Advances in Pure and Applied Mathematics
2026, vol. 17, n° 1, 60-77 pages, DOI: 10.21494/ISTE.OP.2026.1394 ISTE OpenScience

Indecomposable tournaments with minimum Slater index

Les tournois indécomposables à indice de Slater minimal

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ABSTRACT. The Slater index (resp. decomposability index) of a tournament is the minimum number of arcs that must be reversed in that tournament in order to make it a total order (resp. indecomposable (under modular decomposition)). The first author [H. Belkhechine, Decomposability index of tournaments, Discrete Math. 340 (2017) 2986–2994] showed that for every integer $n \geq 5$, the decomposability index of the n-vertex total order equals $\left\lceil \frac{n+1}{4} \right\rceil$. It follows that the Slater index of an indecomposable n-vertex tournament is at least $\left\lceil \frac{n+1}{4} \right\rceil$. This led A. Boussaïri to ask the following question during the thesis defense of the second author on July 2, 2021: what are the indecomposable tournaments T whose Slater index is minimum over all indecomposable tournaments with the same vertex set as T? These tournaments are then the indecomposable tournaments T obtained from a total order by reversing exactly $\left\lceil \frac{v(T)+1}{4} \right\rceil$ arcs, where $v(T)$ is the number of vertices of T. In this paper, we characterize such tournaments by means of so-called irreducible pairings.
2020 Mathematics Subject Classification. 05A18, 05C09, 05C20, 05C35, 05C75, 06A05.
KEYWORDS. Module, pairing, indecomposable, irreducible, decomposability index, Slater index

1. Introduction

The structures we consider are principally tournaments, total orders, and pairings, all of them are finite. We consider the following classical problem: what is the minimum number of arcs that must be reversed in a tournament in order to transform it into a tournament satisfying a given property? This general issue has been considered by several authors under different properties and considerations (see e.g., [1, 2, 5, 11, 12, 14, 17, 20, 24, 27]). In this paper, we consider the problem within two properties: transitivity and indecomposability (under modular decomposition). We identify transitive tournaments with total orders.

The Slater index of a tournament is the minimum number of arcs that must be reversed in that tournament in order to make it a total order. This index was introduced by P. Slater [27] in 1961. It was also extensively studied by several authors under various aspects (combinatorial, algorithmic, etc.), see e.g., [1, 2, 11, 12, 14, 17].

On the other hand, the decomposability index of a tournament (with at least five vertices) was introduced in 2017 by the first author in [4] as the minimum number of arcs that must be reversed in that tournament in order to make it indecomposable. The reason we restrict this index to tournaments with at least five vertices is that tournaments with four vertices are all decomposable. This index has also been studied by the first two authors in [5], and by the second author in [7]. In [5], the authors proved that the maximum value of the decomposability index over the n-vertex tournaments equals $\left\lceil \frac{n+1}{4} \right\rceil$. In [7], the second author characterized the class of tournaments T whose decomposability index is maximum over the tournaments with the same number of vertices as T. In fact, by the result of [5], these tournaments are precisely the tournaments with decomposability index $\left\lceil \frac{v(T)+1}{4} \right\rceil$. Moreover, the total orders (with at least five vertices) are part of these tournaments [5, 7].

The results above give rise to the following relationship between indecomposability and Slater index: given an integer $n \geq 5$, the minimum value of the Slater index over indecomposable n-vertex tournaments is $\left\lceil \frac{n+1}{4} \right\rceil$, which also is the maximum value of the decomposability index over all n-vertex tournaments. Moreover, the indecomposable n-vertex tournaments with minimum Slater index (i.e., whose Slater index is minimum over all indecomposable n-vertex tournaments) are the indecomposable tournaments obtained from n-vertex total orders by reversing exactly $\left\lceil \frac{n+1}{4} \right\rceil$ arcs. These results were presented by the second author during her thesis defense on July 2, 2021, which led A. Boussaïri, who was rapporteur for the thesis, to ask about the structure of the indecomposable tournaments with minimum Slater index. In this paper, we describe these tournaments in terms of irreducible pairings of total orders. In our previous work [6], we established fundamental results that underpin this study. Indeed, the primary motivation for [6] was to lay the necessary groundwork for the current manuscript.

The paper is organized as follows. In Section 2, we present our characterization of indecomposable tournaments T with minimum Slater index by describing them according to the residue of $v(T)$ modulo 4 (Theorems 2.1–2.4). Further notions and tools related to indecomposability are presented in Section 3. We prove Theorems 2.1–2.2 in Section 4, Theorem 2.3 in Section 5, and Theorem 2.4 in Section 6.

2. Preliminaries and main results

A *tournament* $T = (V(T), A(T))$ consists of a finite set $V(T)$ of *vertices* together with a set $A(T)$ of ordered pairs of distinct vertices, called *arcs*, such that for every $x \neq y \in V(T), (x, y) \in A(T)$ if and only if $(y, x) \notin A(T)$. The *size* of T, denoted by $v(T)$, is that of $V(T)$. Given a tournament T, the *subtournament* of T induced by a subset X of $V(T)$ is the tournament $T[X] := (X, A(T) \cap (X \times X))$. For $X \subseteq V(T)$, the subtournament $T[V(T) \setminus X]$ is also denoted by $T - X$, and by $T - x$ when X is the singleton $\{x\}$. Two tournaments T and T' are *isomorphic*, written $T \cong T'$, if there exists an *isomorphism* from T onto T', i.e., a bijection f from $V(T)$ onto $V(T')$ such that for every $x \neq y \in V(T), (x, y) \in A(T)$ if and only if $(f(x), f(y)) \in A(T')$. This paper is based on two specific types of tournaments: total orders and indecomposable tournaments.

A *total order* is a *transitive* tournament, that is, a tournament T such that for every $x, y, z \in V(T)$, if $(x, y) \in A(T)$ and $(y, z) \in A(T)$, then $(x, z) \in A(T)$. We identify a transitive tournament T with the set $V(T)$ totally ordered as follows: for every $x, y \in V(T)$, $x < y$ when $(x, y) \in A(T)$. Given a totally ordered set V, when the total ordering $\leq$ on V is implicitly understood, the total order $(V, \{(x, y) \in V \times V : x < y\})$ is denoted by $\underline{V}$. In this context, V and $\underline{V}$ are often used interchangeably. Since we only consider finite structures, we may assume that V is a subset of $\mathbb{N}$ totally ordered by the natural order on integers. When $V = \{0, \ldots, n-1\}$ for some positive integer n, the total order $\underline{V}$ is simply denoted by $\underline{n}$. Note that for purely technical reasons, we sometimes need to extend $\underline{n}$ by inserting rational numbers between consecutive integers. In this instance, the resulting extension $\underline{W}$ consists of a subset W of $\mathbb{Q}_+$ with the natural order on rational number induced by W.

The notion of indecomposability relies on that of a module. The notion of module generalizes to all tournaments the usual notion of interval in a total order. Recall that an *interval* of a totally ordered set V is a subset I of V such that every element v in $V \smallsetminus I$ is greater than all the elements of I or smaller than all of them. Analogously, we define a *module* of a tournament T to be a subset M of $V(T)$ such that for every vertex v in $V(T) \smallsetminus M$, we have $\{v\} \times M \subseteq A(T)$ or $M \times \{v\} \subseteq A(T)$. Observe that the notions of module and interval clearly coincide for total orders. The notion of module generalizes also to other combinatorial structures such as graphs and digraphs [19], binary relational structures [26], 2-structures [15], and hypergraphs [9]. It appears in the literature under various names such as interval [19], convex subset [16], partitive subset [30], autonomous set [10], clan [15], and, of course, module [13].

We now come to the notion of indecomposability. Let T be a tournament. The empty set $\varnothing$, the entire vertex set $V(T)$, and its singleton subsets are clearly modules of T, called *trivial modules*. The tournament T is *indecomposable* [30] (or *prime* [13] or *primitive* [15] or *simple* [16]) if all its modules are trivial; otherwise it is *decomposable*. Let us consider some examples. The tournaments of sizes at most 2 are clearly indecomposable. Up to isomorphism, the 3-vertex tournaments are the total order $\underline{3}$, which is decomposable, and the 3-*cycle* $C_3 := (\{0,1,2\}, \{(0,1),(1,2),(2,0)\})$, which is indecomposable. Up to isomorphism, there are exactly four 4-vertex tournaments, all of them are decomposable (see e.g., [23]). For sizes at least 5, it is well-known that there exist indecomposable n-vertex tournaments for every $n \geq 5$ (see e.g., [4]). In fact, Erdős et al. [16] proved that almost all tournaments are indecomposable. However, the total orders of sizes at least 3 are all decomposable. Let T be an indecomposable tournament. A vertex x of T is said to be *critical* if $T - x$ is decomposable. The *support* of T is the set of its noncritical vertices; it is denoted by $\mathrm{supp}(T)$.

