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On the locally compact vector groups

Sur les groupes vectoriels localement compacts

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ABSTRACT. This paper presents a necessary and sufficient condition for a topological vector group to be locally compact. We also introduce several sufficient conditions that ensure the local compactness of topological vector groups. Furthermore, we establish a sufficient condition for a topological vector group to be first countable.

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1. Introduction

A topological vector group (or TVG) is defined as a real or complex vector space X equipped with a group topology such that scalar multiplication by any fixed scalar λ (from the scalar field) is continuous. A locally convex vector group is a TVG where the local base at zero consists of convex sets. Similarly, a locally compact vector group is a topological vector group that also possesses the property of local compactness.

The concept of topological vector groups was initially introduced by Raikov in [3] and further explored by Kenderov in [10]. For foundational properties and the motivations behind studying topological vector groups, refer to [13].

It's worth noting that the condition of a group topology on a vector space being locally balanced inherently guarantees the continuity of the mapping $x \to \lambda x$.

Throughout this paper, we assume that every TVG is both locally balanced and Hausdorff.

In Section 2, we establish a necessary and sufficient condition for a topological vector group to be locally compact. While topological vector groups generalize topological vector spaces, we often leverage a well-known property of topological vector spaces: a topological vector space is locally compact if and only if it is finite-dimensional. Additionally, our research presents several sufficient conditions for a topological vector group to achieve local compactness. A fundamental condition explored in this paper is the existence of a bounded open neighborhood at the origin. Finally, we introduce the F_A-topology on the space of real-valued continuous functions on X. This topology is a locally balanced group topology, and we investigate the specific conditions under which it exhibits local compactness.

For fundamental definitions and results concerning topological groups and local compactness, readers are directed to [1], [4], and [6]. Our terminology for topological vector spaces aligns with that used in [14]. We denote a local base of a topological vector group (X, τ) as $\mathcal{N}(0)$, where elements of $\mathcal{N}(0)$ are symmetric and balanced. We recall that in a topological vector group (X, τ), a bounded set is a subset B of X such that for every neighborhood U of 0 in X there is a positive real number r such that $B \subseteq rU = \{rx : x \in U\}$.

Remark 1.1. Given that in this work we assumed that every TVG is locally balanced, the preceding definition of bounded set implies the following one: for every neighborhood U of 0 in X there is a positive real number r such that $B \subseteq tU$ for all $t > r$.

Definition 1.2. ([2]) Let τ be a group topology on a real or complex vector space X. We call $a \in X$ an *absorbed* element, if for every τ-neighborhood U at zero, there is positive real number r such that $a \in rU = \{rx : x \in U\}$. We denote the set of all absorbed elements of X by $A(X)$.

Theorem 1.3. *([2]) If E is a topological vector group, then $A(X)$ is a closed linear subspace of E.*

Remark 1.4. Although the authors in [2] specifically assumed TVGs to be real vector spaces, the proof of the preceding theorem is independent of the scalar field, applying equally to real or complex fields.

2. Locally compact vector groups

It is well-known that every locally bounded topological vector space has a countable local base. The following Theorem is a generalization of this statement.

Theorem 2.1. *If U is a bounded open neighborhood of zero in a topological vector group (X, τ), then $span(U) = A(X)$. Furthermore, in this case (X, τ) has a countable local base. In particular, if U is also convex, then (X, τ) is locally convex.*

Proof. It is easy to verify that $A(X)$ is contained in $span(U)$. Suppose that V is an arbitrary open neighborhood of the origin in X and $0 \neq x \in span(U)$. Then $x = \Sigma_{i=1}^{m} r_i u_i$ where $u_i \in U$ and $0 \neq r_i \in \mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$)for every i. Since U is balanced and $r_i = |r_i| \frac{r_i}{|r_i|}$, we can replace u_i by $\frac{r_i}{|r_i|} u_i$ for each i. We can also suppose that $r_i > 0$. Choose another balanced open neighborhood W at zero with $\overbrace{W + W + ... + W}^{m \ times} \subseteq U$. By hypothesis, we have $U \subseteq tW$ for some positive real number $t > 1$. Put

$$M = \max(1, r_1, r_2, ..., r_m)$$

and so $\frac{1}{t} U \subseteq W$. We have also $\frac{r_i}{M} \leq 1$ for every $i = 1, 2, ..., m$. Consequently every $\frac{1}{M} r_i u_i$ is in U, which means that

$$\frac{1}{Mt} x = \frac{1}{Mt} \sum_{i=1}^{m} r_i u_i \in \overbrace{W + W + ... + W}^{m \ times} \subseteq U.$$

