

Geometric Structures on Kenmotsu Manifolds Admitting a DM Semi Symmetric Non Metric Connection

Structures géométriques sur les variétés de Kenmotsu admettant une connexion semi-symétrique non métrique de type DM

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ABSTRACT. The study of manifolds equipped with special linear connections has become an important topic in contemporary Riemannian geometry. In this work, we investigate Kenmotsu manifolds endowed with a class of semi-symmetric non-metric connections that generalize several previously known connections in the literature. Particular attention is devoted to the geometric behavior of symmetric and skew-symmetric parallel tensor fields associated with such connections. We further examine the conditions under which these manifolds satisfy Ricci-symmetric, weakly symmetric, and weakly Ricci-symmetric properties. In addition, the existence and structure of Ricci solitons related to these non-metric connections are studied within the setting of Kenmotsu geometry.

2010 Mathematics Subject Classification: 53B15, 53B20.

KEYWORDS. Kenmotsu manifold, second order parallel tensor fields, Einstein space, symmetric spaces, Ricci soliton, DM semi-symmetric non metric connection.

1. Introduction

The notion of a semi-symmetric linear connection on a Riemannian manifold (M, g) was first introduced by Friedmann and Schouten [18]. Such a connection is determined by a torsion tensor T that satisfies a specific algebraic condition :

$$T(U, V) = \eta(V)U - \eta(U)V, \quad (1.1)$$

where $U, V \in \Gamma(TM)$ and η is a 1-form dual to the vector field P on M i.e.,

$$\eta(U) = g(U, P). \quad (1.2)$$

A semi-symmetric connection ∇ is called a semi-symmetric metric connection if it preserves the metric g , that is,

$$\nabla g = 0. \quad (1.3)$$

A metric connection whose torsion vanishes is called the Levi-Civita connection, while a metric connection with non-zero torsion is known as a Hayden connection [20]. Therefore, metric connections are classified as either Levi-Civita and Hayden connections.

Agashe and Chafle [2], introduced the notion of a semi-symmetric non-metric connection on a Riemannian manifold (M, g) . A semi-symmetric connection ∇ is said to be a semi-symmetric non-metric connection if

$$\nabla g \neq 0. \quad (1.4)$$

Agashe and Chafle [3] studies some properties of submanifolds of a Riemannian manifold with semi-symmetric non-metric connections. Sengupta, De and Binh in [30], generalizes the semi-symmetric non-metric connection introduced in [2]. They determine the relationships between the curvature tensor R of M , with respect to the semi-symmetric non-metric connection ∇ and the curvature tensor field R^g , with respect to the Riemannian connection.

Diallo and Massamba [16] introduced a new semi-symmetric non-metric connection on a Riemannian manifold, referred to as the DM semi-symmetric non-metric connection. This connection generalizes the semi-symmetric non-metric connections previously defined by Agashe and Chafle [2] and by Sengupta et al. [30]. El Moutar et al. [26] investigated warped product manifolds equipped with the DM semi-symmetric non-metric connection and established several important results. We refer to [27, 28, 1, 35] for others studies wit respect to DM semi-symmetric non-metric connection.

Kenmotsu [22] introduced a distinguished class of almost contact metric manifolds, now known as Kenmotsu manifolds, and established that any Kenmotsu manifold satisfying the condition $R(X, Y) \cdot R = 0$ necessarily has constant negative curvature -1 , where R denotes the Riemannian curvature tensor of type $(1, 3)$, and $R(X, Y)$ acts as a derivation on the tensor algebra of each tangent space $T_p M$. Since then, Kenmotsu manifolds have been extensively studied, and their structural properties analyzed by numerous authors [4, 6, 13, 7, 8, 9, 10, 11, 14, 12, 15]

Eisenhart [17], was among the first to investigate second-order parallel symmetric tensors, showing that a positive-definite Riemannian manifold admitting such a tensor, distinct from a constant multiple of the metric, must be reducible. Later, Levy [23] proved that any non-degenerate second-order parallel symmetric $(0, 2)$ -tensor in a space of constant curvature is necessarily proportional to the metric. While these studies focused on local properties, Sharma [31] extended the analysis globally, employing Ricci identities in complex space forms. Since then, Eisenhart-type problems have been investigated in a variety of geometric contexts, including almost contact metric manifolds [32, 33]

In recent years, the study of Kenmotsu manifolds has been further developed through generalized connections and curvature tensors. In particular, the generalized Tanaka–Webster connection, the Schouten–Van Kampen connection, the Q -curvature tensor, and semi-symmetric non-metric connections have been

successfully employed to obtain new geometric characterizations and properties of Kenmotsu and statistical manifolds. Moreover, semi-symmetric non-metric connections have also been investigated in the context of para-Sasakian manifolds and their submanifolds, yielding interesting results on geometric solitons [24]. These recent advances [34, 35, 36, 37, 38] Motivated by these studies, the present work investigates Eisenhart-type problems on Kenmotsu manifolds equipped with a DM semi-symmetric non metric connection, focusing on second-order parallel symmetric and skew-symmetric tensors.

The paper is organized as follows. Section 2 reviews fundamental properties of Kenmotsu manifolds and we introduce the concept of Ricci solitons. Section 3 examines some properties on DM semi-symmetric non-metric connections on Kenmotsu manifolds and we establish that Ricci solitons relative to such connections are necessarily expanding. Section 4 studies second-order parallel symmetric tensors, establishing several significant results, while Section 5 addresses second-order parallel skew-symmetric tensors.