This paper relies on two reversal indices for tournaments, based on transitivity and indecomposability. We use the following notation. Given a set V, for a nonnegative integer k, we denote by $\binom{V}{k}$ the set of all k-element subsets of V. To every tournament T with a subset P of $\binom{V(T)}{2}$, we associate the tournament $\mathrm{Inv}(T,P)$ obtained from T by reversing P, i.e., by reversing all the arcs $(x,y) \in A(T)$ such that $\{x,y\} \in P$. Thus, $\mathrm{Inv}(T,P) = (V(T), A(T) \smallsetminus (\{(x,y) \in A(T) : \{x,y\} \in P\}) \cup \{(x,y) : \{x,y\} \in P \text{ and } (x,y) \notin A(T)\})$. For example, the *dual* tournament of T is the tournament $T^\star := \mathrm{Inv}(T, \binom{V(T)}{2})$. Note that $(\mathrm{Inv}(T,P))^\star = \mathrm{Inv}(T^\star, P)$. Moreover T and $T^\star$ have the same modules; in particular T is indecomposable if and only if $T^\star$ is indecomposable. In the proof of Theorem 2.4, these remarks justify that a tournament can be interchanged with its dual.

Let T be a tournament. The *Slater index* of T is the smallest integer m for which there exists a subset P of $\binom{V(T)}{2}$ such that $|P| = m$ and $\mathrm{Inv}(T,P)$ is a total order. The resulting total order $\mathrm{Inv}(T,P)$ is called a *median order* of T. When $v(T) \neq 4$, we also define the *decomposability index* of the tournament T as the smallest integer m for which there exists a subset P of $\binom{V(T)}{2}$ such that $|P| = m$ and the tournament $\mathrm{Inv}(T,P)$ is indecomposable. The Slater index and the decomposability index of the tournament T are denoted by $s(T)$ and $\delta(T)$, respectively. The Slater index is clearly well-defined for all tournaments. But the decomposability index is well-defined only for tournaments with sizes other than 4, because, as observed above, the set of integers n for which there exist indecomposable n-vertex tournaments is $\mathbb{N} \smallsetminus \{4\}$. For convenience, we omit the few tournaments with sizes at most 3, and we consider the decomposability index only for tournaments with at least five vertices. Let n be an integer with $n \geq 5$. The first author [4] proved that $\delta(\underline{n}) = \left\lceil \frac{n+1}{4} \right\rceil$. Now let $s(n)$ be the minimum of $s(T)$ over the indecomposable tournaments T of size n. Clearly $s(n) = \delta(\underline{n})$. Since $\delta(\underline{n}) = \left\lceil \frac{n+1}{4} \right\rceil$, we obtain

$$s(n) = \delta(\underline{n}) = \left\lceil \frac{n+1}{4} \right\rceil. \tag{2.1}$$

We are interested in the indecomposable tournaments with minimum Slater index, that is, the indecomposable tournaments T whose Slater index is minimum over the indecomposable tournaments with the same size as T. By (2.1), these tournaments are the indecomposable tournaments T with Slater index $s(T) = \left\lceil \frac{v(T)+1}{4} \right\rceil$, or, equivalently, the indecomposable tournaments T obtained from a total order by reversing exactly $\left\lceil \frac{v(T)+1}{4} \right\rceil$ arcs. The purpose of this paper is to characterize such tournaments (Theorems 2.1–2.4). Their structure is closely related to the notion of so-called irreducible pairings in total orders. In fact, almost all of them are obtained from total orders by reversing irreducible pairings. Given a totally ordered set V, a partition P of V (or of $\underline{V}$) is *irreducible* [3, 8] (or *connected* [22]) if no nontrivial interval of V is a union of some blocks of P; otherwise it is *reducible*. (Recall that the blocks of a partition are the sets constituting this partition.) The partitions we need are primarily *pairings*, i.e., partitions whose blocks have size 2. However, we also consider another closely related type of partitions in which one block has size 3 and all other blocks have size 2. An *irreducible pairing* [28, 29] is then an irreducible partition whose blocks are unordered pairs. The study of irreducible pairings goes back at least to the 1950s. It seems that Touchard [31, 32] was the first author to consider and study these configurations, which he called proper systems. This name is no longer used, but the notion has been reconsidered by several authors under other names such as irreducible pairings [28, 29], irreducible diagrams [21], and linked diagrams [3, 25]. For example, Kleitman [21] found that the proportion of irreducible pairings among all pairings of $\underline{2n}$ is asymptotically e^{-1} (see also [18]).

Given a set V (of even or odd size), a pairing of a subset (of even size) of V is called a *partial pairing* of V. Note that a partial pairing P is a pairing of the union $\cup P$ of all the blocks of P. Given an integer $m \geq 5$, when $m \equiv 3$ or $2 \pmod 4$, we will see that all the m-vertex indecomposable tournaments with minimum Slater index are, up to isomorphism, obtained from $\underline{m}$ by reversing some irreducible partial pairings of $\underline{m}$ (Theorems 2.1–2.2). Since our description of indecomposable tournaments with minimum Slater index is based on irreducible pairings of certain vertex subsets, it is practical to construct these tournaments on vertex sets which are initially totally ordered. Such tournaments, i.e. tournaments whose vertex sets are totally ordered, are called *ordered tournaments*. For this reason, it is convenient for the vertex sets to be subsets of $\mathbb{N}$, implicitly ordered by the natural total order on integers. We then introduce the following notations. Let p and q be two nonnegative integers. We denote by $[\![p,q]\!]$, $[\![p,q[\![$, $]\!]p,q]\!]$, and $]\!]p,q[\![$ the intervals of $\mathbb{N}$ defined as follows: $[\![p,q]\!] := \{i \in \mathbb{N} : p \leq i \leq q\}$; $[\![p,q[\![:= [\![p,q]\!] \setminus \{q\}$; $]\!]p,q]\!] := [\![p,q]\!] \setminus \{p\}$; and $]\!]p,q[\![:= [\![p,q]\!] \setminus \{p,q\}$. Note that when $p > q$, the interval $[\![p,q]\!]$ is the empty set. For $X \subseteq \mathbb{N}$, the set of even (resp. odd) integers in X is denoted by X_{even} (resp. X_{odd}).

For every integer $n \geq 5$, we denote by $\mathcal{T}_n$ the (finite) set of indecomposable n-vertex tournaments T whose vertex set is $V(T) := [\![0, n-1]\!]$ and whose Slater index is minimum over the tournaments of size $v(T)$ (or, equivalently, over the tournaments with vertex set $V(T)$). By (2.1), $\mathcal{T}_n$ is the family of indecomposable tournaments with vertex set $[\![0, n-1]\!]$ and Slater index $\left\lceil \frac{n+1}{4} \right\rceil$. We also consider the subset of $\mathcal{T}_n$, which we denote by $\mathcal{T}_{\underline{n}}$, whose elements are the tournaments of $\mathcal{T}_n$ for which $\underline{n}$ is a median order. In other words,

Remark 2.1. $\mathcal{T}_{\underline{n}}$ is the family of indecomposable tournaments obtained from $\underline{n}$ by reversing exactly $\delta(\underline{n})$ arcs, that is, exactly $\left\lceil \frac{n+1}{4} \right\rceil$ arcs (see (2.1)).

Clearly, every tournament of the family $\mathcal{T}_n$ is isomorphic to a tournament of its subfamily $\mathcal{T}_{\underline{n}}$. We therefore elect to characterize the tournaments of the families $\mathcal{T}_{\underline{m}}$, $m \geq 5$. These tournaments are shown in Figure 1 for small values of m ($m \in \{5, 6, 7\}$).

We describe the families $\mathcal{T}_{\underline{m}}$, $m \geq 5$, according to the modulo 4 residue of m. We begin with $m \equiv 3 \pmod 4$. We obtain the following theorem.

Convention. Let T be a tournament. For $X \subseteq V(T)$, $\overline{X}$ denotes $V(T) \setminus X$.

Theorem 2.1. *For every positive integer n, the tournaments of the family $\mathcal{T}_{\underline{4n+3}}$ are the tournaments $\mathrm{Inv}(\underline{4n+3}, P)$, where P is an irreducible pairing of $[\![0, 4n+2]\!]_{\text{even}}$. Moreover, for every tournament $T \in \mathcal{T}_{\underline{4n+3}}$ and for every $e \subseteq [\![1, 4n+1]\!]_{\text{odd}}$ such that $|\overline{e}| \geq 5$, the tournament $T - e$ is indecomposable.*

For example, $\mathrm{Inv}(\underline{7}, \{\{0, 4\}, \{2, 6\}\})$ is the unique tournament of the family $\mathcal{T}_{\underline{7}}$ (see Figure 1).

