Sur la normalité asymptotique d'un estimateur à noyau du quantile conditionnel pour des données censurées et associées

On the asymptotic normality of a kernel conditional quantile estimator for censored and associated data

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RÉSUMÉ. L'objet du présent article est d'établir la normalité asymptotique d'un estimateur à noyau du quantile conditionnel dans un modèle de censure droite, pour lequel les durées de vie ainsi que les covariables sont supposées satisfaire une dépendance de type association.

ABSTRACT. This paper aims at establishing the asymptotic normality for a kernel conditional quantile estimator in a right censorship model for which, the lifetime observations and the covariates are assumed to satisfy an association dependency type.

MOTS-CLÉS. Association, données censurées, estimateur de Kaplan-Meier, normalité asymptotique, quantile conditionnel.

KEYWORDS. Association, asymptotic normality, censored data, conditional quantile, Kaplan-Meier estimator.

Introduction

In classical statistical inference, the observed random variables (rv's) of interest are generally assumed to be independent and identically distributed (iid). However, it is more common to have dependent variables in some real life situations. Dependent variables are present in several backgrounds such as medicine, biology and social sciences. Associated rv's are of considerable interest when dealing with survival analysis, reliability problems, percolation theory and some models in statistical mechanics.

In the literature, two kinds of dependency are widely used : mixing [DOUKHAN 1994] and association [ESARY et al. 1967]. These two concepts are not completely dissociated. In fact, we can find sequences that are associated but not mixing, associated and mixing, and mixing but not associated.

Recall that a set of finite family of rv's $(T_1, \dots, T_n)$ which are defined on a common probability space $(\Omega, \mathscr{A}, \mathbb{P})$ are said to be associated if for every pair of non-decreasing componentwise functions $\Psi_1(\cdot)$ and $\Psi_2(\cdot)$ from $\mathbb{R}^n$ to $\mathbb{R}$,

$$cov(\Psi_1(T_1, \dots, T_n), \Psi_2(T_1, \dots, T_n)) \geq 0,$$

whenever the covariance exists. An infinite family of rv's is associated if any finite sub-family is a set of associated rv's. The notion of association was firstly introduced by [ESARY et al. 1967] mainly for an application in reliability and its main advantage compared to mixing is that the conditions of limit theorems are easier to verify : indeed, a covariance is much easier to compute than a mixing coefficient.

Let $\{T_n, n \geq 1\}$ be a strictly stationary sequence of associated rv's of interest having an unknown absolutely continuous distribution function (df) F. This variable can be considered as a lifetime under biomedical studies. The major characteristic of survival time is the incompleteness. Random right censoring is a well-known phenomenon in survival analysis since, the lifetime data may not be completely observable if the patient is still alive at the end of study, or is dead for another reason. Hence, the avai-

lable data provide partial information. In this case, the variable of interest T is subject to right censoring by another non-negative rv C. We assume that the censoring lifetimes are independent and identically distributed (iid) and possess an unknown df G. We take in consideration the presence of a strictly stationary and associated covariate $\mathbf{X}$ taking values in $\mathbb{R}^d$. Under this model, the observable sequence is $\{(Y_i = \min(T_i, C_i), \delta_i = \mathbb{1}_{\{T_i \leq C_i\}}, \mathbf{X}_i); 1 \leq i \leq n\}$ where $\mathbb{1}_A$ denotes the indicator function of the event A. In order to ensure the identifiability of the model we assume that the censoring times $\{C_i, 1 \leq i \leq n\}$ are independent of $\{(\mathbf{X}_i, T_i), 1 \leq i \leq n\}$.
Under the model we consider here, we establish the asymptotic normality of the kernel conditional quantile estimator defined in (6). The rest of the paper is organized as follows : Section 1 gives the required notations as well as the estimators. The assumptions needed to state our results are gathered in Section 2 while Section 3 is devoted to the proofs of the main results.

1. Estimators and notations

The conditional df $F(t|x)$ can be written

$$F(t|x) = \frac{F_1(x,t)}{l(x)}, \qquad [1]$$

where

$$F_1(x,t) = \frac{\partial F(x,.)}{\partial x} := \frac{\partial^d F(x,.)}{\partial x_1 ... \partial x_d},$$

with $F(.,.)$ the joint df. The conditional quantile of T given $X = x$ for $p \in (0,1)$ is given by

$$\xi_p(x) = \inf\{t, F(t|x) \geq p\}. \qquad [2]$$

In the complete data case (no censoring), the traditional kernel estimator of $F(t|\mathbf{x})$ is given by

$$F_{\mathbb{1},n}(t|\mathbf{x}) = \sum_{i=1}^{n} \omega_{in}(\mathbf{x}) \mathbb{1}_{\{T_i \leq t\}}, \qquad [3]$$

where $\omega_{in}(.)$ are measurable functions. These functions called weights were introduced by Nadaraya-Watson in the context of the kernel regression and defined by

$$\omega_{in}(\mathbf{x}) = \frac{K_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_{n,1}}\right)}{\sum_{j=1}^{n} K_d\left(\frac{\mathbf{x}-\mathbf{X}_j}{h_{n,1}}\right)} = \frac{\frac{1}{nh_{n,1}^d} K_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_{n,1}}\right)}{l_n(\mathbf{x})},$$

with the convention $0/0 = 0$. Here K_d is a kernel function on $\mathbb{R}^d$ such that for any $\mathbf{x}_l = (x_l^1, x_l^2, \ldots, x_l^d)$, we have

$$K_d\left(\frac{\mathbf{x}-\mathbf{x}_l}{h_{n,1}}\right) = \prod_{i=1}^{d} K\left(\frac{x^i - x_l^i}{h_{n,1}}\right),$$

whereas $h_{n,1}$ is a positive sequence of bandwidths tending to 0 along with n and $l_n(.)$ is the Parzen-Rosenblatt kernel estimator of $l(.)$.
In the sequel, the weights are adapted to the censoring case, that is

$$\omega_{in}(\mathbf{x}) = \frac{1}{nh_{n,1}^d}\frac{\delta_i}{\overline{G}(Y_i)l_n(\mathbf{x})}K_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_{n,1}}\right). \qquad [4]$$

The censoring df G is usually unknown and its appropriate nonparametric estimator is the well-known Kaplan-Meier one, viz

$$G_n(t) = 1 - \prod_{i=1}^{n}\left[1-\frac{1-\delta_{(i)}}{n-i+1}\right]^{\mathbb{1}\{Y_{(i)}\leq t\}},$$

where $Y_{(1)} \leq Y_{(2)} \leq ...... \leq Y_{(n)}$ are the order statistics of Y_i and $\delta_{(1)}, \delta_{(2)}, \ldots, \delta_{(n)}$ are the corresponding indicators of non censoring.
Using the weights defined in (4), [OULD SAID 2006] established a strong uniform consistency rate for the estimator in (3) in the iid case and d=1. The smoothed version of $F_{\mathbb{1},n}(\cdot|\cdot)$, known as « double kernel estimation approach » is

$$F_n(t|\mathbf{x}) = \frac{\frac{1}{nh_{n,1}^d}\sum_{i=1}^{n}\frac{\delta_i}{\overline{G}_n(Y_i)}K_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_{n,1}}\right)H\left(\frac{t-Y_i}{h_{n,2}}\right)}{\frac{1}{nh_{n,1}^d}\sum_{i=1}^{n}K_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_{n,1}}\right)} =: \frac{F_{1,n}(\mathbf{x},t)}{l_n(\mathbf{x})}. \qquad [5]$$

The strong consistency and the asymptotic normality of this estimator was studied in the iid case and under α-mixing condition by [OULD SAID and SADKI 2008, 2011]. Here, the bandwidth $h_{n,2}$ is not necessarily equal to $h_{n,1}$ and they will be denoted by $h_1 := h_{n,1}$ and $h_2 := h_{n,2}$.
Note that the estimator in (5) is an adapted version of that of [YU and JONES 1998] to the censoring case. Originally, this smooth estimate for complete data was proposed and discussed by the last authors mainly to avoid the crossing problem which occurs when using an indicator function instead of a continuous df. It follows that, in view of (5), a natural estimator of (2) can be computed by

$$\xi_{p,n}(\mathbf{x}) = \inf\{t, F_n(t|\mathbf{x}) \geq p\}. \qquad [6]$$

Recall that censored-associated data was studied for the first time by [FERRANI et al. 2016] while the strong uniform consistency of the estimator under study was stated by [DJELLADJ and TATACHAK 2019]. Some supporting evidence show that in presence of outliers and for heavy-tailed or asymetric distributions, conditional quantiles can be helpful.
To overcome some difficulties encountered in computing some operators related to our estimators, the following pseudo estimate will be helpful in further calculations, namely

$$\tilde{F}_n(t|x) =: \frac{\tilde{F}_{1,n}(x,t)}{l_n(x)} = \frac{\frac{1}{nh_1^d}\sum_{i=1}^{n}\frac{\delta_i}{\overline{G}(Y_i)}K_d\left(\frac{x-X_i}{h_1}\right)H\left(\frac{t-Y_i}{h_2}\right)}{\frac{1}{nh_1^d}\sum_{i=1}^{n}K_d\left(\frac{x-X_i}{h_1}\right)}.$$

Thereafter, we will use the conditional pdf $f(t|.)$ defined by $f(t|.) = \dfrac{\partial}{\partial t}F(t|.)$. Hence, the corresponding estimators are given by

$$f_n(t|x) = \frac{\frac{\partial}{\partial t}F_{1,n}(x,t)}{l_n(x)} \quad \text{and} \quad \tilde{f}_n(t|x) = \frac{\frac{\partial}{\partial t}\tilde{F}_{1,n}(x,t)}{l_n(x)}, \qquad [7]$$

where

$$\frac{\partial}{\partial t}F_{1,n}(x,t) =: f_n(\mathbf{x},t) = \frac{1}{nh_1^d h_2}\sum_{i=1}^{n}\frac{\delta_i}{\overline{G}_n(Y_i)}K_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_1}\right)H^{(1)}\left(\frac{t-Y_i}{h_2}\right)$$

and

$$\frac{\partial}{\partial t}\tilde{F}_{1,n}(x,t) =: \tilde{f}_n(\mathbf{x},t) = \frac{1}{nh_1^d h_2}\sum_{i=1}^{n}\frac{\delta_i}{\overline{G}(Y_i)}K_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_1}\right)H^{(1)}\left(\frac{t-Y_i}{h_2}\right)$$

where $H^{(1)}$ denotes the first derivative of H. Applying Taylor expansion of $F_n(.|.)$ in the neighborhood of ξ_p, we obtain

$$F_n(\xi_{p,n}(x)|x) - F_n(\xi_p(x)|x) = (\xi_{p,n}(x) - \xi_p(x))f_n(\xi^*_{p,n}(x)|x).$$

Here, $\xi^*_{p,n}$ lies between ξ_p and $\xi_{p,n}$. This expansion permits to state asymptotic results for $(\xi_{p,n}(x)-\xi_p(x))$ through those stated for $(F_n(\xi_{p,n}(x)|x) - F_n(\xi_p(x)|x))$.