2. Preliminaries

A $(2n + 1)$ -dimensional differentiable manifold M of class C^∞ is said to have a (ϕ, P, η) -structure or almost contact structure if it admits a field ϕ of endomorphisms of the tangent spaces, a vector field P and a 1-form η satisfying

$$\eta(P) = 1 \quad \text{and} \quad \phi^2 = -I + \eta \otimes P, \quad (2.1)$$

where I denotes the identity transformation [5]. From (2.1), it can be easily shown that $\phi P = 0$, $\eta\phi = 0$ and $\text{rank } \phi = 2n$. A Riemannian metric g of type $(0, 2)$ is said to be compatible with the almost contact structure (ϕ, P, η) if the relation

$$g(U, V) = g(\phi U, \phi V) + \eta(U)\eta(V) \quad (2.2)$$

holds, for arbitrary vector fields U and V on M . An almost contact structure (ϕ, P, η) equipped with a compatible Riemannian metric g is known as an almost contact metric structure (ϕ, P, η, g) and the manifold M endowed with almost contact metric structure is called an almost contact metric manifold. If moreover,

$$\nabla_U P = U - \eta(U)P, \quad (2.3)$$

holds for all $U \in \Gamma(TM)$, where $\Gamma(TM)$ denotes the set of all the differentiable vector fields of M , then the manifold M is said to be a Kenmotsu manifold [22]. Here ∇ denotes the Levi-Civita connection of the metric g . For proving our results in next sections, we are going to recall some basic results on Kenmotsu manifolds.

$$(\nabla_U \eta)(Y) = g(U, Y) - \eta(U)\eta(Y), \quad (2.4)$$

$$\eta(R(U, V)W) = \eta(V)g(U, W) - \eta(U)g(V, W), \quad (2.5)$$

$$R(U, V)P = \eta(V)U - \eta(U)V, \quad (2.6)$$

$$R(P, U)V = \eta(V)U - g(U, V)P, \quad (2.7)$$

$$\text{Ric}^g(U, P) = -2n\eta(U). \quad (2.8)$$

A Ricci soliton on a Riemannian manifold is a natural generalization of Einstein metric and it has many applications in physics and other branches of science. A triple (g, U, λ) on a Riemannian manifold (M, g) is said to be a Ricci soliton if it satisfies the condition

$$\frac{1}{2}L_U g + \text{Ric}^g + \lambda g = 0. \quad (2.9)$$

Definition 2.1. Let (M, g) be an n -dimensional Riemannian manifold equipped with a semi symmetric non metric connection ∇ . A triple (g, U, λ) on (M, g) is said to be a Ricci soliton with respect to the connection ∇ if it satisfies the expression

$$L_U g + 2\text{Ric} + 2\lambda g = 0. \quad (2.10)$$

Here L and Ric denote the Lie derivative operator and the Ricci tensor with respect to the connection ∇ , respectively. Here we suppose that Ric is symmetric on M .

3. On DM semi-symmetric non -metric connection

Let M be a $(2n + 1)$ -dimensional Kenmotsu manifold and ∇^g denotes the Levi-Civita connection. We consider a semi-symmetric connection ∇ on M whose torsion tensor T of type $(1, 2)$ satisfies

$$T(U, V) = \eta(V)U - \eta(U)V, \quad (3.1)$$

for all $U, V \in \Gamma(TM)$. A semi-symmetric connection satisfying the relation

$$(\nabla_U g)(V, W) = f(g(P_1, V)g(U, W) + g(P_1, W)g(U, V)) \quad (3.2)$$

is called a semi-symmetric non metric connection.

Diallo and Massamba [26] defined a new semi-symmetric non-metric connection, referred as DM semi-symmetric non-metric connection (briefly DM SSNMC). Let ∇^g be the Levi-Civita connection of a Riemannian manifold (M, g) . Then the DM SSNMC ∇ is defined by

$$\nabla_U V = \nabla_U^g V + g(P, V)U - g(U, V)(P + fP_1), \quad (3.3)$$

for all $U, V \in \Gamma(TM)$.

With the help of equations (2.1), (2.3) and (3.3), we can easily observe that

$$(\nabla_U \eta)(V) = (\nabla_U^g \eta)(V) - \eta(U)\eta(V) + g(U, V) + fg(U, V)g(P, P_1). \quad (3.4)$$

Remark 3.1. If $P = P_1$, we have :

$$(\nabla_U \eta)(V) = (\nabla_U^g \eta)(V) - \eta(U)\eta(V) + (1 + f)g(U, V).$$

If R and R^g are the curvature tensors of ∇ and ∇^g on (M, g) , respectively, and $\xi = P + fP_1$, then R and R^g are related by:

$$\begin{aligned} R(U, V)W &= R^g(U, V)W - \alpha(V, W)U + \alpha(U, W)V \\ &- g(V, W)FU + g(U, W)FV \end{aligned} \quad (3.5)$$

for $U, V, W \in \Gamma(TM)$, where

$$\alpha(U, V) = g(BU, V) = (\nabla_U^g \eta)V - \eta(U)\eta(V), \quad (3.6)$$

is a symmetric tensor of type $(0, 2)$ on M , and

$$FU = \nabla_U^g \xi + g(P, \xi)X - g(U, \xi)\xi \quad (3.7)$$

In consequence of (2.1), (2.4) and (3.4), equation (3.5) takes the form

$$\begin{aligned} R(U, V)W &= R^g(U, V)W - 2g(V, W)U + 2g(U, W)V \\ &+ 2\eta(W)\eta(V)U - 2\eta(W)\eta(U)V \\ &+ 2g(V, W)g(U, \xi)\xi - 2g(U, W)g(V, \xi)\xi \\ &+ g(U, W)g(P, \xi)V - g(V, W)g(P, \xi)U. \end{aligned} \quad (3.8)$$

The contraction of the equation (3.8) along the vector field U gives

$$\begin{aligned} \text{Ric}(V, W) &= \text{Ric}^g(V, W) - 4ng(V, W) + 4n\eta(V)\eta(W) \\ &+ 2g(V, W)g(\xi, \xi) - 2g(\xi, W)g(V, \xi) \\ &- 2ng(V, W)g(P, \xi) \end{aligned} \quad (3.9)$$

for all $V, W \in \Gamma(TM)$

Remark 3.2. if $f = 0$

$$\text{Ric}(V, W) = \text{Ric}^g(V, W) - 2(3n - 1)g(V, W) + 2(2n - 1)\eta(V)\eta(W)$$

Remark 3.3. If $P = P_1$, we get

$$\begin{aligned} \text{Ric}(V, W) &= \text{Ric}^g(V, W) \\ &+ (-6n + 2 - 2nf)g(V, W) \\ &+ (4n - 2)(1 + f)^2g(V, P)g(W, P). \end{aligned}$$

Remark 3.4. If $P = P_1$ and $f = -1$ ($\xi = 0$), we have

$$\text{Ric}(V, W) = \text{Ric}^g(V, W) - 4ng(V, W) + 4n\eta(V)\eta(W) \quad (3.10)$$

which gives by making use of (3.9), we have

$$r = r^g + (2n + 1)(-6n + 2 - 2nf) + (4n - 2)(1 + f)^2, \quad (3.11)$$

where r and r^g denote the scalar curvatures corresponding to the semi-symmetric non-metric connection and the Levi-Civita connection, respectively.