Since U is bounded, we can find $c > 0$ such that $U \subseteq cV$. It follows that

$$x \in MtU \subseteq (Mtc)V.$$

The second assertion follows from the fact that $\left\{\frac{1}{n} U\right\}_{n \in \mathbb{N}}$ is a local base at zero. Also the convexity of U implies the convexity of $\frac{1}{n} U$ for each n. $\square$

Remark 2.2. • The preceding Theorem shows that the linear span of all bounded open neighborhood of 0 are identical and equal to $A(X)$.

•• The previous proof follows that $span(U) = \cup_{r>0} rU$ for every bounded open neighborhood of X. It is valuable to note that the condition of boundedness can be replaced by convexity. In other words, if U is a convex open neighborhood of zero in (X, τ), then $span(U) = \cup_{r>0} rU$. To prove this, suppose that U is a convex open neighborhood of $0 \in X$. Let x, u_i and r_i be the same as in the proof of the previous theorem. Put

$$N = r_1 + r_2 + ... + r_m.$$

Since $\Sigma_{i=1}^{m} \frac{1}{N} = 1$, we have

$$\frac{1}{N} x = \Sigma_{i=1}^{m} \frac{1}{N} r_i u_i \in U$$

or, equivalently that $x \in NU$.

Corollary 2.3. *If* (X, τ) *is a locally convex TVG, then*

$$\cap_{U \in \mathcal{N}(0)} (span(U)) = A(X)$$

where every element of $\mathcal{N}(0)$ *assumed to be convex.*

Proof. By the previous remark we have $span(U) = \cup_{r>0} rU$ for all $U \in \mathcal{N}(0)$. Then, obviously $\cap_{U \in \mathcal{N}(0)} (span(U)) \subseteq A(X)$. If $x \in A(X)$, then $x \in \cup_{r>0} rU = span(U)$ for every $U \in \mathcal{N}(0)$, which completes the proof. ☐

Proposition 2.4. *Let* (X, τ) *be a TVG with a bounded set* $U \in \mathcal{N}(0)$. *Then* (X, τ) *is locally convex if and only if* $A(X)$ *is normable with respect to the subspace topology.*

Proof. By [2], Corollary 4.16, $A(X)$ is a topological vector space. It is well-known that every TVS is normable if and only if it is locally convex and locally bounded (see [14]). If X is locally bounded and locally convex, then so is $A(X)$, which implies that $A(X)$ is normable. Conversely, if $A(X)$ is normable, then it is locally bounded and locally convex. Since $U \in \mathcal{N}(0)$ is bounded, so by Theorem 2.1 we have $\cup_{r>0} rU = A(X)$. Given that the set $\left\{\frac{1}{n} U\right\}_{n \in \mathbb{N}}$ is a local base for $A(X)$, we can suppose the open sets $\frac{1}{n} U$ are convex. Since $\left\{\frac{1}{n} U\right\}_{n \in \mathbb{N}}$ is also a local base at zero for X, the result follows. ☐

Fact: Every closed subgroup of a locally compact abelian group is locally compact.

Theorem 2.5. *Let* (V, τ) *be a topological vector group. Suppose there exists a bounded set* $U \in \mathcal{N}(0)$. *Then* (V, τ) *is locally compact if and only if the linear span of U is finite dimensional.*

Proof. Assume that $X = span(U)$ has finite dimension. By Theorem 2.1 we have $X = A(V)$, which implies that for every neighborhood B of zero in V, $B \cap X$ is an absorbing set in X. It follows that the subspace topology κ on X induced by τ is locally balanced, locally absorbing group topology. Hence (X, κ) is a finite dimensional topological vector space. Thus (X, κ) is locally compact and complete ([14]). Consequently $\bar{U}$ is also compact in (V, τ). Conversely Suppose that (V, τ) is locally compact. Apply Theorem 1.3 and Theorem 2.1 to conclude that X is closed. Now, the previous fact implies that X is locally compact. We also see that X is locally absorbing and locally balanced. It follows that X is a locally compact topological vector space with respect to the subspace topology and so it is finite dimensional. ☐

The following results follow directly from Theorems 2.1 and 2.5.