2. Assumptions and main results

In the sequel, m_1 stands for a positive constant taking different values and τ will denote a positive real number satisfying $\tau < \tau_F < \tau_G$ where, for any df W, $\tau_W := \sup\{y; W(y) < 1\}$. Define $\Omega_0 = \{\mathbf{x} \in \mathbb{R}^d / l(\mathbf{x}) \geq m_0 := \inf_x l(\mathbf{x}) > 0\}$ and let Ω and $\mathcal{C}$ be compact sets included in Ω_0 and $[0,\tau]$, respectively. The main results will be stated using the following assumptions

A1. The bandwidths h_1 and h_2 satisfy
(i) $h_1 \to 0$, $nh_1^{2\alpha+d(1-\alpha)} \to +\infty$ with $\alpha \in (0,1)$ and $\frac{\log^5 n}{nh_1^d} \to 0$ as $n \to +\infty$,
(ii) $h_2 \to 0$, $nh_1^d h_2 \to +\infty$ as $n \to +\infty$,
(iii) $nh_1^d h_2^4 \to 0$ and $nh_1^{d+4} \to 0$ as $n \to +\infty$,
(iv) $v_s h_1^d \to 0$ with v_s a sequence of real numbers ;

A2. The kernel K_d is a bounded pdf, compactly supported and satisfies :
(i) K_d is Hölder continuous of order α,
(ii) $\int_{\mathbb{R}^d} u_j K_d(\mathbf{u})\mathbf{du} = 0$, for all $j = 1, ..., d$, where $\mathbf{u} = (u_1, ..., u_d)^\top$,
(iii) The kernel K_d has bounded partial first derivatives ;

A3. The function H in (5) is of class $\mathcal{C}^1$. Furthermore, its derivative $H^{(1)}$ is assumed to be compactly supported and satisfies the properties of a second order kernel and
(i) $\int_{\mathbb{R}} |t|^\alpha H^{(1)}(t)dt < +\infty$;

A4. The marginal density $l(.)$ is bounded and twice differentiable with $\sup_{\mathbf{x}\in\Omega}\left|\dfrac{\partial^k l(\mathbf{x})}{\partial x_i \partial x_j^{k-1}}(x)\right| < \infty$ for $i,j = 1,\dots,d$ and $k = 1,2$;

A5. The joint pdf $f(.,.)$ is bounded and differentiable up to order 3, moreover
(i)The conditional df $F(t|x)$ satrisfies the lipschitz condition of order u_1 and u_2 with respect to $\mathbf{x}$ and t and
$\forall(\mathbf{x}_1, \mathbf{x}_2) \in \Omega^2, \forall(t_1, t_2) \in \mathbb{R}^2, |F(t_1|\mathbf{x}_1) - F(t_2|\mathbf{x}_2)| \leq m_1(|\mathbf{x}_1 - \mathbf{x}_2|^{u_1} + |t_1 - t_2|^{u_2}),$
(ii) $\forall \mathbf{x} \in \Omega$, we have that $\int_{\mathbb{R}} |t| f(t|\mathbf{x}) dt < +\infty$;

A6. The joint pdf $l_{i,j}(.,.)$ of $(\mathbf{X}_i, \mathbf{X}_j)$ is bounded ;

A7. The joint pdf $f(.,.,.,.)$ of $(\mathbf{X}_i, Y_i, \mathbf{X}_j, Y_j)$ is bounded ;

A8. Let us define Λ_{ij} as follows

$$\Lambda_{ij} := \sum_{k=1}^{d} \sum_{l=1}^{d} Cov(X_i^k, X_j^l) + 2 \sum_{k=1}^{d} Cov(X_i^k, Y_j) + Cov(Y_i, Y_j),$$

with X_i^k the k-th component of $\mathbf{X}_i$, such that for all $j \geq 1$ and $r > 0$

$$\sup_{i:|j-i|\geq r} \Lambda_{ij} =: \rho(r) \leq \gamma_0 e^{-\gamma r}, \text{ for all } \gamma_0, \gamma > 0;$$

A9. The function $\varsigma(\mathbf{x}) = \int_{\mathbb{R}} \frac{1}{\overline{G}(v)} f(\mathbf{x}, v) dv$ is bounded, continuously differentiable and $\sup_{\mathbf{x} \in \Omega} \left| \frac{\partial \varsigma}{\partial x_i}(\mathbf{x}) \right| < \infty$ for $i = 1, ..., d$.

We need the following assumptions for the asymptotic normality

N1. The survival df G of censored rv's has a bounded first derivative g ;

N2. Let $0 < p_n < n$, $0 < q_n < n$ be integers tending to ∞ with n such that $p_n + q_n \leq n$. Let k_n be the largest integer for which $k_n(p_n + q_n) \leq n$ and
(i) $\frac{k_n p_n}{n} \to 1$ (ii)$p_n h_1^d \to 0$ and $\frac{p_n^2}{n h_1^d} \to 0$ (iii)$\frac{e^{-\gamma q_n}}{h_1^{d+2} h_2^2} \to 0$ as $n \to \infty$.

REMARK 1– *Assumption **A1** gives a classical choice of the bandwidths in functional estimation. For the sake of simplicity, many authors consider that $h_1 = h_2$ which is not justified in general. Note that the condition **A1** (ii) implies the first condition in **A1** (i) if $d \geq 2$. For $d = 1$, the comparison is not straightforward and depends upon the order of magnitude of h_2 with respect to h_1^{α}. Assumption **A2** is quite usual in kernel estimation. Assumptions **A3**-**A7** are classical in nonparametric estimation under dependency and **A5**(i) and (ii) are used in the calculation of the variance term in asymptotic normality while **A8** is used for covariance calculation under association structure. Furthermore, this assumption gives a progressive trend to asymptotic independence of "past" and "future". Note that Assumption **A9** is mainly technical. Assumption **N1** is used in technical calculations of the asymptotic normality. Finally, Assumption **N2** is especially usefull in the case of dependent data when studying the asymptotic normality and is used when dealing with the technique of big and small blocks, **N2**(i) and the fact that $\frac{k_n(p_n+q_n)}{n} \to 1$ imply that $\frac{k_n q_n}{n} \to 0$. Remark that $\frac{k_n q_n/n}{k_n p_n/n} \to 0$ gives that $q_n < p_n$. Add to this the first part of Assumption **N2**(ii), we get that $q_n h_n^d \to 0$.*

THEOREM 1– *Under assumptions **A1**-**A8** and **N1**-**N2**, for any $x \in \Omega_0$ and $l(\boldsymbol{x}) > 0$, we have*

$$\sqrt{n h_1^d}(F_n(t|\boldsymbol{x}) - F(t|\boldsymbol{x})) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2(\boldsymbol{x}, t)) \quad \text{as } n \to +\infty$$

with D the convergence in distribution. The variance is such that

$$\sigma^2(\boldsymbol{x}, t) = \frac{\kappa F(t|\boldsymbol{x})(1 - \overline{G}(t) F(t|\boldsymbol{x}))}{l(\boldsymbol{x}) \overline{G}(t)} \qquad [8]$$

and $\kappa = \int_{\mathbb{R}^d} K_d^2(\boldsymbol{u})d\boldsymbol{u} < +\infty$.

COROLLARY 1– *Assume that* $p \in (0,1)$. *Under assumptions of Theorem 1 and for any* $\boldsymbol{x} \in \Omega_0$ *such that* $f(\xi_p(\boldsymbol{x})|\boldsymbol{x}) \neq 0$, *we have*

$$\sqrt{nh_1^d}(\xi_{p,n}(\boldsymbol{x}) - \xi_p(\boldsymbol{x})) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \sigma_\xi^2(\boldsymbol{x}, \xi_p(\boldsymbol{x}))\right) \quad \text{as } n \to +\infty$$

with

$$\sigma_\xi^2(\boldsymbol{x}, \xi_p(\boldsymbol{x})) = \frac{\sigma^2(\boldsymbol{x}, \xi_p(\boldsymbol{x}))}{f^2(\xi_p(\boldsymbol{x})|\boldsymbol{x})}$$

REMARK 2– *If we replace* $l(.)$, $f(.|.)$, $\overline{G}(.)$ *and* $\xi_p(.)$ *by their estimators* $l_n(.)$, $f_n(.|.)$, $\overline{G}_n(.)$ *and* $\xi_{p,n}(.)$, *respectively then a plug-in type convergent estimator denoted* $\sigma_{\xi,n}^2(\boldsymbol{x}, \xi_{p,n}(\boldsymbol{x}))$ *of* $\sigma_\xi^2(\boldsymbol{x}, \xi_p(\boldsymbol{x}))$ *is easily obtained. Note that*

$$\sigma_{\xi,n}^2(\boldsymbol{x}, \xi_{p,n}(\boldsymbol{x})) = \frac{\kappa F_n(t|\boldsymbol{x})(1 - \overline{G}_n(t)F_n(t|\boldsymbol{x}))}{l_n(\boldsymbol{x})\overline{G}_n(t)f_n^2(\xi_{p,n}(\boldsymbol{x})|\boldsymbol{x})}$$

Using Corollary 1, the approximate $(1-\vartheta)$ *confidence interval is then given by*

$$\left[\xi_{p,n} - z_{1-\vartheta/2}\frac{\sigma_{\xi,n}(\boldsymbol{x}, \xi_{p,n}(\boldsymbol{x}))}{\sqrt{nh_1^d}}, \xi_{p,n} + z_{1-\vartheta/2}\frac{\sigma_{\xi,n}(\boldsymbol{x}, \xi_{p,n}(\boldsymbol{x}))}{\sqrt{nh_1^d}}\right]$$

with $z_{1-\vartheta/2}$ *stands for the* $(1-\vartheta/2)$*-quantile of the standard normal distribution.*

3. Proofs of the main results

We first deal with the uniform a.s. convergence of the conditional pdf estimator $f_n(t|\mathbf{x})$ to $f(t|\mathbf{x})$ defined in (7).

PROPOSITION 1– *Under assumptions* ***A1***-***A8***, *we have*

$$\sup_{\boldsymbol{x}\in\Omega}\sup_{t\in\mathcal{C}} |f_n(t|\boldsymbol{x}) - f(t|\boldsymbol{x})| \to 0 \;\text{ a.s. as }\; n \to +\infty. \qquad [9]$$

3.1. *Proof of Proposition 1*

To prove the convergence of the underlying pdf, we use the following decomposition

$$f_n(t|\mathbf{x}) - f(t|\mathbf{x}) = \left(f_n(t|\mathbf{x}) - \tilde{f}_n(t|\mathbf{x})\right) + \left(\tilde{f}_n(t|\mathbf{x}) - f(t|\mathbf{x})\right). \qquad [10]$$

We firstly deal with the left hand side of (10). Using Assumption **A3**, we find

$$\left|f_n(t|\mathbf{x}) - \tilde{f}_n(t|\mathbf{x})\right| \;=\; \frac{1}{l_n(\mathbf{x})}\left|f_n(\mathbf{x},t) - \tilde{f}_n(\mathbf{x},t)\right|$$

$$\leq \frac{m_1}{l_n(\mathbf{x})} \frac{\sup_{t\in\mathcal{C}} |\overline{G}_n(t) - \overline{G}(t)|}{\overline{G}_n(\tau_F)\overline{G}(\tau_F)} \frac{1}{nh_1^d h_2} \sum_{i=1}^n K_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_1}\right) H^{(1)}\left(\frac{t-Y_i}{h_2}\right).$$

Then, we get immediately the convergence of the left hand side of (10). As for the right hand side of (10), we have

$$\begin{aligned}
\sup_{\mathbf{x}\in\Omega}\sup_{t\in\mathcal{C}} \left|\tilde{f}_n(t|\mathbf{x}) - f(t|\mathbf{x})\right| &\leq \frac{1}{\inf_{\mathbf{x}\in\Omega}(l_n(\mathbf{x}))} \left\{ \sup_{\mathbf{x}\in\Omega}\sup_{t\in\mathcal{C}} \left|\tilde{f}_n(\mathbf{x},t) - \mathbb{E}\left[\tilde{f}_n(\mathbf{x},t)\right]\right| \right. \\
&\quad + \sup_{\mathbf{x}\in\Omega}\sup_{t\in\mathcal{C}} \left|\mathbb{E}\left[\tilde{f}_n(\mathbf{x},t)\right] - f_n(\mathbf{x},t)\right| \\
&\quad \left. + \sup_{\mathbf{x}\in\Omega} |l_n(\mathbf{x}) - l(\mathbf{x})| \sup_{\mathbf{x}\in\Omega}\sup_{t\in\mathcal{C}} |f(t|\mathbf{x})| \right\} \\
&=: \frac{1}{m_0 - \sup_{\mathbf{x}\in\Omega} |l_n(\mathbf{x}) - l(\mathbf{x})|} \left[I_1 + I_2 + I_3\right].
\end{aligned}$$

Let us deal with each term of the right hand side. As for I_1, we make choice of the following expression

$$\begin{aligned}
Q_i(\mathbf{x},t) &= \frac{1}{nh_1^d h_2} \frac{\delta_i}{\overline{G}(Y_i)} K_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_1}\right) H^{(1)}\left(\frac{t-Y_i}{h_2}\right) \\
&- \mathbb{E}\left[\frac{1}{nh_1^d h_2} \frac{\delta_i}{\overline{G}(Y_i)} K_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_1}\right) H^{(1)}\left(\frac{t-Y_i}{h_2}\right)\right]
\end{aligned}$$

such that

$$\sum_{i=1}^n Q_i(\mathbf{x},t) = \tilde{f}_n(\mathbf{x},t) - \mathbb{E}\left[\tilde{f}_n(\mathbf{x},t)\right].$$

The proof uses the covering techniques. The compact subset $\mathcal{C}$ is covered by a finite number μ_n of intervals of length ϱ_n, respectively centered at $t_1, \ldots, t_{\mu_n}$ with $\mu_n \varrho_n \leq c$. There exists t_j for any t such that $|t - t_j| \leq \varrho_n$.

$$\left|\sum_{i=1}^n Q_i(\mathbf{x},t)\right| \leq \left|\sum_{i=1}^n Q_i(\mathbf{x},t) - \sum_{i=1}^n Q_i(\mathbf{x},t_j)\right| + \left|\sum_{i=1}^n Q_i(\mathbf{x},t_j)\right| \quad [11]$$

It follows from assumptions **A2** and **A3** that

$$\begin{aligned}
\left|\sum_{i=1}^n Q_i(\mathbf{x},t) - \sum_{i=1}^n Q_i(\mathbf{x},t_j)\right| &\leq \frac{2}{nh_1^d h_2} \sum_{i=1}^n \frac{\delta_i}{\overline{G}(Y_i)} K_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_1}\right) \\
&\quad \times \left|H^{(1)}\left(\frac{t-Y_i}{h_2}\right) - H^{(1)}\left(\frac{t_j-Y_i}{h_2}\right)\right| \\
&\leq \frac{m_1}{h_1^d h_2 \overline{G}(\tau_F)} \left|\frac{t-t_j}{h_2}\right| \\
&\leq \frac{m_1}{h_1^d h_2^2 \mu_n}
\end{aligned}$$