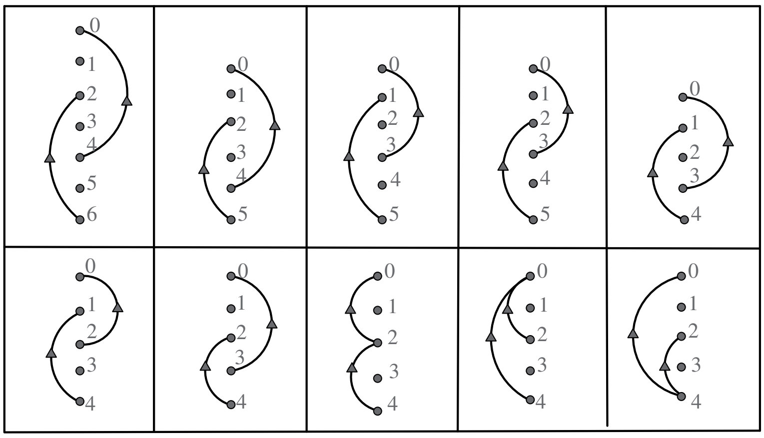

Figure 1. The tournaments of the families $\mathcal{T}_{\underline{5}}$, $\mathcal{T}_{\underline{6}}$, and $\mathcal{T}_{\underline{7}}$ (undrawn arcs belong to $A(\underline{7})$).

We will see that, up to isomorphism, the tournaments of the families $\mathcal{T}_{\underline{4n+2}}$, $\mathcal{T}_{\underline{4n+1}}$ ($n \geq 1$), and $\mathcal{T}_{\underline{4n}}$ ($n \geq 2$) are obtained from those of the family $\mathcal{T}_{\underline{4n+3}}$ (described in Theorem 2.1) by small modifications. These modifications consist of deleting one to three vertices with, in a few cases, a minor adjustment consisting of reversing only one arc. For example, we prove that for every positive integer n, the tournaments of the family $\mathcal{T}_{\underline{4n+2}}$ are, up to isomorphism, obtained from those of the family $\mathcal{T}_{\underline{4n+3}}$ by deleting one odd vertex. These tournaments are isomorphic to the tournaments of the family $\mathcal{T}_{\underline{4n+2}}$, but do not belong to this family. We then need to proceed with a relabeling of the vertices to bring these tournaments into the family $\mathcal{T}_{\underline{4n+2}}$. The situation is similar for the families $\mathcal{T}_{\underline{4n+1}}$ and $\mathcal{T}_{\underline{4n}}$. To describe the appropriate relabelings in a canonical way, we use the following notations.

Notation 2.1 (**Canonizations**). Let $T = (V, A)$ be a tournament, and let $\mathcal{F}$ be a family of subsets of V. Given a bijection π from V onto a set W, we denote by $\pi(T)$ the tournament $(W, \pi(A))$, where $\pi(A) := \{(\pi(x), \pi(y)) : (x, y) \in A\}$. Note that π is an isomorphism from T onto $\pi(T)$. We also denote by $\pi(\mathcal{F})$ the family $\{\pi(F) : F \in \mathcal{F}\}$, where $\pi(F) := \{\pi(x) : x \in F\}$. We mainly need such notations when T is an ordered tournament, especially when $V \subseteq \mathbb{N}$. In this case, let π_T (or π_V) denote the isomorphism from $\underline{V}$ onto $\underline{v(T)}$, i.e. the strictly increasing function from V onto $[\![0, v(T) - 1]\!]$. We then consider the tournament $\pi_T(T)$ (resp. the family $\pi_T(\mathcal{F})$) as the *canonical form* of the tournament T (resp. the family $\mathcal{F}$). We denote the canonical forms $\pi_T(T)$ and $\pi_T(\mathcal{F})$ by $\widetilde{T}$ and $\widetilde{\mathcal{F}}_T$, respectively. For example, given $P \subseteq \binom{V}{2}$, we have $\widetilde{\mathrm{Inv}(T,P)} = \mathrm{Inv}(\widetilde{T}, \widetilde{P}_T)$. Moreover, P is a pairing (resp. an irreducible pairing) of $\cup P$ if and only if $\widetilde{P}_T$ is a pairing (resp. an irreducible pairing) of $\cup \widetilde{P}_T$. Similarly, P is a partition (resp. an irreducible partition) of $\cup P$ if and only if $\widetilde{P}_T$ is a partition (resp. an irreducible partition) of $\cup \widetilde{P}_T$.

We now describe the families $\mathcal{T}_{\underline{4n+2}}$, $n \geq 1$, in the following theorem.

Theorem 2.2. *For every positive integer n, the tournaments of the family $\mathcal{T}_{\underline{4n+2}}$ are the tournaments $\widetilde{T-x}$, where $T \in \mathcal{T}_{\underline{4n+3}}$ and $x \in [\![1, 4n+1]\!]_{\mathrm{odd}}$.*

For example, for $n = 1$, the three tournaments of the family $\mathcal{T}_{\underline{6}}$ are shown in Figure 1.

Observe that for an integer $n \geq 1$, all the tournaments of the family $\mathcal{T}_{\underline{4n+3}}$ are obtained from $\underline{4n+3}$ by reversing some specific partial pairings of $\underline{4n+3}$ (see Theorem 2.1). Similarly, all the tournaments of the families $\mathcal{T}_{\underline{4n+2}}$, $n \geq 1$, are obtained from $\underline{4n+2}$ by reversing some specific partial pairing of $\underline{4n+2}$ (see Theorem 2.2). But this observation no longer holds for the families $\mathcal{T}_{\underline{4n+1}}$ ($n \geq 1$) and $\mathcal{T}_{\underline{4n}}$ ($n \geq 2$). In fact, given a positive integer n, certain tournaments of the family $\mathcal{T}_{\underline{4n+1}}$ (resp. $\mathcal{T}_{\underline{4n}}$, $n \geq 2$) are obtained from $\underline{4n+1}$ (resp. $\underline{4n}$) by reversing quasi-pairings instead of pairings. The notion of quasi-pairing is an extension of that of pairing to sets of odd sizes. Let W be a set of size $2n+1$ with $n \geq 1$. A *quasi-pairing* of W is a cover of W by a family of $n+1$ unordered pairs of W, i.e., a subset Q of $\binom{W}{2}$ such that $|Q| = n+1$ and $\cup Q = W$. When W is a subset of a set V (of odd or even size), a quasi-pairing of W is also called a *partial quasi-pairing* of V. Let Q be a partial quasi-pairing of a totally ordered set V (of size at least 3). We denote by $\hat{v}_Q$, v_Q^-, and v_Q^+ the unique elements of $\cup Q$ such that $\{\hat{v}_Q, v_Q^-\} \in Q$, $\{\hat{v}_Q, v_Q^+\} \in Q$, and $v_Q^- < v_Q^+$. Moreover, the 3-element set $\{\hat{v}_Q, v_Q^-, v_Q^+\}$ is denoted by B_Q.

Let n be a positive integer. We need some additional notations to describe the tournaments of the family $\mathcal{T}_{\underline{4n+1}}$, that we partition into two subfamilies $\mathcal{T}'_{\underline{4n+1}}$ and $\mathcal{T}''_{\underline{4n+1}}$ (see Notation 2.2 and Theorem 2.3). Let V be a set (of even or odd size). With every partial pairing P of V, we associate the involution without fixed points of $\cup P$, which we denote by i_P, defined by $i_P(B) = B$ for every block $B \in P$. This defines a natural one-to-one correspondence between partial pairings of V and involutions without fixed points of subsets (of even sizes) of V. Now let V be a totally ordered set (of size at least 3) and let Q be a partial quasi-pairing of V. Given $x \in V$, the vertex subset $\{y \in \cup Q : \{x,y\} \in Q\}$ is denoted by $\iota_Q(x)$. So $\iota_Q(\hat{v}_Q) = \{v_Q^-, v_Q^+\}$, and for every $x \in (\cup Q) \setminus \{\hat{v}_Q\}$, we have $|\iota_Q(x)| = 1$ and the (unique) element of $\iota_Q(x)$ is denoted by $i_Q(x)$.

Notation 2.2. Let n be a positive integer.

- We denote by $\mathcal{T}'_{\underline{4n+1}}$ the set of tournaments $\widetilde{T-e}$, where $T \in \mathcal{T}_{\underline{4n+3}}$ and $e \in \binom{[\![1,4n+1]\!]_{\mathrm{odd}}}{2}$.
- To every tournament $T \in \mathcal{T}_{\underline{4n+3}}$ with a vertex $x \in [\![1, 4n+1]\!]_{\mathrm{odd}}$, we associate the tournament $T_{(x)} := \mathrm{Inv}(T - \{x, x+1\}, \{\{x-1, i_P(x+1)\}\})$, where P is the pairing of $[\![0, 4n+2]\!]_{\mathrm{even}}$ such that $T = \mathrm{Inv}(\underline{4n+3}, P)$ (see Theorem 2.1). We now denote by $\mathcal{T}''_{\underline{4n+1}}$ the set of tournaments $\widetilde{T_{(x)}}$, where $T \in \mathcal{T}_{\underline{4n+3}}$ and $x \in [\![1, 4n+1]\!]_{\mathrm{odd}}$.

The tournaments of the families $\mathcal{T}''_{\underline{4n+1}}$, $n \geq 1$, can also be expressed in terms of quasi-pairings (see Remark 2.2 below).