In view of equations (2.1), (2.8) and (3.10), we can find

$$\text{Ric}(V, P) = -2n(1 + f)\eta(V) \quad (3.12)$$

Taking covariant derivative of (3.10) and suppose that $P = P_1$, by making use of (2.1), (2.4), (3.3), (3.4) and (3.8), we get

$$\begin{aligned} (\nabla_U \text{Ric})(V, W) &= (\nabla_U \text{Ric}^g)(V, W) - \eta(V)\text{Ric}^g(U, W) - \eta(W)\text{Ric}^g(U, V) \\ &\quad - 8\left(2n - (1 + f)^2\right)\eta(U)\eta(V)\eta(W) \\ &\quad - 6n\left((\eta(W)g(U, V) + \eta(V)g(U, W))\right) \\ &\quad + f\left(2(1 + f)(1 + f - n) - 4n\right)\left((\eta(W)g(U, V) + \eta(V)g(U, W))\right) \end{aligned} \quad (3.13)$$

In view of (2.1), (2.6), (2.7) and (3.7), we can easily calculate the following:

$$\begin{aligned} R(P, V)W &= (2 + f)\eta(W)V - 2(1 - f^2)g(V, W)P \\ &\quad - 2f(2 + f)\eta(V)\eta(W)P \end{aligned} \quad (3.14)$$

and

$$R(U, V)P = (2 + f)\left(\eta(U)V - \eta(V)U\right). \quad (3.15)$$

According to equation (3.12), the manifold M becomes regular when equipped with the specified geometric structure

Remark 3.5. *In the case where $f = 0$ the expression reduces to the curvature associated with a semi-symmetric metric connection*

$$R(P, V)W = 2\left(\eta(W)V - g(V, W)P\right) \quad (3.16)$$

and

$$R(U, V)P = 2\left(\eta(U)V - \eta(V)U\right). \quad (3.17)$$

Remark 3.6. *In the case where $f = -1$ the expression reduces to the curvature associated with a semi-symmetric non metric connection ($R(U, V)P \neq 0$)*

$$R(P, V)W = \eta(W)V + 2\eta(V)\eta(W)P \quad (3.18)$$

and

$$R(U, V)P = \eta(U)V - \eta(V)U. \quad (3.19)$$

From equations (2.1), (2.3), (3.3) and the definition of Lie derivative, we have

$$(\text{L}_P g) = 4\left(g(U, V) - \eta(U)\eta(V)\right) + f(\eta_1(U)a(V) + \eta_1(V)\eta(U)). \quad (3.20)$$

Remark 3.7. If $P = P_1$, we obtain

$$(L_P g) = 4 \left(g(U, V) - \eta(U)\eta(V) \right) + 2f\eta(U)\eta(V). \quad (3.21)$$

In consequence of equations (2.1), (3.8) and (3.13), equation (3.21) takes the form

$$\begin{aligned} 0 &= 2\text{Ric}^g(V, W) + 2 \left(2(1+f)(1+f-n) - 4n \right) g(V, W) \\ &+ 4 \left(2n - (1+f)^2 \right) \eta(V)\eta(W) + 4 \left(g(V, W) - \eta(V)\eta(W) \right) \\ &+ 2f\eta(V)\eta(W) + 2\lambda g(V, W), \end{aligned} \quad (3.22)$$

We obtained,

$$\begin{aligned} 0 &= 2\text{Ric}^g(U, V) - 8ng(U, V) + 8n\eta(U)\eta(V) \\ &+ 2(1+f)(2-2n+f)g(U, V) - 2(1+f)(2+f)\eta(U)\eta(V) \\ &+ 2(f-2)4\eta(U)\eta(V) + 2(2+\alpha)g(U, V). \end{aligned} \quad (3.23)$$

Setting $V = W = P$ and using equations (2.1) and (3.10), we obtain the result $\lambda = 4n(1+f) - f$, we have

$$\begin{aligned} \lambda(f) &> 0 \quad \text{if} \quad f > -\frac{4n}{4n-1}, \\ \lambda(f) &= 0 \quad \text{if} \quad f = -\frac{4n}{4n-1}, \\ \lambda(f) &< 0 \quad \text{if} \quad f < -\frac{4n}{4n-1}. \end{aligned}$$

Thus, the Ricci soliton with respect to the semi-symmetric non-metric connection on (M, g) may be shrinking, expanding, or steady according to the value of f . Consequently, we state the following theorems.

Theorem 3.8. Let (M, g) be a $(2n+1)$ -dimensional Kenmotsu manifold endowed with a semi symmetric non metric connection ∇ . If (M, g) admits a Ricci soliton (g, V, λ) corresponding to ∇ , then

1. (g, V, λ) is expanding if $f > -\frac{4n}{4n-1}$.
2. (g, V, λ) is steady if $f = -\frac{4n}{4n-1}$.
3. (g, V, λ) is shrinking if $f < -\frac{4n}{4n-1}$.

4. Second order parallel symmetric tensor with respect to a semi-symmetric non metric connection

Definition 4.1. A symmetric tensor α of type $(0, 2)$ on a Riemannian manifold (M, g) of dimension n is said to be a second order parallel tensor with respect to the semi-symmetric non metric connection ∇ if $\nabla\alpha = 0$.