By means of (11) and choosing $\mu_n = O(n)$, we get

$$I_1 \leq \frac{m_1}{nh_1^d h_2^2} + \max_{1\leq j\leq \mu_n} \left| \sum_{i=1}^{n} Q_i(\mathbf{x}, t_j) \right|$$

It is now to be shown that

$$\mathbb{P}\left(\max_{1\leq j\leq \mu_n} \left| \sum_{i=1}^{n} Q_i(\mathbf{x}, t_j) \right| > \varepsilon \right) \to 0 \quad \text{as } n \to +\infty \qquad [12]$$

As $Q_i(\mathbf{x}, t_j)$ are associated, the use of an exponential inequality due to [DOUKHAN and NEUMANN 2007] is appropriate to bound (12). An analogous framework as in the proof of the consistency above is established. We proceed now to the demonstration of the bias term I_2.
It is easily seen that

$$\mathbb{E}\left[\tilde{f}_n(\mathbf{x}, t)\right] = \frac{1}{h_1^d h_2} \mathbb{E}\left[K_d\left(\frac{\mathbf{x} - \mathbf{X}_1}{h_1}\right) \mathbb{E}\left[\frac{\delta_1}{\overline{G}(Y_1)} H^{(1)}\left(\frac{t - Y_1}{h_2}\right) |\mathbf{X}_1 \right]\right].$$

We have

$$\begin{aligned}\mathbb{E}\left[\frac{\delta_1}{\overline{G}(Y_1)} H^{(1)}\left(\frac{t - Y_1}{h_2}\right) |\mathbf{X}_1 \right] &= \mathbb{E}\left[\mathbb{E}\left[\frac{\delta_1}{\overline{G}(Y_1)} H^{(1)}\left(\frac{t - Y_1}{h_2}\right) |T_1 \right] |\mathbf{X}_1 \right] \\ &= h_2 \int_{\mathbb{R}} H^{(1)}(z) f(t - h_2 z|\mathbf{X}_1) dz.\end{aligned}$$

Therefore

$$\begin{aligned}\mathbb{E}\left[\tilde{f}_n(\mathbf{x}, t)\right] &= \frac{1}{h_1^d h_2} \int_{\mathbb{R}^d} \int_{\mathbb{R}} h_2 K_d\left(\frac{\mathbf{x} - u}{h_1}\right) H^{(1)}(z) f(t - h_2 z|\mathbf{u}) l(\mathbf{u}) d\mathbf{u} dz \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}} K_d(\mathbf{r}) H^{(1)}(z) f(\mathbf{x} - h_1 \mathbf{r}, t - h_2 z) dr dz.\end{aligned}$$

with $f(\mathbf{x} - h_1\mathbf{r}, t - h_2 z) = F_1'(\mathbf{x} - h_1\mathbf{r}, t - h_2 z)$. Assumption **A5** and a Taylor expansion of $f(.,.)$ around $(\mathbf{x}, t)$ gives

$$\begin{aligned}f(\mathbf{x} - rh_1, t - zh_2) &= f(\mathbf{x}, t) - h_1\left[r_1 \frac{\partial f(\mathbf{x}, t)}{\partial x_1} + \cdots + r_d \frac{\partial f(\mathbf{x}, t)}{\partial x_d} \right] - h_2\left[z \frac{\partial f(\mathbf{x}, t)}{\partial t} \right] \\ &+ \frac{h_1^2}{2}\left[r_1^2 \frac{\partial^2 f(\mathbf{x}^*, t)}{\partial x_1^2} + \cdots + r_d^2 \frac{\partial^2 f(\mathbf{x}^*, t)}{\partial x_d^2} + 2\sum_{i\neq j} r_i r_j \frac{\partial^2 f(\mathbf{x}^*, t)}{\partial x_i \partial x_j} \right] \\ &+ \frac{h_2^2}{2}\left[z^2 \frac{\partial^2 f(\mathbf{x}, t^*)}{\partial t^2} \right] + h_1 h_2 \left[r_1 z \frac{\partial^2 f(\mathbf{x}^*, t)}{\partial x_1 \partial t} + \cdots + r_d z \frac{\partial^2 f(\mathbf{x}^*, t)}{\partial x_d \partial t} \right].\end{aligned}$$

Note that $\mathbf{x}^*$ lies between $\mathbf{x} - rh_1$ and $\mathbf{x}$ and t^* between $t - zh_2$ and t. By assumptions **A2** and **A5** $I_2 = O(h_1^2 + h_2^2)$. As for I_3, under assumptions **A1**, **A2**, **A4**, **A6** and **A8**, it follows from Lemma 3 in [MENNI and TATACHAK 2018] that the kernel estimator $l_n(\mathbf{x}) \to l(\mathbf{x})$ a.s. as $n \to +\infty$. Add to this the first part of Assumption **A5** then we get $I_3 = o(1)$. This point ends the proof of Proposition 1. □

3.2. *Proof of Theorem 1*

We have the following decomposition

$$F_n(t|\mathbf{x}) - F(t|\mathbf{x}) = (F_n(t|\mathbf{x}) - \tilde{F}_n(t|\mathbf{x})) + (\tilde{F}_n(t|\mathbf{x}) - F(t|\mathbf{x})).$$

And

$$\begin{aligned} \tilde{F}_n(t|\mathbf{x}) - F(t|\mathbf{x}) &= \frac{l(\mathbf{x})}{l_n(\mathbf{x})}\left[\frac{\tilde{F}_{1,n}(\mathbf{x},t) - \mathbb{E}\left(\tilde{F}_{1,n}(\mathbf{x},t)\right) - F(t|\mathbf{x})(l_n(\mathbf{x}) - \mathbb{E}(l_n(\mathbf{x})))}{l(\mathbf{x})}\right] \\ &- \left[\frac{F_1(\mathbf{x},t) - \mathbb{E}\left(\tilde{F}_{1,n}(\mathbf{x},t)\right) - F(t|\mathbf{x})(l(\mathbf{x}) - \mathbb{E}(l_n(\mathbf{x})))}{l_n(\mathbf{x})}\right] \\ &=: \frac{l(\mathbf{x})}{l_n(\mathbf{x})} Z_n(\mathbf{x},t) - R_n(\mathbf{x},t). \end{aligned} \quad [13]$$

Note that

$$\sqrt{nh_1^d}(F_n(t|\mathbf{x}) - F(t|\mathbf{x})) =: \frac{l(\mathbf{x})}{l_n(\mathbf{x})}\left[\sqrt{nh_1^d} Z_n(\mathbf{x},t)\right] + \sqrt{nh_1^d} U_n(\mathbf{x},t) \quad [14]$$

with

$$U_n(\mathbf{x},t) =: (F_n(t|\mathbf{x}) - \tilde{F}_n(t|\mathbf{x})) - R_n(\mathbf{x},t). \quad [15]$$

The idea is to give asymptotic results of $\sqrt{nh_1^d} Z_n(\mathbf{x},t)$. For this puspose, we give the convergence in probability of the negligible term $\sqrt{nh_1^d} U_n(\mathbf{x},t)$ to zero as shown in Lemma 1 and determine the asymptotic variance that appears in (8) (see Lemma 3). Finally, we use the Bernstein's procedure of big blocks and small blocks to obtain the asymptotic normality of the principal term $Z_n(\mathbf{x},t)$. Regarding the term $\frac{l(\mathbf{x})}{l_n(\mathbf{x})}$, it converges obviously to 1.
The next lemma deals with the second part of (14).

LEMMA 1– *Assumptions **A1**-**A5** and **A8** yield that*

$$\sqrt{nh_1^d} U_n(\boldsymbol{x},t) = o_{\mathbb{P}}(1) \quad \text{as} \quad n \to +\infty,$$

3.3. *Proof of Lemma 1*

Observe that (15) can be written

$$\sqrt{nh_1^d} U_n(\mathbf{x},t) = \frac{\sqrt{nh_1^d}}{l_n(\mathbf{x})}\left[(F_{1,n}(\mathbf{x},t) - \tilde{F}_{1,n}(\mathbf{x},t)) - (F_1(\mathbf{x},t) - \mathbb{E}(\tilde{F}_{1,n}(\mathbf{x},t))\right]$$

$$\begin{aligned} &+ \quad \frac{\sqrt{nh_1^d}}{l_n(\mathbf{x})}\left[F(t|\mathbf{x})(l(\mathbf{x}) - \mathbb{E}(l_n(\mathbf{x})))\right] \\ &=: \quad \mathcal{X}_{n,1} + \mathcal{X}_{n,2} + \mathcal{X}_{n,3}. \end{aligned} \quad [16]$$

The convergence rate of each term in (16) follows thanks to lemmas 5.6, 5.5 and 5.4 in [DJELLADJ and TATACHAK 2019], respectively. In fact, using **A1**(iii),(iv) we show that

$\mathcal{X}_{n,1} = o_{\mathbb{P}}\left(\sqrt{n^{1-2\theta}h_1^d}\right) =: o_{\mathbb{P}}\left(\sqrt{v_s h_1^d}\right) = o_{\mathbb{P}}(1)$

$\mathcal{X}_{n,2} = O\left(\sqrt{nh_1^d}(h_1^2 + h_2^2)\right) = o(1)$

$\mathcal{X}_{n,3} = O\left(\sqrt{nh_1^{d+4}}\right) = o(1).$

This achieves the proof of Lemma 1. □

Let us deal with the asymptotic normality of the main term $Z_n(\mathbf{x}, t)$. For this we first set

$$\Psi_i(\mathbf{x}, t) = K_{d,i}\left[H_i\frac{\delta_i}{\overline{G}(Y_i)} - F(t|\mathbf{x})\right] - \mathbb{E}\left[K_{d,i}\left(H_i\frac{\delta_i}{\overline{G}(Y_i)} - F(t|\mathbf{x})\right)\right]$$

with

$$K_{d,i} := K_d\left(\frac{\mathbf{x} - \mathbf{X}_i}{h_1}\right) \quad \text{and} \quad H_i := H\left(\frac{t - Y_i}{h_2}\right).$$

Then we write

$$Z_n(\mathbf{x}, t) =: \frac{1}{nh_1^d l(\mathbf{x})}\sum_{i=1}^{n}\Psi_i(\mathbf{x}, t) \quad [17]$$

Now, our interest concerns the variance term

$$\begin{aligned} nh_1^d \mathrm{Var} Z_n(\mathbf{x}, t) &= \frac{1}{h_1^d l^2(\mathbf{x})}\mathrm{Var}(\Psi_1(\mathbf{x}, t)) + \frac{1}{nh_1^d l^2(\mathbf{x})}\sum_{i=1}^{n}\sum_{j=1:j\neq i}^{n}\mathrm{cov}\left(\Psi_i(\mathbf{x}, t), \Psi_j(\mathbf{x}, t)\right) \\ &= \sigma_n^2(\mathbf{x}, t) + \Xi_n(\mathbf{x}, t). \end{aligned} \quad [18]$$

The next lemma reports intermediary results used in further calculations related to the variance term.