Remark 2.2. Consider a tournament $T \in \mathcal{T}''_{\underline{4n+1}}$. We have $T = \widetilde{U_{(x)}}$ for some tournament $U := \text{Inv}(\underline{4n+3}, P) \in \mathcal{T}_{\underline{4n+3}}$ and $x \in [\![1, 4n+1]\!]_{\text{odd}}$ (see Notation 2.2). So there exists a unique quasi-pairing Q of $[\![0, 4n]\!]_{\text{even}}$ such that $T = \text{Inv}(\underline{4n+1}, Q)$. Moreover, we have $\hat{v}_Q = x-1$, $v_Q^- = \min(\pi(i_P(x-1)), \pi(i_P(x+1)))$, $v_Q^+ = \max(\pi(i_P(x-1)), \pi(i_P(x+1)))$, and hence $B_Q = \{x-1, \pi(i_P(x-1)), \pi(i_P(x+1))\}$, where $\pi = \pi_{U_{(x)}}$ (see Notation 2.1).

Theorem 2.3. *For every positive integer n, we have $\mathcal{T}_{\underline{4n+1}} = \mathcal{T}'_{\underline{4n+1}} \cup \mathcal{T}''_{\underline{4n+1}}$. More precisely, $\{\mathcal{T}'_{\underline{4n+1}}, \mathcal{T}''_{\underline{4n+1}}\}$ is a partition of $\mathcal{T}_{\underline{4n+1}}$.*

Theorem 2.3 says that for every positive integer n, the family $\mathcal{T}_{\underline{4n+1}}$ consists of tournaments obtained from $\underline{4n+1}$ by reversing some partial pairings of $\underline{4n+1}$ (the tournaments of $\mathcal{T}'_{\underline{4n+1}}$), as well as tournaments obtained from $\underline{4n+1}$ by reversing some quasi-pairings of $[\![0, 4n]\!]_{\text{even}}$ (the tournaments of $\mathcal{T}''_{\underline{4n+1}}$, see Remark 2.2). For $n = 1$, the six tournaments of the family $\mathcal{T}_{\underline{5}}$ are shown in Figure 1.

At present, the families $\mathcal{T}_{\underline{m}}$, $m \geq 5$, are characterized for $m \equiv 3,\ 2$ or $1 \pmod 4$. So it only remains to characterize the families $\mathcal{T}_{\underline{4n}}$, $n \geq 2$. Let n be an integer with $n \geq 2$. As we did with the family $\mathcal{T}_{\underline{4n+1}}$, we will partition the family $\mathcal{T}_{\underline{4n}}$ into two subfamilies $\mathcal{T}'_{\underline{4n}}$ and $\mathcal{T}''_{\underline{4n}}$ (see Notation 2.3 and Theorem 2.4). We first need the following lemma.

Lemma 2.1. *Let T be a tournament of the family $\mathcal{T}''_{\underline{4n+1}}$, where $n \geq 2$, and let P be the quasi-pairing of $[\![0, 4n]\!]_{\text{even}}$ such that $T = \text{Inv}(\underline{4n+1}, P)$ (see Remark 2.2). We have*

$$\text{supp}(T) = \begin{cases} [\![1, 4n-1]\!]_{\text{odd}} \setminus \{\min(B_P)+1\} & \textit{if } \min(B_P)+2 \in B_P, \\ [\![1, 4n-1]\!]_{\text{odd}} \setminus \{\max(B_P)-1\} & \textit{if } \max(B_P)-2 \in B_P, \\ [\![1, 4n-1]\!]_{\text{odd}} & \textit{otherwise}. \end{cases} \tag{2.2}$$

Remark 2.3. To see that (2.2) is well-defined, it suffices to show that $\min(B_P)+2 \in B_P$ and $\max(B_P)-2 \in B_P$ cannot occur simultaneously. So suppose to the contrary that $\min(B_P)+2 \in B_P$ and $\max(B_P)-2 \in B_P$. Since $|B_P| = 3$, it follows that $\min(B_P)+2 = \max(B_P)-2$ and hence $B_P = \{\min(B_P), \min(B_P)+2, \min(B_P)+4\}$. Since $T = \text{Inv}(\underline{4n+1}, P)$ and P is a quasi-pairing of $[\![0, 4n]\!]_{\text{even}}$, it follows that $[\![\min(B_P), \max(B_P)]\!]$ is a module of T, which is nontrivial because $|[\![\min(B_P), \max(B_P)]\!]| = 5$ and $v(T) \geq 9$. This contradicts that T is indecomposable.

Notation 2.3. Let n be an integer with $n \geq 2$.

- We denote by $\mathcal{T}'_{\underline{4n}}$ the set of tournaments $\widetilde{T-e}$, where $T \in \mathcal{T}_{\underline{4n+3}}$ and $e \in \binom{[\![1,4n+1]\!]_{\text{odd}}}{3}$.
- We denote by $\mathcal{T}''_{\underline{4n}}$ the set of tournaments $\widetilde{T-x}$, where $T \in \mathcal{T}''_{\underline{4n+1}}$ and $x \in \text{supp}(T)$ (see Lemma 2.1).

Theorem 2.4. *For every integer $n \geq 2$, we have $\mathcal{T}_{\underline{4n}} = \mathcal{T}'_{\underline{4n}} \cup \mathcal{T}''_{\underline{4n}}$. More precisely, $\{\mathcal{T}'_{\underline{4n}}, \mathcal{T}''_{\underline{4n}}\}$ is a partition of $\mathcal{T}_{\underline{4n}}$.*

3. Basic tools

3.1. *Co-modules, Δ-decompositions, and indecomposability*

The notions of co-module and Δ-decomposition were introduced in [5] as follows. Given a tournament T, a *co-module* of T is a subset M of $V(T)$ such that M or $\overline{M}$ is a nontrivial module of T. A co-module of T is *minimal* if it does not contain any other co-module. We denote by $\mathrm{mc}(T)$ the family of minimal co-modules of T. For example, for every integer $n \geq 3$, we have

$$\mathrm{mc}(\underline{n}) = \{\{0\}, \{n-1\}\} \cup \{\{i, i+1\} : 1 \leq i \leq n-3\}, \tag{3.1}$$

and for any totally ordered set V with $|V| = n$, we have $\mathrm{mc}(\underline{V}) = \pi_V^{-1}(\mathrm{mc}(\underline{n}))$ (see Notation 2.1).

A *co-modular decomposition* of a tournament T is a set of pairwise disjoint co-modules of T. A Δ-*decomposition* of T is a co-modular decomposition of T which is of maximum size. Such a size is called the *co-modular index* of T, and is denoted by $\Delta(T)$. The first two authors [5] showed that for every integer $n \geq 3$, we have

$$\Delta(\underline{n}) = \left\lceil \frac{n+1}{2} \right\rceil. \tag{3.2}$$

Note that we need the notions of co-module and Δ-decomposition only in the particular case of total orders, instead of general tournaments.

Example 3.1. Given an integer $n \geq 3$, it follows from (3.1) and (3.2) that $D_1 := \{\{0\}, \{n-1\}\} \cup \{\{2i-1, 2i\} : 1 \leq i \leq \lfloor \frac{n-2}{2} \rfloor\}$ is a Δ-decomposition of $\underline{n}$. Moreover, when n is odd, $D_2 := \{\{0\}, \{n-1\}\} \cup \{\{2i, 2i+1\} : 1 \leq i \leq \lfloor \frac{n-2}{2} \rfloor\}$ is another Δ-decomposition of $\underline{n}$.

Given a set family $\mathcal{F}$, a *transversal* of $\mathcal{F}$ is any set R that intersects each element of $\mathcal{F}$, that is, such that $F \cap R \neq \varnothing$ for every $F \in \mathcal{F}$. A transversal R of $\mathcal{F}$ is *exact* if $|F \cap R| = 1$ for every $F \in \mathcal{F}$. The following fact shows the involvement of the aforementioned notions in the class of indecomposable tournaments with minimum Slater index.