If $\nabla\alpha = 0$ then it can be easily shown that $R(X, Y) \cdot \alpha = 0$. Thus

$$\alpha(R(U, V)W, X) + \alpha(W, R(U, V)X) = 0, \quad (4.1)$$

for arbitrary vector fields U, V, W and X on (M, g) .

Substituting $U = X = P$ into the above equation and making use of (2.1), (3.14) and (3.15), we obtain

$$\begin{aligned} 0 &= (2 + f)\eta(W)\alpha(V, P) - 2(1 - f^2)g(V, W)\alpha(P, P) \\ &\quad - 2f(2 + f)\eta(V)\eta(W)\alpha(P, P) + (2 + f)\alpha(V, P)\alpha(V, W) \\ &\quad - (2 + f)\eta(V)\alpha(P, W) \end{aligned} \quad (4.2)$$

By substituting W with P in the above equation (4.3), we deduce that ,

$$\begin{aligned} 0 &= (2 + f)\alpha(V, P) - 2(1 - f^2)\eta(V)\alpha(P, P) \\ &\quad - 2f(2 + f)\eta(V)\alpha(P, P) + (2 + f)\alpha(V, P) \\ &\quad - (2 + f)\eta(V)\alpha(P, P), \end{aligned} \quad (4.3)$$

and then using (2.1), we obtain

$$\alpha(V, P) = \beta g(V, P)\alpha(P, P), \quad (4.4)$$

where

$$\beta = \frac{4 + 5f}{4 + 2f}.$$

Taking the covariant derivative of (4.4) with respect to the semi-symmetric non-metric connection ∇ in the direction of the vector field U , we obtain that

$$\begin{aligned} \alpha(V, \nabla_U P) &= \beta \left(fg(P_1, V)g(U, P) + fg(P_1, P)g(U, V) + g(V, \nabla_U P) \right) \alpha(P, P) \\ &\quad + 2\beta \alpha(\nabla_U P, P)g(V, P). \end{aligned} \quad (4.5)$$

With the help of equations (2.1), (2.3), (3.3), (4.1) and (4.5), we conclude that

$$\alpha(U, V) = \beta g(U, V)\alpha(P, P). \quad (4.6)$$

The covariant differentiation of equation (4.6) with respect to the semi-symmetric non-metric connection ∇ along an arbitrary vector field on the Riemannian manifold (M, g) , together with relations (2.1), (2.3), and (2.4), shows that $\alpha(P, P)$ is constant.

Consequently, equation (4.6) implies that the second-order symmetric tensor α , parallel with respect to ∇ on a regular Kenmotsu manifold (M, g) , is a constant multiple of the tensor $g + a \otimes a$, where g denotes the Riemannian metric and a is the 1-form associated with the structure.

More precisely, on a regular Kenmotsu manifold, any such parallel tensor α can be written as

$$\alpha = c(g + a \otimes a), \quad (4.7)$$

Theorem 4.2. *A Kenmotsu metric on a $(2n + 1)$ -dimensional regular Kenmotsu manifold M equipped with a semi-symmetric non metric connection ∇ is irreducible. This means, the tangent bundle of M does not admit a decomposition $TM = E_1 \times E_2$ parallel with respect to the connection ∇ of g .*

In preparation for proving several geometric results, we state the following definition.

Definition 4.3. *A Riemannian manifold (M, g) of dimension n is said to be β -symmetric with respect to the semi-symmetric non metric connection ∇ if the tensor field β of type $(0, 2)$ defined on M satisfies the condition $\nabla\beta = 0$.*

If we replace the tensor field β with the Ricci tensor S , then it becomes Ricci symmetric.

Let us suppose that the Kenmotsu manifold M of dimension $(2n + 1)$ equipped with the connection ∇ is α -symmetric, where α is a symmetric tensor of type $(0, 2)$ defined by (3.6). With the help of (2.1), (2.4) and theorem 4.2, we can find that

$$\alpha(U, V) = \alpha(P, P)g(U, V) \quad (4.8)$$

where $\alpha(P, P) = -1$. Thus we can state:

Corollary 4.4. *If a $(2n + 1)$ -dimensional Kenmotsu manifold M equipped with a semi-symmetric non metric connection ∇ is α -symmetric, then the symmetric tensor α is a constant multiple of the metric tensor g .*

In view of (3.5), (3.6) and the above result, we conclude that

$$R(U, V)W = R^g(U, V)W + 2g(V, W)U - 2g(U, W)V, \quad (4.9)$$

which gives

$$R(U, V)P = 3(\eta(V)U - \eta(U)V), \quad (4.10)$$

where equations (2.1) and (2.6) are used. Thus the manifold M equipped with a semi-symmetric non metric connection ∇ is regular.

Corollary 4.5. *Every $(2n + 1)$ -dimensional Kenmotsu manifold M equipped with a semi-symmetric non metric connection ∇ is regular, provided it is α -symmetric.*

Apart from the conformal curvature tensor, concircular curvature tensor plays an important role in differential geometry and mathematical physics (specially in the theory of relativity and cosmology). A tensor field C of type $(1, 3)$ defined as

$$C(U, V)W = R^g(U, V)W - \frac{r^g}{2n(2n + 1)}(g(V, W)U - g(U, W)V), \quad (4.11)$$

for all $U, V, W \in \Gamma(TM)$, is called a concircular curvature tensor [40] of (M, g) . Now the contraction of (4.9) along the vector U gives

$$\text{Ric}(V, W) = \text{Ric}(V, W) + 4ng(V, W), \quad (4.12)$$

which implies

$$r = r^g + 4n(2n + 1) \quad (4.13)$$

If the scalar curvature with respect to the semi-symmetric metric connection ∇ vanishes on (M, g) , then the equations (4.9), (4.12) and (4.13) imply that

$$R = C$$

and therefore we state the following corollary:

Corollary 4.6. *If a $(2n + 1)$ -dimensional Kenmotsu manifold (M, g) endowed with a semi symmetric non metric connection ∇ is α -symmetric and the scalar curvature with respect to ∇ vanishes, then the curvature tensor with respect to the connection ∇ coincide with the concircular curvature tensor of the manifold.*

If we suppose that the curvature tensor with respect to the semi-symmetric metric connection ∇ vanishes on (M, g) , then equation (4.9) reduces to

$$R^g(U, V)W = -(2g(V, W)U - 2g(U, W)V), \quad (4.14)$$

which shows that the manifold is of constant curvature. Thus we can state:

Corollary 4.7. *If a $(2n + 1)$ -dimensional Kenmotsu manifold (M, g)) equipped with a semisymmetric metric connection ∇ is α -symmetric and the curvature tensor with respect to ∇ vanishes, then the manifold is of constant curvature.*

Let us suppose that the Kenmotsu manifold M equipped with a semi-symmetric metric connection ∇ is Ricci-symmetric with respect to the connection ∇ . Then equations (2.1), (2.8) (3.13), and theorem 4.2 give

$$\text{Ric}(U, V) = \text{Ric}(P, P)g(U, V), \quad (4.15)$$

where

$$\text{Ric}(P, P) = -2n(2 + f). \quad (4.16)$$

Let $\{e_1, e_2, e_3, \dots, e_{2n}, e_{2n+1} = P\}$ be an orthonormal basis of the tangent space at a point p of the manifold M . Setting $X = Y = e_i$ in (4.15) and then summing over $i, 1 \leq i \leq 2n + 1$, we find that

$$r = -2n(2n + 1)(2 + f), \quad (4.17)$$

which is constant, where $r = \sum_{i=1}^{2n+1} \text{Ric}(e_i, e_i)$. From equation (4.15), it is clear that the Ricci tensor with respect to the connection ∇ is a constant multiple of the metric tensor g and hence the manifold is Einstein with respect to ∇ . By considering (4.15), equation (3.9) leads to

$$\begin{aligned}\operatorname{Ric}(V, W) &= \operatorname{Ric}^g(V, W) + \left(2(1+f)(1+f-n) - 4n\right)g(V, W) \\ &+ 2\left(2n - (1+f)^2\right)\eta(V)\eta(W),\end{aligned}\quad (4.18)$$

$$\begin{aligned}\operatorname{Ric}^g(U, V) &= -\left((2-2n+f)g(P, \xi) + 2nf\right)g(U, V) \\ &- 4n\eta(V)\eta(W) + (2\eta(V) + f\eta_1(V))g(W, \xi).\end{aligned}\quad (4.19)$$

If suppose that $P = P_1$, one has

$$\begin{aligned}\operatorname{Ric}^g(U, V) &= -2\left((1+f)(1+f-n) + 2nf\right)g(U, V) \\ &- \left(4n - 2(1+f)^2\right)\eta(U)\eta(V),\end{aligned}\quad (4.20)$$

which shows that the manifold under consideration is an η -Einstein manifold with scalars $a = -2\left((1+f)(1+f-n) + 2nf\right)$ and $b = (4n - 2(1+f)^2)$. It is obvious that the scalars a and b on a Kenmotsu manifold (M, g) satisfy the relation $a+b = -2n$. It is well-known that an η -Einstein manifold with either a or b being constant is an Einstein manifold (see [22]). Thus we can state:

Corollary 4.8. *If a $(2n+1)$ -dimensional regular Kenmotsu manifold M endowed with a semi-symmetric non metric connection ∇ is Ricci symmetric with respect to ∇ , then the manifold is Einstein.*

Corollary 4.9. *Let M be a $(2n+1)$ -dimensional Ricci symmetric Kenmotsu manifold endowed with a semi-symmetric non metric connection ∇ , then the scalar curvature with respect to the connection ∇ is constant.*

In view of equations (4.20), we find that

$$(\nabla_U \operatorname{Ric}^g)(V, W) = -b\left(g(U, V)\eta(W) + g(U, W)\eta(V) - 2\eta(U)\eta(V)\eta(W)\right). \quad (4.21)$$

If we suppose that $P = W$ in (4.21), we have

$$(\nabla_U \operatorname{Ric}^g)(V, P) = -b\left(g(U, V) - \eta(U)\eta(V)\right). \quad (4.22)$$

Interchanging U and V in (4.22), we get

$$(\nabla_V \operatorname{Ric}^g)(U, P) = -b\left(g(U, V) - \eta(U)\eta(V)\right). \quad (4.23)$$

From equations (4.22) and (4.23), we observe that

$$(\nabla_U \operatorname{Ric}^g)(V, P) = (\nabla_V \operatorname{Ric}^g)(U, P).$$

Thus the Ricci tensor is of Codazzi type. Hence we state

Corollary 4.10. *If a $(2n+1)$ -dimensional Kenmotsu manifold (M, g) equipped with a semi symmetric non metric connection ∇ is Ricci symmetric with respect to ∇ , then the Ricci tensor of M is of Codazzi type.*

Now we present the following definitions.

Definition 4.11. [15] A non-flat differentiable manifold of dimension $m > 3$ is said to be a weakly symmetric manifold if its non-vanishing curvature tensor R satisfies

$$\begin{aligned} (\nabla_U^g R)(V, W)X &= A(U)R(V, W)X + B(V)R(U, W)X + C(W)R(V, U)X \\ &+ D(X)R(V, W)U + g(R(V, W)X, U)P_2, \end{aligned} \quad (4.24)$$

for all the vector fields $U, V, W, X \in \Gamma(TM)$, where A, B, C and D are 1-forms (not simultaneously zero) on M and ∇^g denotes the Levi-Civita connection of the manifold.

Definition 4.12. [15] A non-flat differentiable manifold of dimension $m > 3$ is said to be a weakly Ricci symmetric manifold if its non-vanishing Ricci tensor Ric satisfies

$$(\nabla_U^g Ric^g)(V, W) = \rho(U)Ric^g(V, W) + \mu(V)Ric^g(U, W) + \nu(W)Ric^g(U, V), \quad (4.25)$$

for all the vector fields $U, V, W, X \in \Gamma(TM)$, where ρ, μ and ν are 1-forms (not simultaneously zero) on M and ∇^g denotes the Levi-Civita connection of the manifold.