LEMMA 2– *Assumptions **A3**(i), **A5**(i),(ii) and **N1** give that*

$$\mathrm{Var}\left[\frac{\delta_1}{\overline{G}(Y_1)}H\left(\frac{t - Y_1}{h_2}\right)|\mathbf{X}_1\right] \to F(t|\boldsymbol{x})\left[\frac{1}{\overline{G}(t)} - F(t|\boldsymbol{x})\right] \quad \text{as} \quad n \to +\infty.$$

3.4. *Proof of Lemma 2*

We have

$$\mathrm{Var}\left[\frac{\delta_1}{\overline{G}(Y_1)}H\left(\frac{t - Y_1}{h_2}\right)|\mathbf{X}_1\right] = \mathbb{E}\left[\frac{\delta_1}{\overline{G}^2(Y_1)}H^2\left(\frac{t - Y_1}{h_2}\right)|\mathbf{X}_1\right] - \mathbb{E}\left[\frac{\delta_1}{\overline{G}(Y_1)}H\left(\frac{t - Y_1}{h_2}\right)|\mathbf{X}_1\right]^2$$

$$=: \mathcal{L}1 + \mathcal{L}2$$

Integration by parts and change of variable give

$$\begin{aligned}
\mathbb{E}\left[\frac{\delta_1}{\overline{G}(Y_1)}H\left(\frac{t-Y_1}{h_2}\right)|\mathbf{X}_1\right] &= \mathbb{E}\left[\frac{1}{\overline{G}(T_1)}H\left(\frac{t-T_1}{h_2}\right)\mathbb{E}\left[\mathbb{1}_{\{T_1\le C_1\}}|T_1\right]|\mathbf{X}_1\right] \\
&= \int_{\mathbb{R}} H\left(\frac{t-u}{h_2}\right)f(u|\mathbf{X}_1)du \\
&= \int_{\mathbb{R}} H^{(1)}(z)F(t-zh_2|\mathbf{X}_1)dz \\
&= \int_{\mathbb{R}} H^{(1)}(z)(F(t-zh_2|\mathbf{X}_1)-F(t|\mathbf{x}))dz + \int_{\mathbb{R}} H^{(1)}(z)F(t|\mathbf{x})dz
\end{aligned}$$

Clearly, we have $\int_{\mathbb{R}} H^{(1)}(z)F(t|\mathbf{x})dz \to F(t|\mathbf{x})$. Regarding the left hand side, by assumptions **A3(i)** and **A5(i)**, we have that

$$\begin{aligned}
\int_{\mathbb{R}} H^{(1)}(z)(F(t-zh_2|\mathbf{X}_1)-F(t|\mathbf{x}))dz &\le \int_{\mathbb{R}} H^{(1)}(z)m_1(|\mathbf{X}_1-\mathbf{x}|^{u_1}+|zh_2|^{u_2})dz \\
&= m_1|\mathbf{X}_1-\mathbf{x}|^{u_1}\int_{\mathbb{R}} H^{(1)}(z)dz \\
&\quad + m_1|h_2|^{u_2}\int_{\mathbb{R}} |z|^{u_2}H^{(1)}(z)dz \\
&= O(h_2^{\min(u_1,u_2)}).
\end{aligned}$$

Hence $\mathcal{L}_2 \to F^2(t|\mathbf{x})$ as $n \to +\infty$. Regarding $\mathcal{L}_1$, by a change of variable

$$\begin{aligned}
\mathcal{L}_1 &= \mathbb{E}\left[\frac{1}{\overline{G}^2(T_1)}H^2\left(\frac{t-T_1}{h_2}\right)\mathbb{E}[\mathbb{1}_{\{T_1\le C_1\}}|T_1]|\mathbf{X}_1\right] \\
&= \int_{\mathbb{R}} \frac{1}{\overline{G}(y)}H^2\left(\frac{t-y}{h_2}\right)f(y|\mathbf{X}_1)dy \\
&= \int_{\mathbb{R}} h_2\frac{1}{\overline{G}(t-zh_2)}H^2(z)f(t-zh_2|\mathbf{X}_1)dz.
\end{aligned}$$

Now, we use a Taylor expansion around t. We get

$$\begin{aligned}
\mathcal{L}_1 &= \frac{h_2}{\overline{G}(t)}\int_{\mathbb{R}} H^2(z)f(t-zh_2|\mathbf{X}_1)dz - \frac{h_2^2}{\overline{G}^2(t)}g(t^*)\int_{\mathbb{R}} zH^2(z)f(t-zh_2|\mathbf{X}_1)dz \\
&=: \mathcal{A}_1 - \mathcal{A}_2.
\end{aligned}$$

Bounding $\mathcal{A}_2$ gives

$$\mathcal{A}_2 \leq \frac{h_2^2 \sup_{t\in\mathcal{C}} g(t)}{\overline{G}^2(\tau_F)} \int_{\mathbb{R}} |z| f(t - zh_2|\mathbf{X}_1) dz$$

Applying assumptions **A5(ii)** and **N1**, we conclude that $\mathcal{A}_2 = O(h_2^2)$. By an integration by parts on the term $\mathcal{A}_1$, we get

$$\begin{aligned}
\mathcal{A}_1 &= \frac{1}{\overline{G}(t)} \int_{\mathbb{R}} 2H^{(1)}(z)H(z)F(t - zh_2|\mathbf{X}_1)dz \\
&= \frac{1}{\overline{G}(t)} \int_{\mathbb{R}} 2H^{(1)}(z)H(z)(F(t - zh_2|\mathbf{X}_1) - F(t|\mathbf{x}))dz + \frac{1}{\overline{G}(t)} \int_{\mathbb{R}} 2H^{(1)}(z)H(z)F(t|\mathbf{x})dz
\end{aligned}$$

As well as above, we use Assumption **A5(i)**. Then

$$\begin{aligned}
\int_{\mathbb{R}} 2H^{(1)}(z)H(z)(F(t - zh_2|\mathbf{X}_1) - F(t|\mathbf{x}))dz &\leq \int_{\mathbb{R}} m_1(|\mathbf{X}_1 - \mathbf{x}|^{u_1} + |zh_2|^{u_2})[H(z)^2]'dz \\
&= O(h_2^{\min(u_1,u_2)}).
\end{aligned}$$

Moreover

$$\int_{\mathbb{R}} 2H^{(1)}(z)H(z)F(t|\mathbf{x})dz = F(t|\mathbf{x}).$$

Hence $\mathcal{A}_1 \to \dfrac{F(t|\mathbf{x})}{\overline{G}(t)}$ as $n \to \infty$. We deduce that

$$\mathrm{Var}\left[\frac{\delta_1}{\overline{G}(Y_1)} H\left(\frac{t - Y_1}{h_2}\right) |\mathbf{X}_1\right] \to \frac{F(t|\mathbf{x})}{\overline{G}(t)} - O(h_2^2) - F(t|\mathbf{x})^2$$

which concludes the proof of Lemma 2. □

LEMMA 3– *Under assumptions **A2-A4** and **A6-A7**, we have*

$$\sigma_n^2(\boldsymbol{x}, t) \to \sigma^2(\boldsymbol{x}, t) \text{ as } n \to +\infty \quad \text{and} \quad \Xi_n(\boldsymbol{x}, t) \to 0 \text{ as } n \to +\infty.$$

3.5. *Proof of Lemma 3*

From (18), we define

$$\sigma_n^2(\mathbf{x}, t) = \frac{1}{h_1^d l^2(\mathbf{x})} \mathbb{E}[\Psi_1^2(\mathbf{x}, t)]$$

$$
\begin{aligned}
&= \frac{1}{h_1^d l^2(\mathbf{x})}\mathbb{E}\left[K_{d,1}^2\left(H_1\frac{\delta_1}{\overline{G}(Y_1)} - F(t|\mathbf{x})\right)^2\right] \\
&\quad -\frac{1}{h_1^d l^2(\mathbf{x})}\left\{\mathbb{E}\left[K_{d,1}\left(H_1\frac{\delta_1}{\overline{G}(Y_1)} - F(t|\mathbf{x})\right)\right]\right\}^2 \\
&= \mathcal{R}_1(\mathbf{x},t) - \mathcal{R}_2(\mathbf{x},t).
\end{aligned}
$$

As for $\mathcal{R}_2(\mathbf{x},t)$, we can write

$$
\begin{aligned}
\mathcal{R}_2(\mathbf{x},t) &= \frac{1}{h_1^d l^2(\mathbf{x})}\left[\mathbb{E}\left(\frac{\delta_1}{\overline{G}(Y_1)}K_{d,1}H_1\right) - \mathbb{E}(K_{d,1}F(t|\mathbf{x}))\right]^2 \\
&= \frac{1}{h_1^d l^2(\mathbf{x})}\left[h_1^d\mathbb{E}(\tilde{F}_{1,n}(\mathbf{x},t)) - h_1^d F(t|\mathbf{x})\,\mathbb{E}(l_n(\mathbf{x}))\right]^2 \\
&= \frac{1}{l^2(\mathbf{x})}\left[\mathbb{E}(\tilde{F}_{1,n}(\mathbf{x},t)) - F(t|\mathbf{x})\,\mathbb{E}(l_n(\mathbf{x}))\right]^2.
\end{aligned}
$$

We point out that

$$
\mathbb{E}(\tilde{F}_{1,n}(\mathbf{x},t)) - F(t|\mathbf{x})\,\mathbb{E}(l_n(\mathbf{x})) \to F_1(\mathbf{x},t) - F(t|\mathbf{x})\,\mathbb{E}(l_n(\mathbf{x})) \;\text{ as }\; n\to\infty.
$$

Then

$$
F_1(\mathbf{x},t) - F(t|\mathbf{x})\,\mathbb{E}(l_n(\mathbf{x})) = F(t|\mathbf{x})[l(\mathbf{x}) - \mathbb{E}(l_n(\mathbf{x}))] \;\to 0 \;\text{ as }\; n\to\infty.
$$

On the other hand

$$
\begin{aligned}
\mathcal{R}_1(\mathbf{x},t) &= \frac{1}{h_1^d l^2(\mathbf{x})}\,\mathbb{E}\left[\mathbb{E}\left[K_{d,1}^2\left(H_1\frac{\delta_1}{\overline{G}(Y_1)} - F(t|\mathbf{x})\right)^2|\mathbf{X}_1\right]\right] \\
&= \frac{1}{h_1^d l^2(\mathbf{x})}\,\mathbb{E}\left[K_{d,1}^2\mathrm{Var}\left(H_1\frac{\delta_1}{\overline{G}(Y_1)}|\mathbf{X}_1\right)\right] \\
&\quad +\frac{1}{h_1^d l^2(\mathbf{x})}\mathbb{E}\left[K_{d,1}^2\mathbb{E}\left(\left(H_1\frac{\delta_1}{\overline{G}(Y_1)}|\mathbf{X}_1\right) - F(t|\mathbf{x})\right)^2\right].
\end{aligned}
$$

Working as in lemma 2, the second part in brackets tends to zero as n tends to infinity. In terms of the first part, we write

$$
\begin{aligned}
\mathbb{E}\left[K_{d,1}^2\mathrm{Var}\left(H_1\frac{\delta_1}{\overline{G}(Y_1)}|\mathbf{X}_1\right)\right] &= \mathbb{E}\left[K_{d,1}^2F(t|\mathbf{x})\left(\frac{1}{\overline{G}(t)} - F(t|\mathbf{x})\right)\right] \\
&= F(t|\mathbf{x})\left(\frac{1-\overline{G}(t)F(t|\mathbf{x})}{\overline{G}(t)}\right)\int\limits_{\mathbb{R}^d}K_d^2\left(\frac{\mathbf{x}-\mathbf{u}}{h_1}\right)l(\mathbf{u})d\mathbf{u}.
\end{aligned}
$$

Using a change of variable, Assumption **A3** and Taylor expansion around $\mathbf{x}$, we get

$$
\int\limits_{\mathbb{R}^d}K_d^2\left(\frac{\mathbf{x}-\mathbf{u}}{h_1}\right)l(\mathbf{u})d\mathbf{u} \;=\; h_1^d\int\limits_{\mathbb{R}^d}K_d^2(\mathbf{z})l(\mathbf{x}-h_1\mathbf{z})d\mathbf{z}
$$

Then

$$\frac{1}{h_1^d l^2(\mathbf{x})}\mathbb{E}\left[K_{d,1}^2\mathrm{Var}\left(H_1\frac{\delta_1}{\overline{G}(Y_1)}|\mathbf{X}_1\right)\right] \rightarrow \frac{1}{l(\mathbf{x})}F(t|\mathbf{x})\left(\frac{1-\overline{G}(t)F(t|\mathbf{x})}{\overline{G}(t)}\right)\int_{\mathbb{R}^d} K_d^2(\mathbf{z})d\mathbf{z}$$
$$= \frac{\kappa F(t|\mathbf{x})(1-\overline{G}(t)F(t|\mathbf{x}))}{l(\mathbf{x})\overline{G}(t)} = \sigma^2(\mathbf{x},t).$$

Now we deal with the second part of (18). We have

$$\Xi_n(\mathbf{x},t) = \frac{1}{nh_1^d l^2(\mathbf{x})}\sum_{i=1}^{n}\sum_{j=1:j\neq i}^{n}\mathrm{cov}(\Psi_i(\mathbf{x},t),\Psi_j(\mathbf{x},t))),$$

which we rewrite

$$\begin{aligned}\Xi_n(\mathbf{x},t) &= \frac{1}{nh_1^d l^2(\mathbf{x})}\sum_{i=1}^{n}\sum_{B_1}\mathrm{cov}(\Psi_i(\mathbf{x},t),\Psi_j(\mathbf{x},t))) + \frac{1}{nh_1^d l^2(\mathbf{x})}\sum_{i=1}^{n}\sum_{B_2}\mathrm{cov}(\Psi_i(\mathbf{x},t),\Psi_j(\mathbf{x},t)) \\ &=: \mathcal{M}_1+\mathcal{M}_2\end{aligned}$$

with

$$B_1 = \{(i,l); 1\le |i-l| \le \eta_n\} \text{ and } B_2 = \{(i,l); \eta_n+1 \le |i-l| \le n-1\}.$$

Regarding $\mathcal{M}_1$, considering that the covariance is computed according to formula

$$\begin{aligned}\mathrm{cov}(\Psi_i(\mathbf{x},t),\Psi_j(\mathbf{x},t))) &= \mathbb{E}\left[\Psi_i(\mathbf{x},t).\Psi_j(\mathbf{x},t))\right] \\ &\le \mathbb{E}\left[K_{d,i}K_{d,j}\left(H_i\frac{\delta_i}{\overline{G}(Y_i)}-F(t|\mathbf{x})\right)\left(H_j\frac{\delta_j}{\overline{G}(Y_j)}-F(t|\mathbf{x})\right)\right] \\ &\quad +\mathbb{E}\left[K_{d,1}\left(H_1\frac{\delta_1}{\overline{G}(Y_1)}-F(t|\mathbf{x})\right)\right]^2.\end{aligned}$$