Fact 3.1. *Let V be a finite totally ordered set with $|V| = n \geq 5$, and let P be a subset of $\binom{V}{2}$.*

1. *If the tournament $\mathrm{Inv}(\underline{V}, P)$ is indecomposable, then $\cup P$ is a transversal of the family of all co-modules of $\underline{V}$, and hence a transversal of $\mathrm{mc}(\underline{V})$.*
2. *Let $q \in \mathbb{N} \setminus \{0\}$ and $r \in \{0,1,2,3\}$ be the (unique) integers such that $n = 4q + r$. If $\widetilde{\mathrm{Inv}(\underline{V}, P)} \in \mathcal{T}_{\underline{n}}$, then*

$$|\cup P| = \begin{cases} 2q+2 & \text{if } r \in \{2,3\}, \\ 2q+1 \text{ or } 2q+2 & \text{if } r \in \{0,1\}. \end{cases} \tag{3.3}$$

Proof. We may assume $V = [\![0, n-1]\!]$ and $\underline{V} = \underline{n}$. If $(\cup P) \cap M = \varnothing$ for a co-module M of $\underline{n}$, then M remains a co-module in $\mathrm{Inv}(\underline{n}, P)$, and hence $\mathrm{Inv}(\underline{n}, P)$ is decomposable. Therefore, the first assertion holds. For the second assertion, suppose $\mathrm{Inv}(\underline{n}, P) \in \mathcal{T}_{\underline{n}}$. In this instance, $|P| = \lceil \frac{n+1}{4} \rceil$ (see Remark 2.1) and hence $|\cup P| \leq 2\lceil \frac{n+1}{4} \rceil$. Now consider a Δ-decomposition D of $\underline{n}$. Since $|D| = \Delta(\underline{n}) = \lceil \frac{n+1}{2} \rceil$ (see (3.2)), and the elements of D are pairwise disjoint co-modules of $\underline{n}$, it follows from the first assertion that $|\cup P| \geq \lceil \frac{n+1}{2} \rceil$. Thus $\lceil \frac{n+1}{2} \rceil \leq |\cup P| \leq 2\lceil \frac{n+1}{4} \rceil$, and hence (3.3) holds. □

3.2. *Indecomposability and irreducibility*

We obtained Theorems 3.1–3.3 below in [6]. These theorems form basic tools in our proofs of Theorems 2.1–2.4. Theorem 3.1 provides a characterization of indecomposable tournaments obtained from total orders by reversing partial pairings. Theorems 3.2 and 3.3 are analogues of Theorem 3.1 for quasi-pairings under the following mode of irreducibility. Let Q be a quasi-pairing of a totally ordered set V of odd size at least 3. We denote by Q_{part} the partition of V obtained from Q by merging $\{\hat{v}_Q, v_Q^-\}$ and $\{\hat{v}_Q, v_Q^+\}$, that is, $Q_{\text{part}} := (Q \smallsetminus \{\{\hat{v}_Q, v_Q^-\}, \{\hat{v}_Q, v_Q^+\}\}) \cup \{B_Q\}$. The quasi-pairing Q is *irreducible* if the partition Q_{part} is irreducible; otherwise it is *reducible*.

Theorem 3.1 ([6]). *Let V be a finite totally ordered set such that $|V| \geq 5$, and let P be a partial pairing of V. The following assertions are equivalent.*

1. *The tournament $\text{Inv}(\underline{V}, P)$ is indecomposable.*
2. *The partial pairing P of V is an irreducible pairing of a transversal of $\text{mc}(\underline{V})$.*

Theorem 3.2 ([6]). *Let V be a finite totally ordered set such that $|V| \geq 6$, and let Q be a partial quasi-pairing of V. Consider the tournament $T := \text{Inv}(\underline{V}, Q)$. The following assertions are equivalent.*

1. *The partial quasi-pairing Q of V is an irreducible quasi-pairing of a transversal of $\text{mc}(\underline{V})$.*
2. *At least one of the tournaments T, $T - v_Q^-$, or $T - v_Q^+$ is indecomposable.*

Moreover, the second assertion still implies the first one when $|V| = 5$.

In Theorem 3.2, the first condition is necessary but not sufficient for the tournament T to be indecomposable, necessitating additional conditions. This leads us to the next theorem characterizing indecomposable tournaments obtained from total orders by reversing partial quasi-pairings.

Theorem 3.3 ([6]). *Given an integer m with $m \geq 5$, consider a partial quasi-pairing Q of $\underline{m}$. The tournament $\text{Inv}(\underline{m}, Q)$ is indecomposable if and only if the following conditions are satisfied.*

(C1) *The partial quasi-pairing Q of $\underline{m}$ is an irreducible quasi-pairing of a transversal of $\text{mc}(\underline{m})$.*

(C2) $v_Q^+ \geq v_Q^- + 2$.

(C3) *Given $v \in V(\underline{m})$, if $\{\{v, v+2\}, \{v+1, v+3\}\} \subseteq Q$, then $\hat{v}_Q \in \{v, v+3\}$.*

(C4) *Given $v \in V(\underline{m})$, if $\{v, v+1\} \in Q$, then $\hat{v}_Q \in \{v, v+1\}$ and $\{\hat{v}_Q - 1, \hat{v}_Q + 1\} \subseteq \cup Q$ (in particular $\hat{v}_Q \notin \{0, m-1\}$).*

4. Proofs of Theorems 2.1 and 2.2

We begin with the following simple but useful fact.

Fact 4.1. *Let n be an integer with $n \geq 3$.*

1. *If n is odd, then $[\![0, n-1]\!]_{\text{even}}$ is a transversal of $\text{mc}(\underline{n})$.*
2. *Given $X \subseteq V(\underline{n})$, if X is a transversal of $\text{mc}(\underline{n})$, then X is also a transversal of $\text{mc}(\underline{n} - e)$ for every subset e of $\overline{X}$.*

Proof. The first assertion follows directly from (3.1). For the second one, consider a transversal X of $\mathrm{mc}(\underline{n})$, and let e be a subset of $\overline{X}$. Since X is a transversal of $\mathrm{mc}(\underline{n})$, we have $\{0, n-1\} \subseteq X$ (see (3.1)), and hence $\{\{0\}, \{n-1\}\} \subseteq \mathrm{mc}(\underline{n} - e)$. Now let $C \in \mathrm{mc}(\underline{n} - e)$. We have to prove that $X \cap C \neq \varnothing$. If $C \in \mathrm{mc}(\underline{n})$, then $X \cap C \neq \varnothing$ because X is a transversal of $\mathrm{mc}(\underline{n})$. Now suppose $C \in \mathrm{mc}(\underline{n} - e) \setminus \mathrm{mc}(\underline{n})$. Note that $\underline{n} - e \cong \underline{n - |e|}$, and hence (3.1) applies to $\underline{n} - e$. Since $\{\{0\}, \{n-1\}\} \subseteq \mathrm{mc}(\underline{n} - e)$, it follows from (3.1) applied to $\underline{n} - e$ that $C = \{p, q\}$ with $1 \leq p \leq q - 2 \leq n - 4$. Moreover $p + 1 \notin X$. Since $\{p, p+1\} \cap X \neq \varnothing$ because X is a transversal of $\mathrm{mc}(\underline{n})$, it follows that $p \in X$ and hence $X \cap C \neq \varnothing$. Therefore, X is a transversal of $\mathrm{mc}(\underline{n} - e)$. □

Proof of Theorem 2.1. Let n be a positive integer and let P be an irreducible pairing of $[\![0, 4n+2]\!]_{\text{even}}$. Consider the tournament $T := \mathrm{Inv}(\underline{4n+3}, P)$. Let e be a (possibly empty) subset of $[\![1, 4n+1]\!]_{\text{odd}}$ such that $|\overline{e}| \geq 5$. By Fact 4.1, $[\![0, 4n+2]\!]_{\text{even}}$ is a transversal of $\mathrm{mc}(\underline{4n+3} - e)$. Since $T - e = \mathrm{Inv}(\underline{4n+3} - e, P)$ and P is an irreducible pairing of the transversal $[\![0, 4n+2]\!]_{\text{even}}$ of $\mathrm{mc}(\underline{4n+3} - e)$, it follows from Theorem 3.1 that the tournament $T - e$ is indecomposable. In particular, T is indecomposable. Since $|P| = n + 1$, it follows that $T \in \mathcal{T}_{\underline{4n+3}}$ (see Remark 2.1).

Conversely, let n be a positive integer and let $T \in \mathcal{T}_{\underline{4n+3}}$. There exists $P \subseteq \binom{V(\underline{4n+3})}{2}$ such that $|P| = n+1$ and $T = \mathrm{Inv}(\underline{4n+3}, P)$ (see Remark 2.1). We will prove that P is a pairing of $[\![0, 4n+2]\!]_{\text{even}}$, and that this pairing is irreducible. By Fact 3.1(2), we have $|\cup P| = 2n + 2$. On the other hand, $D_1 := \{\{0\}, \{4n+2\}\} \cup \{\{2i-1, 2i\} : 1 \leq i \leq 2n\}$ and $D_2 := \{\{0\}, \{4n+2\}\} \cup \{\{2i, 2i+1\} : 1 \leq i \leq 2n\}$ are Δ-decompositions of $\underline{4n+3}$ (see Example 3.1). Since T is indecomposable, then by Fact 3.1(1), the union $\cup P$ is a transversal of $\mathrm{mc}(\underline{4n+3})$ and hence of $D_1 \cup D_2$. Since $|D_1| = |D_2| = |\cup P| = 2n + 2$, it follows that