We consider a weakly symmetric and a weakly Ricci symmetric Kenmotsu manifolds, respectively, and prove the following results.

Theorem 4.13. Let (g, P, λ) be a Ricci soliton on a $(2n + 1)$ -dimensional weakly symmetric Kenmotsu manifold (M, g) admitting a semi-symmetric non metric connection ∇ . If $\mathcal{L}_P g + 2Ric$ is parallel with respect to the connection ∇ , then (g, P, λ) on (M, g) is.

1. (g, P, λ) is expanding if $f > -\frac{4n}{2n-1}$.
2. (g, P, λ) is steady if $f = -\frac{4n}{2n-1}$.
3. (g, P, λ) is shrinking if $f < -\frac{4n}{2n-1}$.

Proof. Let (M, g) be a weakly symmetric Kenmotsu manifold of dimension $(2n + 1)$. It is noticed in [15] that

$$\begin{aligned} Ric^g(V, W) &= \frac{1}{1 + D(P)} \left((2nA(V) + B(V) + \rho(V))\eta(W) \right. \\ &\quad \left. - (2n + \rho(\xi))g(V, W) + (2nC(W) - B(W))\eta(V) \right) \end{aligned} \quad (4.26)$$

and

$$A(P) + C(P) + D(P) = 0, \quad (4.27)$$

under the assumptions $1 + D(\xi) \neq 0$ and $\rho(X) = g(X, P_2)$, and by using (3.9), equation (4.26) can be expressed as

$$Ric(V, W) = \frac{1}{1 + D(P)} \left((2nA(V) + B(V) + \rho(V))\eta(W) - (2n + \rho(P))g(V, W) \right)$$

$$\begin{aligned}
& + \left(2nC(W) - B(W) \right) \eta(V) \Big) - 4ng(V, W) + 4n\eta(V)\eta(W) \\
& + \left(2 - 2n + f \right) g(V, W)g(P, \xi) - \left(2\eta(V) + f\eta_1(V) \right) g(W, \xi).
\end{aligned} \tag{4.28}$$

Let define $\theta(V, W) = (L_P g + 2\text{Ric})(V, W)$. Then equations (4.28) and (3.20) give

$$\begin{aligned}
\theta(V, W) &= \frac{2}{1 + D(P)} \left(\left(2nA(V) + B(V) + \rho(V) \right) \eta(W) - (2n + \rho(P))g(V, W) \right. \\
&+ \left. \left(2nC(W) - B(W) \right) \eta(V) \right) - 8ng(V, W) + 8n\eta(V)\eta(W) \\
&+ 2 \left(2 - 2n + f \right) g(V, W)g(P, \xi) - 2 \left(2\eta(V) + f\eta_1(V) \right) g(W, \xi) \\
&+ 4 \left(g(V, W) - \eta(V)\eta(W) \right) + f(\eta_1(V)a(W) + \eta_1(W)\eta(V).
\end{aligned} \tag{4.29}$$

Suppose that $P = P_1$; then

$$\begin{aligned}
\theta(V, W) &= \frac{2}{1 + D(P)} \left(\left(2nA(V) + B(V) + \rho(V) \right) \eta(W) - (2n + \rho(P))g(V, W) \right. \\
&+ \left. \left(2nC(W) - B(W) \right) \eta(V) \right) - 8ng(V, W) + 8n\eta(V)\eta(W) \\
&+ 2 \left(2 - 2n + f \right) \left(1 + f \right) g(V, W) - 2 \left(2 + f \right) (1 + f) \eta(V)\eta(W) \\
&+ 4 \left(g(V, W) - \eta(V)\eta(W) \right) + 2f\eta(V)\eta(W).
\end{aligned} \tag{4.30}$$

If $L_P g + 2\text{Ric}$ is parallel with respect to the semi-symmetric metric connection ∇ , from Theorem 4.2 we conclude that

$$L_P g + 2S = \theta(P, P)g. \tag{4.31}$$

From equations (2.10) and (4.31), it follows that

$$\theta(P, P) = -2\lambda. \tag{4.23}$$

By taking $V = W = P$ in (4.30) and applying (2.1), we deduce that

$$\theta(P, P) = \frac{4n}{1 + D(P)} \left(A(P) + C(P) - 1 \right) - 4n(1 + f) + 2f. \tag{4.32}$$

As a consequence of (4.27), the above equation can be written as

$$\lambda = 2n(2 + f) - f. \tag{4.33}$$

We obtain

$$\begin{cases} \lambda(f) > 0 & \text{if } f > -\frac{4n}{2n-1}, \\ \lambda(f) = 0 & \text{if } f = -\frac{4n}{2n-1}, \\ \lambda(f) < 0 & \text{if } f < -\frac{4n}{2n-1}. \end{cases}$$

Hence, the sign of λ depends on f . In particular, if $f > -\frac{4n}{2n-1}$, then $\lambda > 0$, and the Ricci soliton (g, ξ, λ) on (M, g) is expanding. If $f = -\frac{4n}{2n-1}$, then $\lambda = 0$ and the Ricci soliton is steady. If $f < -\frac{4n}{2n-1}$, then $\lambda < 0$ and the Ricci soliton is shrinking. This completes the proof. $\square$

Theorem 4.14. *Let (M, g) be a $(2n + 1)$ -dimensional weakly Ricci symmetric Kenmotsu manifold endowed with a semi-symmetric metric connection $\bar{\nabla}$. Then the Ricci soliton (g, ξ, λ) on (M, g) is classified according to the function f as follows:*

1. **Expanding** if $f > -\frac{4n}{n-1}$.
2. **Steady** if $f = -\frac{4n}{n-1}$.
3. **Shrinking** if $f < -\frac{4n}{n-1}$.