Clearly, we have

$$\left|H_i\frac{\delta_i}{\overline{G}(Y_i)}-F(t|\mathbf{x})\right| \le \frac{1}{\overline{G}(\tau_F)}+1.$$

Consequently

$$\mathrm{cov}(\Psi_i(\mathbf{x},t),\Psi_j(\mathbf{x},t))) \le \left(\frac{1}{\overline{G}(\tau_F)}+1\right)^2\mathbb{E}(K_{d,i}K_{d,j}) + O(1)\,\mathbb{E}(K_{d,1})^2.$$

Assumptions **A2** and **A6** give

$$\begin{aligned}\mathbb{E}(K_{d,i}K_{d,j}) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K_d\left(\frac{\mathbf{x}-\mathbf{u}}{h_1}\right)K_d\left(\frac{\mathbf{x}-\mathbf{v}}{h_1}\right)l_{i,j}(\mathbf{u},\mathbf{v})d\mathbf{u}d\mathbf{v} \\ &= h_1^{2d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K_d(\mathbf{s})K_d(\mathbf{z})l_{i,j}(\mathbf{x}-h_1\mathbf{s},\mathbf{x}-h_1\mathbf{z})d\mathbf{s}d\mathbf{z}\end{aligned}$$

$$= \quad O(h_1^{2d}).$$

Again, by a Taylor expansion and assumptions **A2(i)** and **A4**, we have that $\mathbb{E}(K_{d,1}) = O(h_1^d)$. Bounding $\mathcal{M}_1$ gives

$$\begin{aligned} \mathcal{M}_1 &\leq \frac{1}{nh_1^d l^2(\mathbf{x})} \sum_{i=1}^{n} \sum_{B_1} \left[m_1 \left(\frac{1}{\overline{G}(\tau_F)} + 1 \right)^2 h_1^{2d} + O(1) h_1^{2d} \right] \\ &= m_1 \frac{h_1^d \eta_n}{l^2(\mathbf{x})} + O(1) \frac{h_1^d \eta_n}{l^2(\mathbf{x})}. \end{aligned}$$

Supposing $h_1^d \eta_n \to 0$ as $n \to +\infty$, we get $\mathcal{M}_1 = o(1)$.
The second term can be written

$$\begin{aligned} \mathcal{M}_2 &= \frac{1}{nh_1^d l^2(\mathbf{x})} \sum_{i=1}^{n} \sum_{B_2} \text{cov} \left(K_{d,i} H_i \frac{\delta_i}{\overline{G}(Y_i)} - K_{d,i} F(t|\mathbf{x}), K_{d,j} H_j \frac{\delta_j}{\overline{G}(Y_j)} - K_{d,j} F(t|\mathbf{x}) \right) \\ &=: \frac{1}{nh_1^d l^2(\mathbf{x})} \sum_{i=1}^{n} \sum_{B_2} \mathcal{W}. \end{aligned}$$

We have

$$\begin{aligned} \mathcal{W} &= \text{cov} \left(K_{d,i} H_i \frac{\delta_i}{\overline{G}(Y_i)}, K_{d,j} H_j \frac{\delta_j}{\overline{G}(Y_j)} \right) - F(t|\mathbf{x}) \text{cov} \left(K_{d,i} H_i \frac{\delta_i}{\overline{G}(Y_i)}, K_{d,j} \right) \\ &- F(t|\mathbf{x}) \text{cov} \left(K_{d,i}, K_{d,j} H_j \frac{\delta_j}{\overline{G}(Y_j)} \right) + F(t|\mathbf{x})^2 \text{cov}(K_{d,i}, K_{d,j}) \\ &=: \mathcal{W}_1 - F(t|\mathbf{x}) \mathcal{W}_2 - F(t|\mathbf{x}) \mathcal{W}_3 + F(t|\mathbf{x})^2 \mathcal{W}_4. \end{aligned}$$

Note that

$$\begin{aligned} \frac{1}{nh_1^d l^2(\mathbf{x})} \sum_{i=1}^{n} \sum_{B_2} \mathcal{W}_1 &\leq \frac{1}{nh_1^d l^2(\mathbf{x})} \sum_{i=1}^{n} \sum_{B_2} m_1 h_1^d h_2^{\frac{2}{d+1}} \rho^{\frac{d}{2d+2}} (|i-j|) \\ &\leq \frac{m_1}{l^2(\mathbf{x})} \sum_{B_2} h_2^{\frac{2}{d+1}} \gamma_0^{\frac{d}{2d+2}} e^{-\frac{\gamma |i-j| d}{2d+2}} \\ &\leq \frac{m_1 h_2^{\frac{2}{d+1}}}{l^2(\mathbf{x})} \int_{\eta_n}^{n} e^{-\frac{\gamma u d}{2d+2}} du \\ &\leq O(h_2^{\frac{2}{d+1}} e^{-\frac{\gamma \eta_n d}{2d+2}}). \end{aligned} \qquad [19]$$

Choosing $\eta_n = O(h_2^{\nu-1})$, $0 < \nu < 1$, (19) is of order $o(h_2^{\frac{2}{d+1}})$.
As for $\mathcal{W}_2$ and $\mathcal{W}_3$, we have

$$\mathcal{W}_2 = \mathbb{E} \left[K_{d,i} K_{d,j} H_i \frac{\delta_i}{\overline{G}(Y_i)} \right] - \mathbb{E} \left[K_{d,i} H_i \frac{\delta_i}{\overline{G}(Y_i)} \right] \mathbb{E}[K_{d,j}]$$

$$=: \mathcal{W}_2' - \mathcal{W}_2''.$$

Using conditional expectation techniques, we get

$$\begin{aligned}
\mathcal{W}_2' =& \mathbb{E}\left[K_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_1}\right)K_d\left(\frac{\mathbf{x}-\mathbf{X}_j}{h_1}\right)\mathbb{E}\left[H\left(\frac{t-Y_i}{h_2}\right)\frac{\delta_i}{\overline{G}(Y_i)}|\mathbf{X}_i,\mathbf{X}_j\right]\right]\\
=& \mathbb{E}\left[K_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_1}\right)K_d\left(\frac{\mathbf{x}-\mathbf{X}_j}{h_1}\right)\mathbb{E}\left[\mathbb{E}\left(\frac{\mathbb{1}_{\{T_i\leq C_i\}}}{\overline{G}(Y_i)}H\left(\frac{t-Y_i}{h_2}\right)|T_i\right)|\mathbf{X}_i,\mathbf{X}_j\right]\right]\\
=& \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}} K_d\left(\frac{\mathbf{x}-\mathbf{u}}{h_1}\right)K_d\left(\frac{\mathbf{x}-\mathbf{v}}{h_1}\right)H\left(\frac{t-s}{h_2}\right)f(\mathbf{u},\mathbf{v},s)d\mathbf{u}d\mathbf{v}ds
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{W}_2'' =& \mathbb{E}\left[K_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_1}\right)\mathbb{E}\left(\frac{\delta_i}{\overline{G}(Y_i)}H\left(\frac{t-Y_i}{h_2}\right)|\mathbf{X}_i\right)\right]\mathbb{E}\left[K_d\left(\frac{\mathbf{x}-\mathbf{X}_j}{h_1}\right)\right]\\
=& \mathbb{E}\left[K_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_1}\right)\mathbb{E}\left(\mathbb{E}\left(\frac{\mathbb{1}_{\{T_i\leq C_i\}}}{\overline{G}(Y_i)}H\left(\frac{t-Y_i}{h_2}\right)|T_i\right)|\mathbf{X}_i\right)\right]\mathbb{E}\left[K_d\left(\frac{\mathbf{x}-\mathbf{X}_j}{h_1}\right)\right]\\
=& \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}} K_d\left(\frac{\mathbf{x}-\mathbf{u}}{h_1}\right)H\left(\frac{t-s}{h_2}\right)f(\mathbf{u},s)d\mathbf{u}ds\int\limits_{\mathbb{R}^d}K_d\left(\frac{\mathbf{x}-\mathbf{v}}{h_1}\right)l(\mathbf{v})d\mathbf{v}.
\end{aligned}$$

Using assumptions **(A2)**, **(A7)**, a change of variables and the fact that H is bounded by 1, subtracting gives

$$\begin{aligned}
\mathcal{W}_2' - \mathcal{W}_2'' &= \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}} K_d\left(\frac{\mathbf{x}-\mathbf{u}}{h_1}\right)K_d\left(\frac{\mathbf{x}-\mathbf{v}}{h_1}\right)H\left(\frac{t-s}{h_2}\right)[f(\mathbf{u},\mathbf{v},s)-f(\mathbf{u},s)l(\mathbf{v})]d\mathbf{u}d\mathbf{v}ds\\
&= O(h_1^{2d}h_2).
\end{aligned}$$

In the same way, by assumptions **(A2)**, **(A4)** and **A6** we get

$$\begin{aligned}
\mathcal{W}_4 &= \mathbb{E}[K_{d,i}K_{d,j}] - \mathbb{E}[K_{d,i}]\,\mathbb{E}[K_{d,j}]\\
&= \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} K_d\left(\frac{\mathbf{x}-\mathbf{u}}{h_1}\right)K_d\left(\frac{\mathbf{x}-\mathbf{v}}{h_1}\right)[l(\mathbf{u},\mathbf{v})-l(\mathbf{u})l(\mathbf{v})]d\mathbf{u}d\mathbf{v}\\
&= O(h_1^{2d}).
\end{aligned}$$

All these steps conclude that

$$\Xi_n(\mathbf{x},t) \to 0 \text{ as } n \to +\infty$$

which achives the proof of Lemma 3. □

By the means of (14) and Lemma 3, to justify the Theorem 1, it suffices to show that

$$\sqrt{nh_1^d}(Z_n(\mathbf{x},t)) \xrightarrow{\mathcal{D}} \mathcal{N}(0,\sigma^2(\mathbf{x},t)) \quad \text{as } n \to +\infty. \qquad [20]$$

To do that, as already mentioned above, we use the Bernstein's procedure based on small and big blocks. In the sequel, for the sake of simplify, we will normalize ψ_i. Therefore, from (17), we get

$$\begin{aligned}\sqrt{nh_1^d}(Z_n(\mathbf{x},t)) &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\Psi_i(\mathbf{x},t)}{\sqrt{h_1^d}l(\mathbf{x})}\\ &=: \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\tilde{\Psi}_i(\mathbf{x},t)\\ &=: \frac{\tilde{S}_n}{\sqrt{n}}.\end{aligned}$$

Moreover, it can be checked from (18) and Lemma 3 that

$$\mathrm{var}(\tilde{\psi_i}) = \frac{\mathrm{var}(\psi_i)}{h_1^d l^2(\mathbf{x})} =: \sigma_n^2(\mathbf{x},t) \text{ and } \mathrm{var}(\tilde{\psi_i}) \to \sigma^2(\mathbf{x},t) \text{ as } n \to +\infty.$$

Note also that

$$\sum_{i=1}^{n}\sum_{j=1:j\neq i}^{n}\mathrm{cov}\left(\tilde{\Psi}_i(\mathbf{x},t),\tilde{\Psi}_j(\mathbf{x},t)\right) = \frac{1}{h_1^d l^2(\mathbf{x})}\sum_{i=1}^{n}\sum_{j=1:j\neq i}^{n}\mathrm{cov}\left(\Psi_i(\mathbf{x},t),\Psi_j(\mathbf{x},t)\right) = o(1).$$

Clearly, (20) is equivalent to

$$\frac{\tilde{S}_n}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,\sigma^2(\mathbf{x},t)) \quad \text{as } n \to +\infty. \qquad [21]$$

The main goal is to establish the asymptotic normality of $\frac{\tilde{S}_n}{\sqrt{n}}$. To this end we use the Bernstein's blocking technique. Let us partition the set $\{1,\dots,n\}$ into $2k_n+1$ subsets and for $m = 1,\dots,k_n$, we set

$$\begin{aligned}I_m &= \{i; i = (m-1)(p_n+q_n)+1,\dots,(m-1)(p_n+q_n)+p_n\}\\ J_m &= \{j; j = (m-1)(p_n+q_n)+p_n+1,\dots,m(p_n+q_n)\}.\end{aligned}$$