$$\cup P \text{ is an exact transversal of } D_1 \cup D_2. \tag{4.1}$$

Moreover $\cup P \subseteq \cup D_1$. In particular $4n + 1 \notin \cup P$. Since the elements of P are pairwise disjoint because $|\cup P| = 2|P|$, to prove that P is a pairing of $[\![0, 4n+2]\!]_{\text{even}}$, it suffices to show that $\cup P = [\![0, 4n+2]\!]_{\text{even}}$. So suppose for a contradiction that $\cup P \neq [\![0, 4n+2]\!]_{\text{even}}$. Since $|\cup P| = |[\![0, 4n+2]\!]_{\text{even}}|$, it follows that the intersection $X := [\![1, 4n+1]\!]_{\text{odd}} \cap (\cup P)$ is nonempty. So let p denote $\max(X)$. Note that $\{p+1, p+2\} \in D_2$. Since $4n + 1 \notin \cup P$, we have $p < 4n + 1$ and $\{p, p+1\} \in D_1$. Now since $4n + 2 \in \cup P$ (see (4.1)), the intersection $Y := [\![p+1, 4n+2]\!]_{\text{even}} \cap (\cup P)$ is nonempty. So let q denote $\min(Y)$. If $q = p + 1$, we obtain $\{p, p+1\} \subseteq \cup P$, which contradicts (4.1) because $\{p, p+1\} \in D_1$. If $q \neq p + 1$, i.e. $q \geq p + 3$, we obtain $\{p+1, p+2\} \cap (\cup P) = \varnothing$, which again contradicts (4.1) because $\{p+1, p+2\} \in D_2$. Therefore, P is a pairing of $[\![0, 4n+2]\!]_{\text{even}}$. Moreover, since T is indecomposable, it follows from Theorem 3.1 that the pairing P is irreducible. □

Proof of Theorem 2.2. Let n be a positive integer. Consider $T \in \mathcal{T}_{\underline{4n+3}}$, and let $x \in [\![1, 4n+1]\!]_{\text{odd}}$. We have to show that $\widetilde{T - x} \in \mathcal{T}_{\underline{4n+2}}$. By Theorem 2.1, there exists an irreducible pairing P of $[\![0, 4n+2]\!]_{\text{even}}$ such that $T = \mathrm{Inv}(\underline{4n+3}, P)$. Moreover, the tournament $T - x$ is indecomposable. Since $T - x = \mathrm{Inv}(\underline{4n+3} - x, P)$ and $\widetilde{\underline{4n+3} - x} = \underline{4n+2}$, we have $\widetilde{T - x} = \mathrm{Inv}(\underline{4n+2}, \widetilde{P}_{T-x})$ (see Notation 2.1). Since $|\widetilde{P}_{T-x}| = |P| = \delta(\underline{4n+2})$, and $\widetilde{T - x}$ is indecomposable because $T - x$ is, it follows that $\widetilde{T - x} \in \mathcal{T}_{\underline{4n+2}}$ (see Remark 2.1).

Conversely, let $U \in \mathcal{T}_{\underline{4n+2}}$. We have to prove that $U = \widetilde{T - x}$ for some $T \in \mathcal{T}_{\underline{4n+3}}$ and $x \in [\![1, 4n+1]\!]_{\text{odd}}$. Since $U \in \mathcal{T}_{\underline{4n+2}}$, there exists $P \subseteq \binom{V(\underline{4n+2})}{2}$ such that $|P| = n+1$ and $U = \mathrm{Inv}(\underline{4n+2}, P)$ (see Remark 2.1).

Claim 1. *There exists* $p \in [\![0, 2n]\!]$ *such that* P *is an irreducible pairing of* $[\![0, 2p]\!]_{\text{even}} \cup [\![2p+1, 4n+1]\!]_{\text{odd}}$.

Proof of Claim 1. Since $|\cup P| = 2n+2 = 2|P|$ (see Fact 3.1(2)), then P is a pairing of $\cup P$. Since U is indecomposable, then by Theorem 3.1, the pairing P is irreducible. To show that $\cup P = [\![0, 2p]\!]_{\text{even}} \cup [\![2p+1, 4n+1]\!]_{\text{odd}}$ for some $p \in [\![0, 2n]\!]$, which completes the proof, we consider the Δ-decomposition $D := \{\{0\}, \{4n+1\}\} \cup \{\{2i-1, 2i\} : 1 \leq i \leq 2n\}$ of $\underline{4n+2}$ (see Example 3.1). By Fact 3.1(1), $\cup P$ is a transversal of $\text{mc}(\underline{4n+2})$ and hence of D. Since $|D| = |\cup P| = 2n+2$, it follows that

$$\cup P \text{ is an exact transversal of } D. \tag{4.2}$$

In particular $\{0, 4n+1\} \subseteq \cup P$. Now consider the nonempty intersection $X := [\![1, 4n+1]\!]_{\text{odd}} \cap (\cup P)$. We have $\min(X) = 2p+1$ for some $p \in [\![0, 2n]\!]$. Take $x \in [\![0, 2p]\!]_{\text{even}} \setminus \{0\}$. Since $\{x-1, x\} \in D$, and $x-1 \notin \cup P$ because $\min(X) = 2p+1$, then $x \in \cup P$ by (4.2). Since $0 \in \cup P$ and $\min(X) = 2p+1$, it follows that $[\![0, 2p]\!] \cap (\cup P) = [\![0, 2p]\!]_{\text{even}}$. Now suppose for a contradiction that the intersection $Y := [\![2p+2, 4n]\!]_{\text{even}} \cap (\cup P)$ is nonempty. Let q denote $\min(Y)$. If $q = 2p+2$, then $\{2p+1, q\} \in D$ and $\{2p+1, q\} \subseteq \cup P$, which contradicts (4.2). Therefore $q \geq 2p+4$. Since $\{q-1, q\} \in D$ and $q \in \cup P$, then by (4.2), $q-1 \notin \cup P$. Since $\{q-2, q-1\} \cap (\cup P) \neq \varnothing$ because $\cup P$ is a transversal of $\text{mc}(\underline{4n+2})$ (see (3.1)), it follows that $q-2 \in \cup P$. As $q \geq 2p+4$, this implies $q-2 \in Y$, contradicting $q = \min(Y)$. Therefore $Y = \varnothing$. Now the equality $\cup P = [\![0, 2p]\!]_{\text{even}} \cup [\![2p+1, 4n+1]\!]_{\text{odd}}$ follows directly from the following facts: $\cup P \in \binom{[\![0,4n+1]\!]}{2n+2}$, $[\![0, 2p]\!] \cap (\cup P) = [\![0, 2p]\!]_{\text{even}}$, and $Y = \varnothing$. □

We now consider the tournament Γ obtained from U by adding one new vertex $(2p+\frac{1}{2})$ in the following manner: $\Gamma = \text{Inv}(\underline{W}, P)$, where $W := [\![0, 4n+1]\!] \cup \{2p+\frac{1}{2}\}$ (see Claim 1). Note that $U = \Gamma - (2p+\frac{1}{2})$. Now consider the tournament $T := \widetilde{\Gamma}$. By construction, we have $U = \widetilde{T-x}$ for $x = 2p+1$. Since $x \in [\![1, 4n+1]\!]_{\text{odd}}$, it suffices to verify that $T \in \mathcal{T}_{\underline{4n+3}}$ to conclude the proof. Since $\Gamma = \text{Inv}(\underline{W}, P)$ and $\widetilde{\underline{W}} = \underline{4n+3}$, then $T = \text{Inv}(\underline{4n+3}, \widetilde{P}_\Gamma)$ (see Notation 2.1). Moreover, since by Claim 1, P is an irreducible pairing of $\cup P$, we obtain that $\widetilde{P}_\Gamma$ is an irreducible pairing of $\cup \widetilde{P}_\Gamma$ (see Notation 2.1). But $\cup \widetilde{P}_\Gamma = [\![0, 4n+2]\!]_{\text{even}}$ because $\cup P = [\![0, 2p]\!]_{\text{even}} \cup [\![2p+1, 4n+1]\!]_{\text{odd}}$ and $\cup \widetilde{P}_\Gamma = \cup\pi(P) = \pi(\cup P)$, where $\pi := \pi_\Gamma$ is the strictly increasing function from $V(\Gamma)$ onto $[\![0, 4n+2]\!]$ (see Notation 2.1). Thus, $\widetilde{P}_\Gamma$ is an irreducible pairing of $[\![0, 4n+2]\!]_{\text{even}}$. It follows from Theorem 2.1 that $T \in \mathcal{T}_{\underline{4n+3}}$. □

5. Proof of Theorem 2.3

We need the following technical lemma.