Proof. Let (M, g) be a $(2n + 1)$ -dimensional weakly Ricci symmetric Kenmotsu manifold equipped with a semi-symmetric non metric connection ∇ . In [15], author proved that a weakly Ricci symmetric Kenmotsu manifold satisfies

$$\text{Ric}^g(U, V) = \frac{2n}{1 + v(P)} \left(\rho(U)\eta(V) + \mu(V)\eta(U) - g(U, V) \right). \quad (4.34)$$

and

$$\rho(P) + \mu(P) + v(P) = 0 \quad (4.35)$$

In view of (3.9), equation (4.34) reduces to

$$\begin{aligned} \text{Ric}(V, W) &= \frac{2n}{1 + v(P)} \left(\rho(U)\eta(V) + \mu(V)\eta(U) - g(U, V) \right) \\ &- 4ng(U, V) + 4n\eta(U)\eta(V) \\ &+ (2 - 2n + f)g(U, V)g(P, \xi) \\ &- (2\eta(U) + f\eta_1(U))g(V, \xi) \end{aligned} \quad (4.36)$$

In the case where $P = P_1$, we have

$$\begin{aligned} \text{Ric}(U, V) &= \frac{2n}{1 + v(P)} \{ \rho(U)\eta(V) + \mu(U)\eta(V) - g(U, V) \} \\ &- 4ng(V, W) + 4n\eta(V)\eta(W) \\ &+ (1 + f)(2 - 2n + f)g(V, W) \\ &- (1 + f)(2 + f)\eta(V)\eta(W) \end{aligned} \quad (4.37)$$

Equations (4.31) and (4.37) give

$$\begin{aligned} \alpha(U, V) &= L_P g(U, V) + 2S(U, V) \\ &= \frac{4n}{1 + v(P)} \{ \rho(U)\eta(V) + \mu(V)\eta(U) - g(U, V) \} \end{aligned}$$

$$\begin{aligned}
& - 8ng(U, V) + 8n\eta(U)\eta(V) \\
& + 2(1+f)(2-2n+f)g(U, V) \\
& - 2(1+f)(2+f)\eta(U)\eta(V) \\
& + 4g(U, V) + 2(f-2)\eta(U)\eta(V)
\end{aligned} \tag{4.38}$$

We suppose that α is symmetric with respect to the semi-symmetric non metric connection ∇ . Therefore by Theorem 4.2 and equation (2.10), we obtain

$$\alpha(U, V) = \alpha(\xi, \xi)g(U, V) \iff -2\lambda = \alpha(\xi, \xi).$$

Replacing U and V with P in (4.38) and using equations (2.1) and (4.26), we conclude that

$$\lambda(f) = 4n + (n-1)f, \quad n > 0.$$

We obtain

$$\begin{cases} \lambda(f) > 0 & \text{if } f > -\frac{4n}{n-1}, \\ \lambda(f) = 0 & \text{if } f = -\frac{4n}{n-1}, \\ \lambda(f) < 0 & \text{if } f < -\frac{4n}{n-1}. \end{cases}$$

Hence, the sign of λ depends on f . In particular, if $f > -\frac{4n}{n-1}$, then $\lambda > 0$, and the Ricci soliton (g, ξ, λ) on (M, g) is expanding. If $f = -\frac{4n}{n-1}$, then $\lambda = 0$ and the Ricci soliton is steady. If $f < -\frac{4n}{n-1}$, then $\lambda < 0$ and the Ricci soliton is shrinking. $\square$

Let L_V be parallel, and let the regular Kenmotsu manifold be Ricci symmetric with respect to the connection ∇ , i.e., $\nabla \text{Ric} = 0$, where L_V denotes the Lie derivative of g along the vector field V with respect to ∇ . We distinguish two cases for the vector field V : first, $V \in \text{Span}\{\xi\}$, and second, $V \perp P$. From an analytical point of view, the second case is more involved; hence, we restrict our attention to the first case, i.e., $V = P$. In a 3-dimensional Kenmotsu manifold, the Ricci tensor Ric takes the form

$$\text{Ric}^g(U, V) = \left(\frac{r}{2} + 1\right)g(U, V) - \left(\frac{r}{2} + 3\right)\eta(U)\eta(V). \tag{4.39}$$

For all $U, V \in \Gamma(TM)$, we consider $\alpha(U, V) = (L_P g + 2\text{Ric})(U, V)$. Then, using equations (2.1), (3.18), (3.19), (3.10) and (4.39), we obtain:

$$\begin{aligned}
(2\text{Ric} + L_P g)(U, V) &= (r^g + 2)g(U, V) - (r^g + 6)\eta(U)\eta(V) \\
&+ (-12n + 8 - 4nf)g(U, V) \\
&+ 2(4n - 2)(1 + f)^2\eta(U)\eta(V) \\
&+ (2f - 4)\eta(U)\eta(V).
\end{aligned}$$

We find

$$\begin{aligned}
(2\text{Ric} + L_P g)(U, V) &= r(g(U, V) - \eta(U)\eta(V)) \\
&+ A(n, f)g(U, V) + B(n, f)\eta(U)\eta(V),
\end{aligned}$$

where

$$\begin{aligned} A(n, f) &= -(2n+1)(-6n+2-2nf) - (4n-2)(1+f)^2 - 12n+10-4nf, \\ B(n, f) &= (2n+1)(-6n+2-2nf) + (12n-6)(1+f)^2 + 2f-10. \end{aligned}$$

Taking $f = -1$

$$\alpha(U, V) = (r+8)g(U, V) - (r-18)\eta(U)\eta(V) \quad (4.40)$$

The covariant derivative of (4.40) with respect to the semi-symmetric non metric connection ∇ along the vector field Z gives

$$(\nabla_Z \alpha)(U, V) = dr(Z)g(\phi U, \phi V) - 2(r-18)\left\{g(\phi U, \phi Z)\eta(V) + g(\phi V, \phi Z)\eta(U)\right\}. \quad (4.41)$$

Let us suppose that α is a symmetric parallel tensor, and take $Z = P$ with $U = V \in (\text{Span}\{\xi\})^\perp$. Then equation (4.41) reduces to $dr(P) = 0$.