The remaining points are defined in the set $\{l; k_n(p_n+q_n)+1 \leq l \leq n\}$ which may be empty and p_n, q_n and k_n are given in Assumption **N2**. Let us define the rv's U_{nm}, U'_{nm} and U''_{nm} as follows

$$U_{nm} = \sum_{i=(m-1)(p_n+q_n)+1}^{(m-1)(p_n+q_n)+p_n}\tilde{\Psi}_i, \quad U'_{nm} = \sum_{j=(m-1)(p_n+q_n)+p_n+1}^{m(p_n+q_n)}\tilde{\Psi}_j, \; U''_{nk} = \sum_{l=k_n(p_n+q_n)+1}^{n}\tilde{\Psi}_l$$

with

$$\begin{aligned}\frac{\tilde{S}_n}{\sqrt{n}} &= \frac{1}{\sqrt{n}}\left[\sum_{m=1}^{k_n}U_{nm} + \sum_{m=1}^{k_n}U'_{nm} + U''_{nk_n}\right]\\ &=: \frac{1}{\sqrt{n}}\left[T_n + {T_n}' + {T_n}''\right].\end{aligned}$$

Then, in order to establish the convergence of (21), we have to show that

$$\frac{T_n}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2(\mathbf{x}, t)) \quad \text{as } n \to +\infty \qquad [22]$$

and

$$\frac{1}{n}\mathbb{E}\left[T_n'^2\right] + \frac{1}{n}\mathbb{E}\left[T_n''^2\right] \to 0 \; as \; n \to +\infty. \qquad [23]$$

In order to prove (22) and (23), we use the following lemmas

LEMMA 4– *Assumption **N2** yields that*

(i) $\frac{k_n}{n}\text{var}(U'_{n1}) \to 0$

(ii) $\frac{1}{n}|\text{cov}(U'_{n1}, U'_{n,l+1})| \le \frac{q_n m_1}{n l^2(\boldsymbol{x})} \frac{1}{h_1^{d+2} h_2^2} \sum_{r=l(p_n+q_n)-(q_n-1)}^{l(p_n+q_n)+(q_n-1)} |\Lambda_{1,r+1}(\boldsymbol{x}, t)|$

(iii) $\frac{1}{n} \sum_{1 \le i < j \le k_n} |\text{cov}|(U'_{ni}, U'_{nj})| \to 0$

3.6. *Proof of Lemma 4*

(i) We have

$$\begin{aligned}
\frac{k_n}{n}\text{var}(U'_{n1}) &= \frac{k_n}{n}\text{var}\left(\sum_{j=p_n+1}^{p_n+q_n} \tilde{\Psi}_j(\mathbf{x}, t)\right) \\
&= \frac{k_n q_n}{n}\text{var}(\tilde{\Psi}_1(\mathbf{x}, t)) + \frac{2k_n}{n} \sum_{1 \le i < j \le q_n} \left|\text{cov}\left(\tilde{\Psi}_i(\mathbf{x}, t), \tilde{\Psi}_j(\mathbf{x}, t)\right)\right| \\
&= \frac{k_n q_n}{n} \frac{1}{h_1^d l^2(\mathbf{x})}\text{var}\left(\Psi_1(\mathbf{x}, t)\right) + \frac{2k_n}{n} \sum_{1 \le i < j \le q_n} \left|\text{cov}\left(\tilde{\Psi}_i(\mathbf{x}, t), \tilde{\Psi}_j(\mathbf{x}, t)\right)\right| \\
&=: \mathcal{J}_1 + \mathcal{J}_2.
\end{aligned}$$

As for the first term, we apply Assumption **N2(i)**, (18) and Lemma 3. Then

$$\mathcal{J}_1 = \frac{k_n q_n}{n}\sigma_n^2(\mathbf{x}, t) \to 0 \text{ as } n \to +\infty.$$

Regarding the second term, by stationarity we can write

$$\begin{aligned}
\mathcal{J}_2 &= \frac{2k_n}{n}\sum_{l=1}^{q_n-1}(q_n - l)\left|\text{cov}\left(\tilde{\Psi}_1(\mathbf{x}, t), \tilde{\Psi}_{l+1}(\mathbf{x}, t)\right)\right| \\
&\le \frac{2k_n q_n}{n}\sum_{l=1}^{q_n-1}\left|\text{cov}\left(\tilde{\Psi}_1(\mathbf{x}, t), \tilde{\Psi}_{l+1}(\mathbf{x}, t)\right)\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{2k_nq_n}{n}q_n\frac{1}{h_1^dl^2(\mathbf{x})}\left|\text{cov}\left(\Psi_1(\mathbf{x},t),\Psi_{l+1}(\mathbf{x},t)\right)\right| \\
&\leq m_1\frac{k_nq_n}{n}q_nh_1^d.
\end{aligned}
$$

From the second part of Lemma 3 and assumptions **N2(i)-(ii)**, we get that $\mathcal{J}_2 \to 0$ as $n \to +\infty$.
(ii) Using stationarity and Theorem 5.3, p.89 in [BULINSKI and SHASHKIN 2007] we have

$$
\begin{aligned}
\frac{1}{n}|\text{cov}(U'_{n1},U'_{n,l+1})| &= \frac{1}{n}\left|\sum_{i=p_n+1}^{p_n+q_n}\sum_{i=l(p_n+q_n)+p_n+1}^{(l+1)(p_n+q_n)}\text{cov}\left(\tilde{\Psi}_i(\mathbf{x},t),\tilde{\Psi}_j(\mathbf{x},t)\right)\right| \\
&= \frac{1}{n}\left|\sum_{r=1}^{q_n}\sum_{j=l(p_n+q_n)+p_n+1}^{(l+1)(p_n+q_n)}\text{cov}\left(\tilde{\Psi}_{p_n+r}(\mathbf{x},t),\tilde{\Psi}_j(\mathbf{x},t)\right)\right| \\
&= \frac{1}{n}\left|\sum_{r=1}^{q_n}(q_n-r+1)\text{cov}\left(\tilde{\Psi}_1(\mathbf{x},t),\tilde{\Psi}_{l(p_n+q_n)+r}(\mathbf{x},t)\right)\right. \\
&\quad\left.+\sum_{r=1}^{q_n-1}(q_n-r)\text{cov}\left(\tilde{\Psi}_{r+1}(\mathbf{x},t),\tilde{\Psi}_{l(p_n+q_n)+1}(\mathbf{x},t)\right)\right| \\
&= \frac{1}{n}\left|\sum_{r=1}^{q_n}(q_n-r+1)\text{cov}\left(\tilde{\Psi}_1(\mathbf{x},t),\tilde{\Psi}_{l(p_n+q_n)+r}(\mathbf{x},t)\right)\right. \\
&\quad\left.+\sum_{r=1}^{q_n-1}(q_n-r)\text{cov}\left(\tilde{\Psi}_1(\mathbf{x},t),\tilde{\Psi}_{l(p_n+q_n)-r+1}(\mathbf{x},t)\right)\right| \\
&\leq \frac{q_n}{n}\sum_{r=l(p_n+q_n)+1}^{l(p_n+q_n)+q_n}\left|\text{cov}\left(\tilde{\Psi}_1(\mathbf{x},t),\tilde{\Psi}_r(\mathbf{x},t)\right)\right| \\
&\quad+\frac{q_n}{n}\sum_{r=l(p_n+q_n)-(q_n-2)}^{l(p_n+q_n)}\left|\text{cov}\left(\tilde{\Psi}_1(\mathbf{x},t),\tilde{\Psi}_r(\mathbf{x},t)\right)\right| \\
&= \frac{q_n}{n}\sum_{r=l(p_n+q_n)-(q_n-2)}^{l(p_n+q_n)+q_n}\left|\text{cov}\left(\tilde{\Psi}_1(\mathbf{x},t),\tilde{\Psi}_r(\mathbf{x},t)\right)\right| \\
&= \frac{q_n}{n}\sum_{r=l(p_n+q_n)-(q_n-1)}^{l(p_n+q_n)+(q_n-1)}\left|\text{cov}\left(\tilde{\Psi}_1(\mathbf{x},t),\tilde{\Psi}_{r+1}(\mathbf{x},t)\right)\right| \\
&= \frac{q_n}{nh_1^dl^2(\mathbf{x})}\sum_{r=l(p_n+q_n)-(q_n-1)}^{l(p_n+q_n)+(q_n-1)}|\text{cov}(\Psi_1(\mathbf{x},t),\Psi_{r+1}(\mathbf{x},t))| \\
&= \frac{q_nM_0}{nh_1^{d+2}h_2^2l^2(\mathbf{x})}\sum_{r=l(p_n+q_n)-(q_n-1)}^{l(p_n+q_n)+(q_n-1)}|\Lambda_{1,r+1}(\mathbf{x},t)|
\end{aligned}
$$

where $M_0 = \max\left\{h_2 Lip(K)\left\|K\right\|_\infty^{d-1}, h_1\left(Lip(H) + h_2\frac{Lip(\overline{G})}{\overline{G}(\tau)}\right)\left\|K_d\right\|_\infty\right\}$ and (ii) is then obtained.
(iii) By stationarity and (ii), we find

$$
\begin{aligned}
\frac{1}{n}\sum_{1\le i<j\le k_n}|\mathrm{cov}(U'_{ni}, U'_{nj})| &= \frac{1}{n}\sum_{l=1}^{k_n-1}(k_n - l)|\mathrm{cov}(U'_{n1}, U'_{n,l+1})| \\
&\le \frac{k_n}{n}\sum_{l=1}^{k_n-1}|\mathrm{cov}(U'_{n1}, U'_{n,l+1})| \\
&\le m_1\frac{q_n k_n}{n}\frac{1}{l^2(\mathbf{x})h_1^{d+2}h_2^2}\sum_{l=1}^{k_n-1}\sum_{r=l(p_n+q_n)-(q_n-1)}^{l(p_n+q_n)+(q_n-1)}|\Lambda_{1,r+1}(\mathbf{x}, t)| \\
&\le m\frac{q_n k_n}{n}\frac{1}{l^2(\mathbf{x})h_1^{d+2}h_2^2}\sum_{r=p_n}^{\infty}|\Lambda_{1,r+1}(\mathbf{x}, t)| \\
&\le m_1\frac{q_n k_n}{n}\frac{1}{l^2(\mathbf{x})}\frac{\gamma_0}{1-e^{-\gamma}}\frac{e^{-\gamma p_n}}{h_1^{d+2}h_2^2} \to 0.
\end{aligned}
$$

The result follows from assumptions **N2(i)** and **(iii)**. □

LEMMA 5– *If Assumption* **N2** *holds, we have*
(i) $\frac{k_n}{n}\mathrm{var}(U_{n1}) \to \sigma^2(\boldsymbol{x}, t)$
(ii) $\frac{1}{n}|\mathrm{cov}(U_{n1}, U_{n,l+1})| \le \frac{p_n m_1}{nl^2(\boldsymbol{x})}\frac{1}{h_1^{d+2}h_2^2}\sum_{r=l(p_n+q_n)-p_n}^{l(p_n+q_n)+p_n}|\Lambda_{1,r+1}(\boldsymbol{x}, t)|$
(iii) $\frac{1}{n}\sum_{1\le i<j\le k_n}|\mathrm{cov}(U_{ni}, U_{nj})| \to 0$
(iv) $\mathrm{var}\left(\frac{T_n}{\sqrt{n}}\right) \to \sigma^2(\boldsymbol{x}, t)$

3.7. *Proof of Lemma 5*

(i) We have

$$
\begin{aligned}
\frac{k_n}{n}\mathrm{var}(U_{n1}) &= \frac{k_n}{n}\mathrm{var}\left(\sum_{i=1}^{p_n}\tilde{\Psi}_i(\mathbf{x}, t)\right) \\
&= \frac{p_n k_n}{n}\frac{1}{h_1^d l^2(\mathbf{x})}\mathrm{var}\left(\Psi_1(\mathbf{x}, t)\right) + \frac{2k_n}{nh_1^d l^2(\mathbf{x})}\sum_{1\le i<j\le p_n}\left|\mathrm{cov}\left(\Psi_i(\mathbf{x}, t), \Psi_j(\mathbf{x}, t)\right)\right|.
\end{aligned}
$$