Corollary 2.6. *If X is a topological vector group with a bounded neighborhood of zero, then:*
(i) $A(X)$ is closed and open,
(ii) $x \in A(X)$ if and only if $\frac{1}{n}x \to 0$ as $n \to \infty$.

Corollary 2.7. *If X, Y are topological vector groups and $f : X \to Y$ is a continuous linear mapping, then $f(A(X)) \subseteq A(Y)$.*

Proof. By Corollary 2.6, we have $\frac{1}{n}f(x) = f(\frac{1}{n}x) \to 0$ for each $x \in A(X)$. ☐

Corollary 2.8. *If X is a locally compact vector group, then so is the quotient space $\frac{X}{A(X)}$. Furthermore, if U is a bounded neighborhood of zero, then the quotient space $\frac{X}{A(X)}$ is discrete.*

Proof. Since $A(X)$ is closed, $\frac{X}{A(X)}$ is Hausdorff. Local compactness of $\frac{X}{A(X)}$ is an elementary property of locally compact groups. Also $[U] = U + A(X)$ is balanced in the quotient topology for all balanced neighborhoods U at $zero$. ☐

Corollary 2.9. *If X is a locally compact vector group, then $A(X)$ has finite dimension.*

Proof. Since $A(X)$ is closed in X, it is itself locally compact. Moreover, $A(X)$ is a topological vector space, and it is a standard result that a locally compact topological vector space is finite-dimensional. Hence, $A(X)$ is finite dimensional. ☐

Theorem 2.10. *If X is a finite dimensional topological vector group, then X is first countable and locally compact.*

Proof. Assume that (X, τ) is non-discrete finite dimensional complex topological vector group . It is enough to show that there is a bounded open neighborhood of zero. The proof will be by induction. If X has one dimension, it is easy to see that τ is discrete or equal to the ordinary topology. Assume that it is fulfilled for all dimensions less or equal than n. Let X be $n + 1$ dimensional with the basis $\{e_i\}_{i=1}^{n+1}$. We have $X = \mathbb{C}e_1 \oplus Y$ where $Y = span(\{e_i\}_{i=2}^{n+1})$. Since Y is closed, it is an n dimensional topological vector subgroup of X. So there is a balanced bounded open neighborhood U of zero in Y with respect to the subspace topology. Therefore there is a balanced open set V in (X, τ) such that $U = Y \cap V$. If $\{0\} \oplus U$ is open in X, then we can suppose $V = \{0\} \oplus U$ which is bounded and there is nothing left to prove. If $\{re_1\} \oplus U \subseteq V$ for some $0 \neq r \in \mathbb{C}$, then $Ce_1 \oplus U \subseteq V$, where $C = \{c \in \mathbb{C}, |c| \leq |r|\}$. Since X is Hausdorff, there is a balanced neighborhood $W \subseteq V$ at zero such that $|r|\, e_1 \notin W \subseteq Ce_1 \oplus U$. We see that W is bounded. Consequently $\left\{\frac{1}{n}W\right\}_{n\in\mathbb{N}}$ is a local base at zero. Locally compactness of X follows immediately from the Theorems 2.1 and 2.5 ☐

Corollary 2.11. *If (X, τ) is a finite dimensional topological vector group with $A(X) = \{0\}$, then τ is discrete.*

Proof. As shown in the proof of Theorem 2.10, $\mathcal{N}(0)$ contains a bounded element U. By Theorem 2.1, $span(U) = A(X)$. But $A(X) = \{0\}$, which completes the proof. ☐

Proposition 2.12. *Let X be a topological vector group with a bounded neighborhood of zero. Then $x \in A(X)$ if and only if the line segment $\{rx : r \in [0, 1]\}$ is connected.*

Proof. Assume that $x \notin A(X)$ and put $S = \{rx : r \in [0,1]\}$. The Corollary 2.6, (i) implies that $A(X)$ is a connected component of X. We have

$$S = (A(X) \cap S) \cup (A(X)^c \cap S).$$

Since $0 \in A(X)$, the sets $A(X) \cap S$ and $A(X)^c \cap S$ are nonempty, which means that S is not connected. The converse is clear. ☐