Lemma 5.1. *Let n be a positive integer and let P be a subset of $\binom{V(\underline{4n+1})}{2}$ such that $\text{Inv}(\underline{4n+1}, P) \in \mathcal{T}_{\underline{4n+1}}$. Recall that $|P| = n+1$ (see Remark 2.1) and $|\cup P| \in \{2n+1, 2n+2\}$ (see Fact 3.1(2)).*

1. *If $|\cup P| = 2n+2$, then there are $p \leq q \in [\![0, 2n-1]\!]$ such that P is an irreducible pairing of $\cup P = [\![0, 2p]\!]_{\text{even}} \cup [\![2p+1, 2q+1]\!]_{\text{odd}} \cup [\![2q+2, 4n]\!]_{\text{even}}$.*
2. *If $|\cup P| = 2n+1$, then P is an irreducible quasi-pairing of $\cup P = [\![0, 4n]\!]_{\text{even}}$.*

Proof. We consider the Δ-decompositions $D_1 := \{\{0\}, \{4n\}\} \cup \{\{2i-1, 2i\} : 1 \leq i \leq 2n-1\}$ and $D_2 := \{\{0\}, \{4n\}\} \cup \{\{2i, 2i+1\} : 1 \leq i \leq 2n-1\}$ of $\underline{4n+1}$ (see Example 3.1). By Fact 3.1(1),

$$\cup P \text{ is a transversal of } D_1 \cup D_2. \tag{5.1}$$

For the first assertion, suppose $|\cup P| = 2n+2$. In this instance, P is a pairing of $\cup P$ because $|\cup P| = 2|P|$. Moreover, since $\mathrm{Inv}(\underline{4n+1}, P)$ is indecomposable, then by Theorem 3.1, the pairing P is irreducible. Recall that $0 \in \cup P$ and $4n \in \cup P$ (see (5.1)). So consider, in $[\![0, 2n]\!]$, the largest integer p such that $[\![0, 2p]\!] \cap (\cup P) = [\![0, 2p]\!]_{\text{even}}$, and the smallest integer v such that $[\![2v, 4n]\!] \cap (\cup P) = [\![2v, 4n]\!]_{\text{even}}$. If $v \leq p$, then $\cup P = [\![0, 4n]\!]_{\text{even}}$ and hence $|\cup P| = 2n+1$, a contradiction. Therefore $0 \leq p \leq q \leq 2n-1$, where $q := v - 1$. We will prove that

$$\cup P = [\![0, 2p]\!]_{\text{even}} \cup [\![2p+1, 2q+1]\!]_{\text{odd}} \cup [\![2q+2, 4n]\!]_{\text{even}}. \tag{5.2}$$

If $2p+1 \notin \cup P$, then $2p+2 \in \cup P$ (see (5.1)) and hence $[\![0, 2p+2]\!] \cap (\cup P) = [\![0, 2p+2]\!]_{\text{even}}$, contradicting the definition of p. Therefore $2p+1 \in \cup P$. Similarly, $2q+1 \in \cup P$. Thus,

$$\begin{cases} [\![0, 2p+1]\!] \cap (\cup P) = [\![0, 2p]\!]_{\text{even}} \cup \{2p+1\} \\ \text{and} \\ [\![2q+1, 4n]\!] \cap (\cup P) = [\![2q+2, 4n]\!]_{\text{even}} \cup \{2q+1\}. \end{cases} \tag{5.3}$$

It follows that (5.2) holds when $q = p$. Now suppose $q \geq p+1$. In this instance, it follows from (5.3) that $[\![0, 2p+1]\!] \cap (\cup P)$ and $[\![2q+1, 4n]\!] \cap (\cup P)$ are disjoint with respective sizes $p+2$ and $2n-q+1$. Since $|\cup P| = 2n+2$, it follows that

$$|[\![2p+2, 2q]\!] \cap (\cup P)| = q - p - 1. \tag{5.4}$$

By (5.3), to obtain (5.2), it suffices to prove that $[\![2p+2, 2q]\!] \cap (\cup P) = [\![2p+2, 2q]\!]_{\text{odd}}$. Since $|[\![2p+2, 2q]\!]_{\text{odd}}| = |[\![2p+2, 2q]\!] \cap (\cup P)| = q-p-1$ (see (5.4)), it suffices to show that $[\![2p+2, 2q]\!]_{\text{even}} \cap (\cup P) = \varnothing$. Let $u \in [\![p+1, q]\!]$. To prove that $2u \notin \cup P$, consider $D := \{d \in D_2 : d \subseteq [\![2p+2, 2u[\![\} \cup \{d \in D_1 : d \subseteq]\!]2u, 2q]\!]\}$. Clearly $|D| = q-p-1$, and the elements of D are pairwise disjoint elements of $D_1 \cup D_2$. Thereby, it follows from (5.1) that $|(\cup D) \cap (\cup P)| \geq q-p-1$. Consequently, since $\cup D = [\![2p+2, 2q]\!] \smallsetminus \{2u\}$, it follows from (5.4) that $2u \notin \cup P$. Therefore $[\![2p+2, 2q]\!]_{\text{even}} \cap (\cup P) = \varnothing$, completing the proof of the first assertion.

For the second assertion, suppose $|\cup P| = 2n+1$. Since $|D_2| = 2n+1$, it follows from (5.1) that

$$\cup P \text{ is an exact transversal of } D_2. \tag{5.5}$$

Moreover $\cup P \subseteq \cup D_2$. In particular $1 \notin \cup P$ since $1 \notin \cup D_2$. On the other hand, P is a quasi-pairing of $\cup P$ because $|P| = n+1$ and $|\cup P| = 2n+1$. Moreover, since $\mathrm{Inv}(\underline{4n+1}, P)$ is indecomposable, then by Theorem 3.2, the quasi-pairing P is irreducible. So to conclude the proof, it remains only to show that $\cup P = [\![0, 4n]\!]_{\text{even}}$. Suppose for a contradiction that the intersection $X := [\![0, 4n]\!]_{\text{odd}} \cap (\cup P)$ is nonempty. Let k denote $\min(X)$. We have $k \geq 3$ because $1 \notin \cup P$. Therefore $\{k-2, k-1\} \in D_1$. Since $k-2 \notin \cup P$ because $k = \min(X)$, we obtain $k-1 \in \cup P$ (see (5.1)). Thus $\{k-1, k\} \subseteq \cup P$. Since $\{k-1, k\} \in D_2$, this contradicts (5.5). Therefore $\cup P \subseteq [\![0, 4n]\!]_{\text{even}}$. Since $|[\![0, 4n]\!]_{\text{even}}| = |\cup P|$, it follows that $\cup P = [\![0, 4n]\!]_{\text{even}}$. □

Proof of Theorem 2.3. Let n be a positive integer. We have to prove that $\{\mathcal{T}'_{\underline{4n+1}}, \mathcal{T}''_{\underline{4n+1}}\}$ is a partition of $\mathcal{T}_{\underline{4n+1}}$. Clearly $\mathcal{T}'_{\underline{4n+1}} \neq \varnothing$, $\mathcal{T}''_{\underline{4n+1}} \neq \varnothing$, and $\mathcal{T}'_{\underline{4n+1}} \cap \mathcal{T}''_{\underline{4n+1}} = \varnothing$. We now prove that $\mathcal{T}_{\underline{4n+1}} = \mathcal{T}'_{\underline{4n+1}} \cup \mathcal{T}''_{\underline{4n+1}}$. Let $U \in \mathcal{T}_{\underline{4n+1}}$. To prove that $U \in \mathcal{T}'_{\underline{4n+1}} \cup \mathcal{T}''_{\underline{4n+1}}$, we consider the subset P of $\binom{V(\underline{4n+1})}{2}$ such that $U = \mathrm{Inv}(\underline{4n+1}, P)$. By Fact 3.1(2), we have $|\cup P| \in \{2n+1, 2n+2\}$.

First suppose $|\cup P| = 2n+2$. By Lemma 5.1(1), there are $p \leq q \in [\![0, 2n-1]\!]$ such that P is an irreducible pairing of $\cup P = [\![0, 2p]\!]_{\text{even}} \cup [\![2p+1, 2q+1]\!]_{\text{odd}} \cup [\![2q+2, 4n]\!]_{\text{even}}$. We now consider the tournament Γ obtained from U by adding two new vertices $2p+\frac{1}{2}$ and $2q+\frac{3}{2}$ in the following manner: $\Gamma = \mathrm{Inv}(\underline{W}, P)$, where $W := [\![0,4n]\!] \cup \{2p+\frac{1}{2}, 2q+\frac{3}{2}\}$. Note that $U = \Gamma - \{2p+\frac{1}{2}, 2q+\frac{3}{2}\}$. Now consider the tournament $T := \widetilde{\Gamma}$. By construction, we have $U = \widetilde{T-e}$ for $e = \{2p+1, 2q+3\}$. Clearly $e \in \binom{[\![1,4n+1]\!]_{\mathrm{odd}}}{2}$. To see that $T \in \mathcal{T}_{\underline{4n+3}}$, observe that $T = \mathrm{Inv}(\underline{4n+3}, \widetilde{P}_\Gamma)$. Moreover, $\widetilde{P}_\Gamma$ is an irreducible pairing of $\cup\widetilde{P}_\Gamma = [\![0,4n+2]\!]_{\mathrm{even}}$ because P is an irreducible pairing of $[\![0,2p]\!]_{\mathrm{even}} \cup [\![2p+1,2q+1]\!]_{\mathrm{odd}} \cup [\![2q+2,4n]\!]_{\mathrm{even}}$ (see Notation 2.1). It follows from Theorem 2.1 that $T \in \mathcal{T}_{\underline{4n+3}}$ and hence $U \in \mathcal{T}'_{\underline{4n+1}}$.