By the same arguments as in Lemma 4(i) and Assumption **N2(i)-(ii)**, we get the result.
(ii) An analogous framework as in Lemma 4(ii) gives

$$
\frac{1}{n}|\mathrm{cov}(U_{n1}, U_{n,l+1})| = \frac{1}{n}\left|\sum_{i=1}^{p_n}\sum_{j=l(p_n+q_n)+1}^{l(p_n+q_n)+p_n}\mathrm{cov}\left(\tilde{\Psi}_i(\mathbf{x}, t), \tilde{\Psi}_j(\mathbf{x}, t)\right)\right|
$$

$$
\begin{aligned}
&= \frac{1}{n}\left|\sum_{r=1}^{p_n}(p_n-r+1)\mathrm{cov}\left(\tilde{\Psi}_1(\mathbf{x},t),\tilde{\Psi}_{l(p_n+q_n)+r}(\mathbf{x},t)\right)\right.\\
&\quad\left.+\sum_{r=1}^{p_n-1}(p_n-r)\mathrm{cov}\left(\tilde{\Psi}_{r+1}(\mathbf{x},t),\tilde{\Psi}_{l(p_n+q_n)+1}(\mathbf{x},t)\right)\right|\\
&= \frac{1}{n}\left|\sum_{r=1}^{p_n}(p_n-r+1)\mathrm{cov}\left(\tilde{\Psi}_1(\mathbf{x},t),\tilde{\Psi}_{l(p_n+q_n)+r}(\mathbf{x},t)\right)\right.\\
&\quad\left.+\sum_{r=1}^{p_n-1}(p_n-r)\mathrm{cov}\left(\tilde{\Psi}_1(\mathbf{x},t),\tilde{\Psi}_{l(p_n+q_n)-r+1}(\mathbf{x},t)\right)\right|\\
&\leq \frac{p_n}{n}\sum_{r=l(p_n+q_n)-(p_n-2)}^{l(p_n+q_n)+p_n}\left|\mathrm{cov}\left(\tilde{\Psi}_1(\mathbf{x},t),\tilde{\Psi}_r(\mathbf{x},t)\right)\right|\\
&= \frac{p_n}{nh_1^d l^2(\mathbf{x})}\sum_{r=l(p_n+q_n)-(p_n-1)}^{l(p_n+q_n)+(p_n-1)}|\mathrm{cov}(\Psi_1(\mathbf{x},t),\Psi_{r+1}(\mathbf{x},t))|\\
&\leq \frac{p_n m_1}{nh_1^{d+2}h_2^2 l^2(\mathbf{x})}\sum_{r=l(p_n+q_n)-p_n}^{l(p_n+q_n)+p_n}|\Lambda_{1,r+1}(\mathbf{x},t)|
\end{aligned}
$$

(iii) By stationarity and item (ii), we have

$$
\begin{aligned}
\frac{1}{n}\sum_{1\leq i<j\leq k_n}|\mathrm{cov}(U_{ni},U_{nj})| &= \frac{1}{n}\sum_{l=1}^{k_n-1}(k_n-l)|\mathrm{cov}(U_{n1},U_{n,l+1})|\\
&\leq \frac{k_n}{n}\sum_{l=1}^{k_n-1}|\mathrm{cov}(U_{n1},U_{n,l+1})|\\
&\leq m_1\frac{p_nk_n}{n}\frac{1}{l^2(\mathbf{x})h_1^{d+2}h_2^2}\sum_{l=1}^{k_n-1}\sum_{r=l(p_n+q_n)-p_n}^{l(p_n+q_n)+p_n}|\Lambda_{1,r+1}(\mathbf{x},t)|\\
&\leq m\frac{p_nk_n}{n}\frac{1}{l^2(\mathbf{x})h_1^{d+2}h_2^2}\sum_{r=p_n}^{\infty}|\Lambda_{1,r+1}(\mathbf{x},t)|\\
&\leq m_1\frac{p_nk_n}{n}\frac{1}{l^2(\mathbf{x})}\frac{\gamma_0}{1-e^{-\gamma}}\frac{e^{-\gamma q_n}}{h_1^{d+2}h_2^2}\to 0.
\end{aligned}
$$

As stated in Lemma 4(ii) and from Assumptions **N2(i)** and **(iii)** , we get the result.
(iv) Observe that

$$
\begin{aligned}
\mathrm{var}\left(\frac{T_n}{\sqrt{n}}\right) &= \frac{1}{n}\mathrm{var}\left(\sum_{m=1}^{k_n}U_{nm}\right)\\
&= \frac{k_n}{n}\mathrm{var}(U_{n1})+\frac{2}{n}\sum_{1\leq i<j\leq k_n}|\mathrm{cov}(U_{ni},U_{nj})|.
\end{aligned}
$$

By items (i) and (iii), we finish the proof of Lemma 5. □

As for the demonstration of (23), we use exactly the same arguments as those employed in Lemma 5(iii)

$$\begin{aligned}\operatorname{var}\left(\frac{T_n'}{\sqrt{n}}\right) &= \frac{1}{n}\operatorname{var}\left(\sum_{m=1}^{k_n} U_{nm}'\right)\\ &= \frac{k_n}{n}\operatorname{var}(U_{n1}') + \frac{2}{n}\sum_{1\le i<j\le k_n} |\operatorname{cov}(U_{ni}', U_{nj}')| \to 0.\end{aligned}$$

This holds by items (i) and (iii) of Lemma 4. And,

$$\begin{aligned}\operatorname{var}\left(\frac{T_n''}{\sqrt{n}}\right) &= \frac{1}{n}\operatorname{var}\left(U_{nk_n}''\right)\\ &= \frac{1}{n}\operatorname{var}\left(\sum_{l=k_n(p_n+q_n)+1}^{n} \tilde{\Psi}_l(\mathbf{x}, t)\right)\\ &= \frac{n-k_n(p_n+q_n)}{n}\operatorname{var}(\tilde{\Psi}_1(\mathbf{x}, t)) + \frac{2}{n}\sum_{k_n(p_n+q_n)+1\le i<j\le n} |\operatorname{cov}(\tilde{\Psi}_i(\mathbf{x}, t)), \tilde{\Psi}_j(\mathbf{x}, t))|\\ &=: \mathcal{I}_{n1} + \mathcal{I}_{n2}.\end{aligned}$$

We may write

$$\begin{aligned}\mathcal{I}_{n1} &= \frac{n-k_n(p_n+q_n)}{n}\frac{1}{h_1^d l^2(\mathbf{x})}\operatorname{var}(\Psi_1(\mathbf{x}, t))\\ &\le \frac{p_n}{n}\sigma_n^2(\mathbf{x}, t).\end{aligned}$$

Employing Assumption **N2** and Lemma 3, we obtain that $\mathcal{I}_{n1} = o(1)$. Remark that $n - k_n(p_n + q_n) \le p_n$. As for the second term, we have

$$\begin{aligned}\mathcal{I}_{n2} &= \frac{2}{n}\sum_{k_n(p_n+q_n)+1\le i<j\le n} |\operatorname{cov}(\tilde{\Psi}_i(\mathbf{x}, t)), \tilde{\Psi}_j(\mathbf{x}, t))|\\ &= \frac{2}{n}\sum_{1\le i<j\le n-k_n(p_n+q_n)} |\operatorname{cov}(\tilde{\Psi}_i(\mathbf{x}, t)), \tilde{\Psi}_j(\mathbf{x}, t))| \quad \text{(by stationarity)}\\ &\le \frac{2}{n}\sum_{1\le i<j\le p_n} |\operatorname{cov}(\tilde{\Psi}_i(\mathbf{x}, t)), \tilde{\Psi}_j(\mathbf{x}, t))|\\ &= \frac{2}{n}\sum_{l=1}^{p_n-1}(p_n - l)|\operatorname{cov}(\tilde{\Psi}_1(\mathbf{x}, t)), \tilde{\Psi}_{l+1}(\mathbf{x}, t))|\\ &\le \frac{2p_n}{n}\sum_{l=1}^{p_n-1} |\operatorname{cov}(\tilde{\Psi}_1(\mathbf{x}, t)), \tilde{\Psi}_{l+1}(\mathbf{x}, t))|\\ &\le \frac{p_n(p_n-1)}{nh_1^d l^2(\mathbf{x})} m_1 h_1^{2d}\end{aligned}$$

$$= O\left(\frac{p_n}{n}p_n h_1^d\right).$$

Thus, under Assumption **N2** the proof of (23) is achieved.

The next step consists in proving (22), that is by proving first that the rv's $\{U_{nm}, m = 1, \ldots, k\}$ are asymptotically independent. For this purpose, we deal with the characteristic functions and show that

$$\left|\mathbb{E}\left(e^{it\sum_{m=1}^{k_n}\frac{1}{\sqrt{n}}U_{nm}}\right) - \prod_{m=1}^{k_n}\mathbb{E}\left(e^{it\frac{1}{\sqrt{n}}U_{nm}}\right)\right| \to 0 \text{ as } n \to \infty.$$

Indeed, we have

$$\begin{aligned}
I_{k_n}(t) &=: \left|\mathbb{E}\left(e^{it\sum_{m=1}^{k_n}\frac{U_{nm}}{\sqrt{n}}}\right) - \prod_{m=1}^{k_n}\mathbb{E}\left(e^{it\frac{U_{nm}}{\sqrt{n}}}\right)\right| \\
&= \left|\mathbb{E}\left(e^{it\sum_{m=1}^{k_n}\frac{U_{nm}}{\sqrt{n}}}\right) - \prod_{m=1}^{k_n-1}\mathbb{E}\left(e^{it\frac{U_{nm}}{\sqrt{n}}}\right)\left(e^{it\frac{1}{\sqrt{n}}U_{nk_n}}\right)\right| \\
&\leq \left|\mathbb{E}\left(e^{it\sum_{m=1}^{k_n}\frac{U_{nm}}{\sqrt{n}}}\right) - \mathbb{E}\left(e^{it\frac{U_{nk_n}}{\sqrt{n}}}\right)\mathbb{E}\left(e^{it\sum_{m=1}^{k_n-1}\frac{U_{nm}}{\sqrt{n}}}\right)\right| \\
&\quad + \left|\mathbb{E}\left(e^{it\frac{U_{nk_n}}{\sqrt{n}}}\right)\mathbb{E}\left(e^{it\sum_{m=1}^{k_n-1}\frac{U_{nm}}{\sqrt{n}}}\right) - \prod_{m=1}^{k_n-1}\mathbb{E}\left(e^{it\frac{U_{nm}}{\sqrt{n}}}\right)\mathbb{E}\left(e^{it\frac{U_{nk_n}}{\sqrt{n}}}\right)\right| \\
&= \left|\mathbb{E}\left(e^{it\frac{U_{nk_n}}{\sqrt{n}}}\right)\right|\left|\mathbb{E}\left(e^{it\sum_{m=1}^{k_n-1}\frac{U_{nm}}{\sqrt{n}}}\right) - \prod_{m=1}^{k_n-1}\mathbb{E}\left(e^{it\frac{U_{nm}}{\sqrt{n}}}\right)\right| \\
&\quad + \left|\mathbb{E}\left(e^{it\sum_{m=1}^{k_n}\frac{U_{nm}}{\sqrt{n}}}\right) - \mathbb{E}\left(e^{it\frac{U_{nk_n}}{\sqrt{n}}}\right)\mathbb{E}\left(e^{it\sum_{m=1}^{k_n-1}\frac{U_{nm}}{\sqrt{n}}}\right)\right| \\
&= \left|\mathbb{E}\left(e^{it\sum_{m=1}^{k_n-1}\frac{U_{nm}}{\sqrt{n}}}\right) - \prod_{m=1}^{k_n-1}\mathbb{E}\left(e^{it\frac{U_{nm}}{\sqrt{n}}}\right)\right| \\
&\quad + \left|\mathbb{E}\left(e^{it\sum_{m=1}^{k_n}\frac{U_{nm}}{\sqrt{n}}}\right) - \mathbb{E}\left(e^{it\frac{U_{nk_n}}{\sqrt{n}}}\right)\mathbb{E}\left(e^{it\sum_{m=1}^{k_n-1}\frac{U_{nm}}{\sqrt{n}}}\right)\right| \\
&=: I_{k_n-1}(t) + \left|\operatorname{cov}\left(e^{it\sum_{m=1}^{k_n-1}\frac{U_{nm}}{\sqrt{n}}}, e^{it\frac{U_{nk_n}}{\sqrt{n}}}\right)\right|.
\end{aligned}$$

Analogously $I_{k_n-1}(t) \leq I_{k_n-2}(t) + \left|\operatorname{cov}\left(e^{it\sum_{m=1}^{k_n-2}\frac{U_{nm}}{\sqrt{n}}}, e^{it\frac{U_{nk_n-1}}{\sqrt{n}}}\right)\right|$.