Proposition 2.13. *If X is a connected locally convex, locally compact vector group, then X is a finite dimensional topological vector space.*

Proof. By [1], Corollary 3.1.13, the connectedness of X implies that X is algebraically generated by every open neighborhood of zero. Let $U \in \mathcal{N}(0)$ be convex and $0 \neq x \in X$. Then $x = \sum_{i=1}^{n} r_i u_i$, where $r_i \in \mathbb{F}$ and $0 \neq u_i \in U$ for all i. Since U is symmetric and balanced, we can suppose that $0 < r_i$ for every i. Since U is convex, we have $\frac{1}{\sum_{i=1}^{n} r_i} . \sum_{i=1}^{n} r_i u_i \in U$. Therefore

$$x = \sum_{i=1}^{n} r_i \Big(\frac{1}{\sum_{i=1}^{n} r_i} . \sum_{i=1}^{n} r_i u_i \Big)$$

which means that U is absorbing, and so $A(X) = X$. It follows that X is a finite dimensional topological vector space. ☐

Example 2.14. Assume that X is a vector space and $F \subsetneq X$ is a finite dimensional linear subspace. Let τ be the ordinary topology on F with $\mathcal{B}$ a local base at zero. Suppose that κ is the topology on X for which the collection $x + \mathcal{B} = \{x + U : U \in \mathcal{B}\}$ is a local base at $x \in X$. It is easy to verify that (X, κ) is a locally compact, locally convex vector group.

Example 2.15. Let $\mathcal{S}$ be a Hausdorff topological space and $A \subseteq \mathcal{S}$ be nonempty. Let $C(\mathcal{S})$ be the linear space of all real valued continuous functions on $\mathcal{S}$ and $C^+ = \{f \in C(\mathcal{S}) : 0 \leq f\}$. Put

$$F_A = \left\{ f \in C^+ : f(A) \subseteq (r, +\infty),\ \mathit{for\ some}\ r > 0 \right\}.$$

F_A induces a Hausdorff, locally balanced group topology on $C(\mathcal{S})$ as follows.

At first we give some of the elementary properties of the set F_A.

1) If $f, g \in F_A$, then $\inf\{f, g\} \in F_A$.

2) If $f \in F_A$ and $f \leq g$, then $g \in F_A$.

3) F_A closed under addition and positive scalar multiplication.

For every $u \in F_A$, we define

$$U_u = \{f \in C(\mathcal{S}) : u - |f| \in F_A\}$$

where $|f|$ is the absolute value of f. We show that the collection

$$\mathcal{B} = \{g + U_u : g \in C(\mathcal{S}), u \in F_A\}$$

is a base of a locally balanced group topology on $C(\mathcal{S})$. Suppose that $g_1, g_2 \in C(\mathcal{S})$, $u, v \in F_A$ and $z \in (g_1 + U_u) \cap (g_2 + U_v)$. Then there are $h_1 \in U_u, h_2 \in U_v$ such that $z = g_1 + h_1 = g_2 + h_2$. We have also $u - |h_1|, v - |h_2| \in F_A$. Set

$$w = \inf\{u - |h_1|, v - |h_2|\}.$$

We show that

$$z+U_w \subseteq (g_1+U_u)\cap(g_2+U_v).$$

If $l \in U_w$, then we have

$$z+l = g_1+h_1+l = g_2+h_2+l.$$

Also

$$u-|h_1+l| \geq u-(|h_1|+|l|) = (u-|h_1|)-|l| \geq w-|l| \in F_A.$$

It follows that $u-|h_1+l| \in F_A$, which means that $h_1+l \in U_u$. A similar argument implies that $h_2+l \in U_v$. Consequently

$$z+U_w \subseteq (g_1+U_u)\cap(g_2+U_v).$$

We see that U_u is symmetric for every $u \in F_A$. Also it is easy to see that U_u is balanced for each $u \in F_A$. Now, we show that

$$U_{\frac{1}{2}u}+U_{\frac{1}{2}u} \subseteq U_u$$

for each $u \in F_A$. If $f,g \in U_{\frac{1}{2}u}$ then

$$u-|f+g| \geq u-(|f|+|g|) = \frac{1}{2}u-|f|+\frac{1}{2}u-|g| \in F_A.$$

Question: Is the F_A-topology on $C(\mathcal{S})$ (explained above), locally compact?

