branching distributions with a 3-dimensional parameter (λ, q_0, α), where the offspring distribution takes the form

$$G_Y(z) = q_0 + (1-q_0) z^2 \frac{\alpha}{1-(1-\alpha)z}.$$



A B

Figure 1. – A. Example of numerical approximation of the primitive $H(z)$ for cubic branching process. **B.**The inverse function of H yields the pgf $G(5,z) = H^{-1}(H(z)+5),\ z \in [-1,1]$, for a cubic branching at time $t = 5$.

As already pointed out, Y can be written $Y = U(2+W)$ where U and W are independent, U is Bernoulli distributed $B(1-q_0)$ and W is geometrically distributed with probabilities

$$P(W = k) = \alpha(1-\alpha)^k,\ k \geqslant 0.$$

Therefore, we have $m_Y^{[1]} = (1-q_0)(1+\alpha)/\alpha$, and for $k \geqslant 2$

$$m_Y^{[k]} = \alpha(1-q_0)(1-\alpha)^{(k-2)} \sum_{l \geqslant 0} \frac{(k+l)!}{l!}(1-\alpha)^l.$$

In particular, we also have $Var(Y) = (1-q_0)(4+(1-\alpha)(2+\alpha)/\alpha^2)$.

Here $V(z) = \lambda(G_Y(z) - z)$ is a rational function :

$$V(z) = \lambda \frac{q_0 - (1+q_0(1-\alpha))z + (1-q_0\alpha)z^2}{1-(1-\alpha)z}.$$

It can be factorized into :

$$V(z) = \tilde{\lambda}\frac{(z-1)(z-z_v)}{z-z_*},$$

where $\tilde{\lambda} = (\lambda(1-q_0\alpha)/(\alpha-1),\ \ z_V = q_0/(1-\alpha q_0)$ and $z_* = 1/(1-\alpha)$.

In addition to $z_* > 1$, we notice that $\alpha > 0$ and $q_0 \neq 1$ imply also that $z_* \neq z_V$, so that we have to deal with only two cases.

Case $z_V = 1$**.** It corresponds to equality $q_0 = 1/(1+\alpha)$ and means that 1 is a double root. Then setting $H(0) = 0$, leads to

$$H(I) = \frac{1}{\tilde{\lambda}} \int^{I} \frac{dz}{V(z)} = \frac{1}{\tilde{\lambda}} \left(\log |I-1| - \frac{1-z_*}{I-1} \right)$$

H is well defined and increases on $[0,1[$. It equals the convex combination

$$H(I) = -\frac{1}{\lambda(1-q_0\alpha)} \left((1-\alpha)\log(1-I) + \alpha \frac{1}{1-I} \right)$$

and has an inverse on $[0,+\infty[$, which is precisely the extinction probability $P_{1,0}(t)$.

Case $z_V \neq 1$. We have not to distinguish between the extinction and non extension cases corresponding respectively to $z_V > 1$) and $z_V < 1$, since for both we obtain :

$$H(I) = \frac{1}{\tilde{\lambda}} \left[\left(1 - \frac{z_* - z_V}{1-z_V} \right) \log |I-1| + \frac{z_* - z_V}{1-z_V} \log |I - z_V| \right].$$

In particular, on the interval $[0, 1 \wedge z_V[$, the function H is written

$$H(I) = \frac{1}{\tilde{\lambda}} \log \left((z_V - I)^{\theta} (1-I)^{1-\theta} \right),$$

with $\theta = (z_* - z_V)/(1-z_V)$ and $\tilde{\lambda} = \lambda q_0 z_*/z_V$.

For both cases, the inverse H^{-1} is not a classical function but enough simple to be very well approximated as well as are the recursive formulas for transition probabilities.

4.4. *Branching Processes Are Exceptionally Poisson Distributed*

This subsection do not deal with Poisson offspring distributions, but shows that contrarily to the frequently made assumption in applications, the Poisson distributions in branching processes, even inflated in zero, occur only for a very specific and simple dynamics. First, we give the following definition.