Therefore

$$\begin{aligned}
\left|\mathbb{E}\left(e^{it\sum_{m=1}^{k_n}\frac{U_{nm}}{\sqrt{n}}}\right) - \prod_{m=1}^{k_n}\mathbb{E}\left(e^{it\frac{U_{nm}}{\sqrt{n}}}\right)\right| &\leq \left|\operatorname{cov}\left(e^{it\sum_{m=1}^{k_n-1}\frac{U_{nm}}{\sqrt{n}}}, e^{it\frac{U_{nk_n}}{\sqrt{n}}}\right)\right| \\
&\quad + \left|\operatorname{cov}\left(e^{it\sum_{m=1}^{k_n-2}\frac{U_{nm}}{\sqrt{n}}}, e^{it\frac{U_{nk_n-1}}{\sqrt{n}}}\right)\right| \\
&\quad + \ldots + \left|\operatorname{cov}\left(e^{it\frac{U_{n1}}{\sqrt{n}}}, e^{it\frac{U_{n2}}{\sqrt{n}}}\right)\right|. \qquad [24]
\end{aligned}$$

In what follows, we apply the Lemma1 in [BULINSKI 1996] on each term in the right hand side of (24). For this purpose, we need to calculate the first order partial derivatives of the function $\mathcal{V}_m : \mathbb{R}^{p_n(d+1)} \to \mathbb{R}$, $m = 1, \ldots, k_n$ defined by $V_m(\mathbf{x}_l, y_l) = e^{it\frac{U_{nm}}{\sqrt{n}}}$ for all l lying between $(m-1)(p_n+q_n)+1$ and $(m-1)(p_n+q_n)+p_n$, where

$$\mathbf{x}_l = (x^1_{(m-1)(p_n+q_n)+1}, \ldots, x^d_{(m-1)(p_n+q_n)+1}, \ldots, x^1_{(m-1)(p_n+q_n)+p_n}, \ldots, x^d_{(m-1)(p_n+q_n)+p_n})$$

and

$$y_l = (y_{(m-1)(p_n+q_n)+1}, \ldots, y_{(m-1)(p_n+q_n)+p_n}).$$

By the way, the first partial derivative of the function $\mathcal{V}_m$ with respect to y_l for $l = (m-1)(p_n+q_n)+1, \ldots, (m-1)(p_n+q_n)+p_n$ is given by

$$\begin{aligned}
\frac{\partial \mathcal{V}_m}{\partial y_l}(\mathbf{x}_l, y_l) &= \frac{\partial}{\partial y_l}\left(it\frac{U_{nm}}{\sqrt{n}}\right)e^{it\frac{U_{nm}}{\sqrt{n}}} \\
&= \frac{\partial}{\partial y_l}\left(\frac{it}{\sqrt{n}}\sum_{i=(m-1)(p_n+q_n)+1}^{(m-1)(p_n+q_n)+p_n}\tilde{\Psi}_l\right)e^{it\frac{U_{nm}}{\sqrt{n}}} \\
&= \frac{it}{\sqrt{n}}e^{it\frac{U_{nm}}{\sqrt{n}}}\frac{\partial}{\partial y_l}\left[\frac{1}{\sqrt{h_1^d}l(\mathbf{x})}\left(K_{d,l}\left[H_l\frac{\delta_l}{\overline{G}(Y_l)} - F(t|\mathbf{x})\right]\right)\right] \\
&= \frac{it}{\sqrt{nh_1^d}l(\mathbf{x})}e^{it\frac{U_{nm}}{\sqrt{n}}}K_d\left(\frac{\mathbf{x}-\mathbf{x}_l}{h_1}\right)\delta_l\frac{\partial}{\partial y_l}\left[H\left(\frac{t-y_l}{h_2}\right)\frac{1}{\overline{G}(y_l)}\right] \\
&= \frac{it}{\sqrt{nh_1^d h_2^2}l(\mathbf{x})}e^{it\frac{U_{nm}}{\sqrt{n}}}K_d\left(\frac{\mathbf{x}-\mathbf{x}_l}{h_1}\right) \\
&\quad \times\delta_l\left[-h_2\frac{g(y_l)}{\overline{G}^2(y_l)}H\left(\frac{t-y_l}{h_2}\right) - H^{(1)}\left(\frac{t-y_l}{h_2}\right)\frac{1}{\overline{G}(y_l)}\right]
\end{aligned}$$

Moreover, the first partial derivative of the function $\mathcal{V}_m$ with respect to x_l^j for $l = (m-1)(p_n+q_n)+1, \ldots, (m-1)(p_n+q_n)+p_n$ and $j = 1, \ldots d$ is

$$\begin{aligned}
\frac{\partial \mathcal{V}_m}{\partial x_l^j}(\mathbf{x}_l, y_l) &= \frac{it}{\sqrt{nh_1^d}}e^{it\frac{U_{nm}}{\sqrt{n}}}\left[\frac{H_l\delta_l}{\overline{G}(y_l)} - F(t|\mathbf{x})\right]\frac{\partial}{\partial x_l^j}\left[\frac{1}{l(\mathbf{x})}K_d\left(\frac{\mathbf{x}-\mathbf{x}_l}{h_1}\right)\right] \\
&= \frac{it}{\sqrt{nh_1^d}}e^{it\frac{U_{nm}}{\sqrt{n}}}\left[\frac{H_l\delta_l}{\overline{G}(y_l)} - F(t|\mathbf{x})\right] \\
&\quad \times\left[-\frac{l^{(1)}(\mathbf{x})}{l^2(\mathbf{x})}K_d\left(\frac{\mathbf{x}-\mathbf{x}_l}{h_1}\right) + \frac{\partial}{\partial x_l^j}K_d\left(\frac{\mathbf{x}-\mathbf{x}_l}{h_1}\right)\frac{1}{l(\mathbf{x})}\right]
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\frac{\partial \mathcal{V}_m}{\partial x_l^j}(\mathbf{x}_l, y_l) &= \frac{it}{\sqrt{nh_1^{d+2}}}e^{it\frac{U_{nm}}{\sqrt{n}}}\left[\frac{H_l\delta_l}{\overline{G}(y_l)} - F(t|\mathbf{x})\right] \\
&\quad \times\left[-h_1\frac{l^{(1)}(\mathbf{x})}{l^2(\mathbf{x})}K_d\left(\frac{\mathbf{x}-\mathbf{x}_l}{h_1}\right) - \prod_{i=1,i\neq j}^{d}K\left(\frac{x^i-x_l^i}{h_1}\right)K^{(1)}\left(\frac{x^j-x_l^j}{h_1}\right)\right]
\end{aligned}$$

Besides, Applying assumptions **A2**-**A4** and **A7**, there exists $A > 0$ such that

$$\left\| \frac{\partial \mathcal{V}_m}{\partial y_l}(\mathbf{x}_l, y_l) \right\|_\infty \leq \frac{At}{\sqrt{nh_1^d h_2^2}} \quad \text{and} \quad \left\| \frac{\partial \mathcal{V}_m}{\partial x_l^j}(\mathbf{x}_l, y_l) \right\|_\infty \leq \frac{At}{\sqrt{nh_1^{d+2}}}.$$

Thereby, using the Lemma 1 in [BULINSKI 1996] we find

$$\left| \text{cov}\left(e^{it\frac{U_{n2}}{\sqrt{n}}}, e^{it\frac{U_{n1}}{\sqrt{n}}} \right) \right| \leq \frac{A^2 t^2}{nh_1^{d+1} h_2} \sum_{i \in I_1} \sum_{j \in I_2} |\Lambda_{i,j}(\mathbf{x}, t)|.$$

In addition

$$\left| \text{cov}\left(e^{it\sum_{m=1}^{k_n-1}\frac{U_{nm}}{\sqrt{n}}}, e^{it\frac{U_{nk_n}}{\sqrt{n}}} \right) \right| \leq \frac{A^2 t^2}{nh_1^{d+1} h_2} \sum_{i \in I_1 + \ldots + I_{k_n-1}} \sum_{j \in I_{k_n}} |\Lambda_{i,j}(\mathbf{x}, t)|.$$

We conclude

$$\begin{aligned} \left| \mathbb{E}\left(e^{it\sum_{m=1}^{k_n}\frac{U_{nm}}{\sqrt{n}}} \right) - \prod_{m=1}^{k_n} \mathbb{E}\left(e^{it\frac{U_{nm}}{\sqrt{n}}} \right) \right| \quad &\leq \quad \frac{A^2 t^2}{nh_1^{d+1} h_2} \left[\sum_{i \in I_1} \sum_{j \in I_2} |\Lambda_{i,j}(\mathbf{x}, t)| + \sum_{i \in I_1 + I_2} \sum_{j \in I_3} |\Lambda_{i,j}(\mathbf{x}, t)| \right. \\ &\left. + \ldots + \sum_{i \in I_1 + \ldots + I_{k_n-1}} \sum_{j \in I_{k_n}} |\Lambda_{i,j}(\mathbf{x}, t)| \right]. \end{aligned} \quad [25]$$

Using stationarity, the inequality (25) becomes

$$\begin{aligned} \left| \mathbb{E}\left(e^{it\sum_{m=1}^{k_n}\frac{U_{nm}}{\sqrt{n}}} \right) - \prod_{m=1}^{k_n} \mathbb{E}\left(e^{it\frac{U_{nm}}{\sqrt{n}}} \right) \right| \quad &\leq \quad \frac{A^2 t^2}{nh_1^{d+1} h_2} \left[(k_n - 1) \sum_{i \in I_1} \sum_{j \in I_2} |\Lambda_{i,j}(\mathbf{x}, t)| \right. \\ &\left. + \ldots + \sum_{i \in I_1} \sum_{j \in I_{k_n}} |\Lambda_{i,j}(\mathbf{x}, t)| \right]. \end{aligned}$$

Again, by stationarity we have

$$\begin{aligned} \left| \mathbb{E}\left(e^{it\sum_{m=1}^{k_n}\frac{U_{nm}}{\sqrt{n}}} \right) - \prod_{m=1}^{k_n} \mathbb{E}\left(e^{it\frac{U_{nm}}{\sqrt{n}}} \right) \right| \quad &\leq \quad \frac{A^2 t^2}{nh_1^{d+1} h_2} p_n k_n \sum_{j=(k_n-1)(p_n+q_n)-(p_n-2)}^{(k_n-1)(p_n+q_n)+p_n} |\Lambda_{1,j}(\mathbf{x}, t)| \\ &\leq \quad m_1 t^2 \frac{p_n k_n}{n} \frac{1}{h_1^{d+1} h_2} \sum_{j=q_n}^{\infty} |\Lambda_{1,j}(\mathbf{x}, t)| \\ &\leq \quad m_1 t^2 \frac{p_n k_n}{n} \frac{\gamma_0}{1 - e^{-\gamma}} \frac{e^{-\gamma q_n}}{h_1^{d+1} h_2} \to 0. \end{aligned} \quad [26]$$

Note that (26) tends to zero under assumptions **N2(i)** and **(iii)**.
It remains to show that $\frac{T_n}{\sqrt{n}}$ is asymptotically normal, we rely on the standard Lindeberg's condition

$$k_n \mathbb{E}\left(\frac{1}{n} U_{n1}^2 \mathbb{1}_{\left\{ \left| \frac{1}{\sqrt{n}} U_{n1} \right| > \varepsilon\sigma(\mathbf{x}, t) \right\}} \right) \to 0.$$

From assumption **A2**, we can write

$$\begin{aligned} |U_{n1}| &= \left| \sum_{m=1}^{p_n} \frac{1}{\sqrt{h_1^d l(\mathbf{x})}} \left\{ K_{d,i} \left[H_i \frac{\delta_i}{\overline{G}(Y_i)} - F(t|\mathbf{x}) \right] - \mathbb{E} \left[K_{d,i} \left(H_i \frac{\delta_i}{\overline{G}(Y_i)} - F(t|\mathbf{x}) \right) \right] \right\} \right| \\ &\leq \frac{2p_n \|K_d\|_\infty}{\sqrt{h_1^d l(\mathbf{x})} \overline{G}(\tau_F)}. \end{aligned}$$

Therefore, by Tchebytchev's inequality and assumptions **A4** and **N2(ii)** we get

$$\begin{aligned} k_n \mathbb{E} \left(\frac{1}{n} U_{n1}^2 \mathbb{1}_{\left\{ \left| \frac{1}{\sqrt{n}} U_{n1} \right| > \varepsilon \sigma(\mathbf{x},t) \right\}} \right) &\leq \frac{4 k_n p_n^2 \|K_d\|_\infty^2}{n h_1^d l^2(\mathbf{x}) \overline{G}^2(\tau_F)} \mathbb{P} \left(\frac{1}{\sqrt{n}} |U_{n1}| > \varepsilon \sigma(\mathbf{x},t) \right) \\ &\leq \frac{4 \|K_d\|_\infty^2}{l^2(\mathbf{x}) \overline{G}^2(\tau_F)} \frac{k_n \operatorname{Var}(U_{n1})}{n \varepsilon^2 \sigma^2(\mathbf{x},t)} \frac{p_n^2}{n h_1^d} \to 0. \end{aligned}$$

This achieves the proof of Theorem 1. □

3.8. *Proof of Corollary 1*

Making use of a Taylor expansion of $F_n(.|.)$ in the neighborhood of ξ_p, we get

$$\xi_{p,n}(\mathbf{x}) - \xi_p(\mathbf{x}) = \frac{F_n(\xi_{p,n}(\mathbf{x})|\mathbf{x}) - F_n(\xi_p(\mathbf{x})|\mathbf{x})}{f_n(\xi_{p,n}^*(\mathbf{x})|\mathbf{x})}$$

where $\xi_{p,n}^*$ lies between ξ_p and $\xi_{p,n}$. The almost sure convergence of $\xi_{p,n}(\mathbf{x})$ to $\xi_p(\mathbf{x})$ [see DJELLADJ and TATACHAK 2019], Proposition 1 and the continuity of $f(.|.)$ give the convergence in probability of $f_n(\xi_{p,n}^*(\mathbf{x})|\mathbf{x})$ toward $f(\xi_p(\mathbf{x})|\mathbf{x})$. The proof is hence established using Theorem 1. □

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