which gives $r = \text{constant}$. Thus from equations (2.10), (4.39), (4.40) and Theorem 4.2, we can conclude that

$$\alpha(X, Y) = -2\lambda g(X, Y),$$

where $\lambda = 4 > 0$. Therefore, we can conclude the following result:

Corollary 4.15. *Let M^3 be a 3-dimensional Kenmotsu manifold endowed with a semi-symmetric non-metric connection ∇ . Assume that M^3 is Ricci symmetric and that $\mathcal{L}_P g$ is parallel. Then the Ricci soliton (g, P, λ) on M^3 is expanding.*

Definition 4.16. *A vector field $U \in \Gamma(TM)$ on a Riemannian manifold (M, g) is called an affine Killing vector field if*

$$\nabla(L_U g) = 0. \quad (4.42)$$

Analogously to the previous definition, we introduce the following:

Definition 4.17. *[15] A vector field $U \in \Gamma(TM)$ on a $(2n+1)$ -dimensional Kenmotsu manifold endowed with a semi-symmetric non metric connection ∇ is called an affine Killing vector field with respect to ∇ if*

$$\nabla(L_U g) = 0.$$

With the help of 4.2 and above definition, we have

$$(L_U g)(V, W) = cg(V, W), \quad (4.43)$$

where $c = -2g(L_U P, P)$.

Using equation (3.11), we can easily compute that

$$(L_U Q)(P) = 0, \quad (4.44)$$

and hence

$$(\mathcal{L}_U \text{Ric})(P, P) = 0. \quad (4.45)$$

On the other hand,

$$(\mathcal{L}_U \text{Ric})(P, P) = -2\text{Ric}(\mathcal{L}_U P, P) = 4n(1 + f) g(\mathcal{L}_U P, P) = 0, \quad (4.46)$$

which implies

$$g(\mathcal{L}_U P, P) = 0. \quad (4.47)$$

It follows that

$$(\mathcal{L}_U g)(P, \xi) = -2 g(\mathcal{L}_U P, P) = 0, \quad (4.48)$$

and consequently,

$$(\mathcal{L}_U g)(V, W) = 0. \quad (4.49)$$

Thus, the vector field U is a Killing vector field. Hence, we obtain the following result:

Corollary 4.18. *An affine Killing vector field with respect to the semi-symmetric non metric connection ∇ on a $(2n + 1)$ -dimensional regular Kenmotsu manifold (M, g) equipped with ∇ is a Killing vector field.*

5. Second order parallel skew-symmetric tensor with respect to semi-symmetric non metric connection

In this section, we investigate the properties of a second-order skew-symmetric tensor that is parallel with respect to a semi-symmetric non-metric connection ∇ on a regular Kenmotsu manifold. More precisely, we assume that α is a skew-symmetric tensor field of type $(0, 2)$ which is parallel with respect to the semi-symmetric metric connection ∇ , that is,

$$\alpha(U, V) = -\alpha(V, U) \quad \text{and} \quad \nabla \alpha = 0. \quad (5.1)$$

By assuming that $\nabla \alpha = 0$, we obtain $R(X, Y) \cdot \alpha = 0$, and consequently, it follows that

$$\alpha(R(U, V)W, X) + \alpha(W, R(U, V)X) = 0 \quad (5.2)$$

for arbitrary vector fields U, V, W and X on (M, g) . By taking $V = X = P$ in the previous relation and applying (2.1), (3.18), and (3.19), we deduce that

$$\alpha(U, W) = \eta(W)\alpha(U, P) + 2\eta(U)\eta(W)\alpha(P, P) - \eta(U)\alpha(W, P). \quad (5.3)$$

Let A be a $(1, 1)$ -tensor field metrically associated with α , that is,

$$\alpha(U, V) = g(AU, V). \quad (5.4)$$

Combining equations (5.2) and (5.4), we deduce that

$$AU = g(U, AP)P - \eta(U)AP + 2\eta(U)g(AP, P)P. \quad (5.5)$$

Since α is parallel, it follows that the associated $(1, 1)$ -tensor field A is also parallel. Consequently, we obtain

$$\nabla_U(AP) = -2g(AP, U)P, \quad (5.6)$$

where we have used equations (2.1), (2.3), (3.3), and (5.3). In addition, by taking into account (2.1) and (5.3), we obtain

$$g(AU, P) = \alpha(U, P) = g(U, AP) + \eta(U)g(AP, P) \quad (5.7)$$

By setting $U = P$ in (5.7), we obtain

$$g(AP, P) = 0. \quad (5.8)$$

It follows directly from the above equation that

$$g(\nabla_U(AP), AP) = 0, \quad (5.9)$$

which implies that $\|AP\|$ is constant on M . Moreover, the above equations show that

$$A^2P = -\|AP\|^2P. \quad (5.10)$$

Taking the covariant derivative of (5.10) with respect to the semi-symmetric non metric connection ∇ in the direction of a vector field U , and using equations (2.1), (2.3), and (2.10), we obtain

$$A^2P = -\|AP\|^2P. \quad (5.11)$$

Assume now that $\|AP\|^2 \neq 0$. Then the endomorphism $J = \frac{1}{\|AP\|}A$ defines an almost complex structure on M . Moreover, the pair (J, g) induces a Kähler structure on M . Consequently, the fundamental 2-form satisfies

$$g(JX, Y) = \lambda \alpha(X, Y), \quad \text{where } \lambda = \frac{1}{\|AP\|} \text{ is a constant.} \quad (5.12)$$

However, since α satisfies relation (5.1), it is degenerate, which leads to a contradiction. Hence, we must have $\|A\xi\| = 0$, and therefore $\alpha = 0$ on M .

Theorem 5.1. *Let (M, g) be a $(2n + 1)$ -dimensional regular Kenmotsu manifold equipped with a semi-symmetric non metric connection ∇ . Then there does not exist a non-zero parallel skew-symmetric tensor field of type $(0, 2)$ on M .*

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