DEFINITION 2– *An integer valued r.v. X is said to be zero inflated Poisson distributed i.e. $X \sim ZIP(\alpha,\nu), 0 \leqslant \alpha \leqslant 1$ and $\nu > 0$, if*

$$\begin{aligned} P(X=0) &= 1-\alpha \\ P(X=n) &= \frac{\alpha e^{-\nu}}{1-e^{-\nu}} \frac{\nu^n}{n!}. \end{aligned}$$

Consequently, its generating function is written :

$$G_X(z) = 1 - \alpha \frac{e^{\nu(z-1)} - 1}{e^{-\nu} - 1}.$$

Here are characterized branching processes with marginal zero inflated Poisson distributions.

PROPOSITION 5– *A branching process $X(t)$ with infinitesimal generator given by Equation 2.2.2. $(Q_{i,j}(\lambda, \mu, p_k, k \geq 1))$ has its marginal distributions zero inflated Poisson $ZIP(\alpha(t), \nu(t))$ for some non negative smooth functions $0 \leqslant \alpha(t) \leqslant 1$ and $\nu(t) \geqslant 0$ if and only if it corresponds to the degenerate case of a pure death process with death rate $\mu = 1$. In that case we have $X(0) \sim ZIP(\alpha(0), \nu(0))$, $\alpha'_t/\alpha_t = -\nu'_t e^{-\nu_t}/(e^{-\nu_t} - 1)$, and $\nu(t) = \nu(0)e^{-\lambda t}$.*

Proof

For sake of simplicity, let us denote $\alpha_t = \alpha(t)$, and $\nu_t = \nu(t)$ and then assume that the marginal $X(t)$ is $ZIP(\alpha_t, \nu_t)$ distributed with pgf

$$g(t, z) = 1 - \alpha_t \frac{e^{\nu_t(z-1)} - 1}{e^{-\nu_t} - 1}.$$

First, our smoothness assumptions imply that the initial condition $X(0)$ is of the same type with $g(0, \mathbf{z}) = 1 - \alpha_0 \frac{e^{\nu_0(z-1)} - 1}{e^{-\nu_0} - 1}$.

Next, as a pgf of a branching process, $g(t, \mathbf{z})$ satisfies the fundamental equation $\partial_t g(t, \mathbf{z}) = V(z)\, \partial_z g(t, z)$, which thereby is written

$$\frac{\alpha'_t}{\alpha_t}\left(e^{\nu_t(z-1)} - 1\right) + \nu'_t\left((z-1)e^{\nu_t(z-1)} + e^{-\nu_t}\frac{e^{\nu_t(z-1)} - 1}{e^{-\nu_t} - 1}\right) = V(z)\nu_t e^{\nu_t(z-1)}. \quad [25]$$

Setting $\theta_t = \frac{\alpha'_t}{\alpha_t}e^{\nu_t} + \frac{\nu'_t}{e^{-\nu_t} - 1}$, yields

$$V(z) = \frac{1}{\nu_t}\left(\frac{\alpha'_t}{\alpha_t} - \frac{\nu'_t}{e^{-\nu_t} - 1} - \theta_t e^{-\nu_t z} + \nu'_t z\right)$$

The series expansion of $e^{\nu_t z}$ in addition to the canonical form $V(z) = \lambda(F_Y(z) - z)$ given in Equation 7, with $G_Y(z) = q_0 + \sum_{k=2}^{\infty} q_k z^k$, entail that

$$\begin{aligned}
\lambda &= -\frac{\nu'_t + \theta_t \nu_t}{\nu_t} \\
q_0 &= \frac{\alpha'_t}{\alpha_t}\frac{e^{\nu_t} - 1}{\nu'_t + \theta_t \nu_t} \\
q_k &= \frac{\theta_t}{\nu'_t + \theta_t \nu_t}\frac{(-\nu_t)^k}{k!}, \quad k \geq 2
\end{aligned}$$

are all being non negative and independent of t.

However this is possible only if $\theta_t = 0$ for all t, otherwise the q_k would have opposite sign for even and odd k.

As a consequence, we have $\alpha'_t/\alpha_t = -\nu'_t e^{-\nu_t}/(e^{-\nu_t} - 1)$, which taking account of the initial condition finally yields $\alpha_t = \alpha_0(e^{-\nu_t} - 1)/(e^{-\nu_0} - 1)$.

Therefore $q_0 = 1$ and so the death parameter μ of the infinitesimal generator equals 1. Finally this simply means that the process corresponds to a pure death (without progeny) dynamics of the individuals of the initial population with an exponential lifetime parameter $\nu_t = \nu_0 e^{-\lambda t}$. □

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