The following theorem holds.

Theorem 2.16. *If A is an open, finite subset of $\mathcal{S}$, then F_A-topology on $C(\mathcal{S})$ is locally compact and locally convex.*

Proof. We show that the open set U_u is bounded, where $u=\chi_A$ is the characteristic function. If $v \in F_A$, then there is a positive number r such that $r=\inf\{v(x) : x \in A\}$. For a positive number $c<r$, we claim that $cU_u = U_{cu} \subseteq U_v$. We have $u-|f| \in F_A$, if and only if $cu-|cf| \in F_A$, which means that $cU_u = U_{cu}$. If $x \in \mathcal{S}\setminus A$, then $cu(x)=u(x)=0 \leq v(x)$. If $x \in A$, then $cu(x)=c<r\leq r(x)$. It follows that $U_{cu} \subseteq U_v$. Since every element of U_u is 0-valued on $\mathcal{S}\setminus A$ and A is a finite set, so $span(U_u)$ has finite dimension. Now, the result follows from Theorem 2.5. Now, we let $f,g \in U_u$ for some $u \in F_A$ and $0<t<1$. Therefore $t(u-|f|)+(1-t)(u-|g|) \in F_A$. We have also

$$t(u-|f|)+(1-t)(u-|g|) \leq u-(t\,|f|+(1-t)\,|g|) \leq u-|tf+(1-t)g|\,.$$

It follows that $tf+(1-t)g \in U_u$. ☐

The F_A-topology given above inspired by positive filter and link topologies that are have been presented in [11] and [8] respectively. Another topology related to these structures is the C-topology [7]. Also the relations between them have been investigated in [5], [9] and [12].

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References

[1] A. Arhangelskii, and M. Tkachenko, *Topological groups and related structures* , volume 1 of Atlantis Studies in Mathematics. Atlantis Press, Paris. 2008.

[2] B. Yousefipour, S. S. Gashti and H. Myrnouri, *Additive subgroups of real vector spaces and topologies on them*, Topology and its Applications, volume 373,(2025).

[3] D. A. Raikov, *On B-complete topological vector groups*, (Russian) Studia Math. 31,(1968), 295–306.

[4] D. N. Dikranjan, I. Prodanov and L. Stoyanov, *Topological groups: characters, dualities and minimal group topologies*, Monographs and Textbooks in Pure and Apllied Mathematics 130 (Marcel Dekker Inc., New York-Basel), 1989.

[5] F. Jordan and H. Pajoohesh, *Topologies on abelian lattice ordered groups induced by a positive filter and completeness*. Algebra Universalis, 79(62),(2018), 1–18.

[6] G. B. Folland, *A course in abstract harmonic analysis*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, MR1397028 (98c:43001), 1995.

[7] I. Gusic, *A topology on lattice-ordered groups*. Proc Amer Math Soc, 126(9)(1998), 2593–2597.

[8] M. Pourgholamhossein and M. A. Ranjbar. (2019). *On the Topological mass Lattice Groups*, Positivity, volume 23, issue 4, 811–827.

[9] M. Pourgholamhossein and M.A. Ranjbar, . *Positive filters and links in an ℓ-group*. Quaestiones Mathematicae, 45(8)(2022), 1297–1308.

[10] P. S. Kenderov, On topological vector groups, Mat. Sb. 10 (4)(1970), 531–546.

[11] R. Kopperman, H. Pajoohesh and T. Richmond, *Topologies arising from metrics valued in abelian ℓ-groups*. Algebra Universalis, 65(2011), 315–330.

[12] S. Karamdoust, H. Myrnouri and M. Pourgholamhossein, *On the Boolean algebra induced by a unital ℓ-group*, Algeb, Uni, Vol 85. 16.(2024).

[13] W. Banaszczyk, *Additive subgroups of topological vector spaces*, Lecture Notes in Mathematics, 1466, Springer-Verlag, Berlin-Heidelberg, 1991.

[14] W. Rudin, *Functional Analysis*. 2nd ed., International Series in Pure and Applied Mathematics. New York, NY:McGraw-Hill. xviii, 424, (1991).