Second suppose $|\cup P| = 2n+1$. By Lemma 5.1(2), P is an irreducible quasi-pairing of $[\![0,4n]\!]_{\mathrm{even}}$. Let us consider the pairing Π of $[\![0,4n]\!]_{\mathrm{even}} \cup \{\hat{v}_P + \frac{1}{2}\}$ obtained from the quasi-pairing P in the following manner:

$$\Pi = \begin{cases} (P \setminus \{\{\hat{v}_P, v_P^-\}\}) \cup \{\{\hat{v}_P + \frac{1}{2}, v_P^-\}\} & \text{if } v_P^- < \hat{v}_P < v_P^+, \\ (P \setminus \{\{\hat{v}_P, v_P^+\}\}) \cup \{\{\hat{v}_P + \frac{1}{2}, v_P^+\}\} & \text{otherwise.} \end{cases}$$

We then consider the tournament $\Gamma = \mathrm{Inv}(\underline{W}, \Pi)$, where $W := [\![0,4n]\!] \cup \{\hat{v}_P + \frac{1}{3}, \hat{v}_P + \frac{1}{2}\}$. Now consider the tournament $T := \widetilde{\Gamma}$. By construction, we have $T = \mathrm{Inv}(\underline{4n+3}, \widetilde{\Pi}_\Gamma)$, and $U = \widetilde{T_{(x)}}$ for $x = \hat{v}_P + 1 \in [\![1,4n+1]\!]_{\mathrm{odd}}$ (see Notations 2.1 and 2.2). By Theorem 2.1, to prove that $T \in \mathcal{T}_{\underline{4n+3}}$ and hence $U \in \mathcal{T}''_{\underline{4n+1}}$, it suffices to show that $\widetilde{\Pi}_\Gamma$ is an irreducible pairing of $[\![0,4n+2]\!]_{\mathrm{even}}$, or, equivalently, that Π is an irreducible pairing of $\cup\Pi = [\![0,4n]\!]_{\mathrm{even}} \cup \{\hat{v}_P + \frac{1}{2}\}$ (see Notation 2.1). By construction, $\hat{v}_P$ and $\hat{v}_P + \frac{1}{2}$ belong respectively to two distinct blocks B and B' of the pairing Π. Moreover, $B \cup B' = \{\hat{v}_P, \hat{v}_P + \frac{1}{2}, v_P^-, v_P^+\}$, and the pairing $\{B, B'\}$ of $B \cup B'$ is irreducible. Now let S be a nonempty and proper subset of Π. We have to prove that $\cup S$ is not an interval of $\cup\Pi$, which implies that Π is irreducible. We distinguish the following three cases. First suppose $S \cap \{B, B'\} = \varnothing$. In this instance, S is a nonempty and proper subset of P. Since the quasi-pairing P is irreducible, we obtain that $\cup S$ is not an interval of $\cup P$, and hence it is not an interval of $\cup\Pi$. Second suppose $S \cap \{B, B'\} = \{B\}$ or $\{B'\}$. In this instance, $\cup S$ is not an interval of $\cup\Pi$ because the pairing $\{B, B'\}$ of $B \cup B'$ is irreducible. Third suppose $S \cap \{B, B'\} = \{B, B'\}$, i.e. $\{B, B'\} \subseteq S$. In this instance, $(\cup S) \setminus \{\hat{v}_P + \frac{1}{2}\}$ is a union of blocks of P_{part} (including the block $B_P = \{\hat{v}_P, v_P^-, v_P^+\}$). Moreover $(\cup S) \setminus \{\hat{v}_P + \frac{1}{2}\} \subsetneq \cup P$. Therefore, $(\cup S) \setminus \{\hat{v}_P + \frac{1}{2}\}$ is not an interval of $\cup P$ because P is irreducible, and hence $\cup S$ is not an interval of $\cup\Pi$.

Conversely, let $U \in \mathcal{T}'_{\underline{4n+1}} \cup \mathcal{T}''_{\underline{4n+1}}$. We have to prove that $U \in \mathcal{T}_{\underline{4n+1}}$. To begin, suppose $U \in \mathcal{T}'_{\underline{4n+1}}$. There exist $T \in \mathcal{T}_{\underline{4n+3}}$ and $e \in \binom{[\![1,4n+1]\!]_{\mathrm{odd}}}{2}$ such that $U = \widetilde{T-e}$ (see Notation 2.2). Since $|\overline{e}| \geq 5$, so by Theorem 2.1, the tournament U is indecomposable because $T-e$ is indecomposable. Again by Theorem 2.1, $T = \mathrm{Inv}(\underline{4n+3}, P)$ for some pairing P of $[\![0,4n+2]\!]_{\mathrm{even}}$. Thus $T - e = \mathrm{Inv}(\underline{4n+3} - e, P)$, and hence $U = \mathrm{Inv}(\underline{4n+1}, \widetilde{P}_{T-e})$ (see Notation 2.1). Since $|\widetilde{P}_{T-e}| = |P| = n+1$, it follows that $U \in \mathcal{T}_{\underline{4n+1}}$ (see Remark 2.1).

Now suppose $U \in \mathcal{T}''_{\underline{4n+1}}$. There exist $T \in \mathcal{T}_{\underline{4n+3}}$ and $x \in [\![1,4n+1]\!]_{\mathrm{odd}}$ such that $U = \widetilde{T_{(x)}}$ (see Notation 2.2). By Theorem 2.1, $T = \mathrm{Inv}(\underline{4n+3}, P)$ for some irreducible pairing P of $[\![0,4n+2]\!]_{\mathrm{even}}$. Moreover, $T_{(x)} = \mathrm{Inv}(\underline{W}, R)$, where $W := [\![0,4n+2]\!] \setminus \{x, x+1\}$, and where R is the quasi-pairing of $[\![0,4n+2]\!]_{\mathrm{even}} \setminus \{x+1\}$ obtained from the pairing P as follows: $R = (P \setminus \{\{x+1, i_P(x+1)\}\}) \cup \{\{x-1, i_P(x+1)\}\}$ (see Notation 2.2). Thus $U = \mathrm{Inv}(\underline{4n+1}, Q)$, where $Q = \widetilde{R}_{T_{(x)}}$ (see Notation 2.1). Clearly Q is a quasi-pairing of $[\![0,4n]\!]_{\mathrm{even}}$ (see Notation 2.1). Since $|Q| = |P| = n+1$, then by Remark 2.1, to prove that $U \in \mathcal{T}_{\underline{4n+1}}$, which completes the proof, it suffices to prove that the tournament U is indecomposable. Since $U = \mathrm{Inv}(\underline{4n+1}, Q)$, so by Theorem 3.3, one only needs to show that the partial quasi-pairing Q of $\underline{4n+1}$ satisfies Conditions (C1)–(C4) of this theorem with $m = 4n+1$. Because

$\cup Q = [\![0,4n]\!]_{\text{even}}$, the quasi-pairing Q clearly satisfies Conditions (C2)–(C4). Now since $[\![0,4n]\!]_{\text{even}}$ is a transversal of $\mathrm{mc}(\underline{4n+1})$ (see Fact 4.1(1)), to prove that Condition (C1) is satisfied, we only have to show that the quasi-pairing Q of $[\![0,4n]\!]_{\text{even}}$ is irreducible, or, equivalently, that the quasi-pairing R of $[\![0,4n+2]\!]_{\text{even}} \smallsetminus \{x+1\}$ is irreducible (see Notation 2.1). So let S be a nonempty and proper subset of R_{part}. We have to prove that $\cup S$ is not an interval of $\cup R = [\![0,4n+2]\!]_{\text{even}} \smallsetminus \{x+1\}$. Note that $\hat{v}_R = x-1$ and $B_R = \{x-1, i_P(x-1), i_P(x+1)\}$. First suppose $B_R \notin S$. In this instance, S is a nonempty proper subset of the pairing P. Since P is irreducible, it follows that $\cup S$ is not an interval of $\cup P = [\![0,4n+2]\!]_{\text{even}}$. If $x+1 \notin]\!]\min(\cup S), \max(\cup S)[\![$, then since $\cup S$ is not an interval of $\cup P$, $\cup S$ is not an interval of $(\cup P) \smallsetminus \{x+1\}$ either. If $x+1 \in]\!]\min(\cup S), \max(\cup S)[\![$, then since $x-1 \notin \cup S$, we also have $x-1 \in]\!]\min(\cup S), \max(\cup S)[\![$ and hence $\cup S$ is again not an interval of $(\cup P) \smallsetminus \{x+1\}$. Second suppose $B_R \in S$. In this instance, $(\cup S) \cup \{x+1\}$ is a union of blocks of the pairing P. Moreover, $(\cup S) \cup \{x+1\} \subsetneq \cup P$ because $(\cup S) \subsetneq \cup R$. Since P is irreducible, it follows that $(\cup S) \cup \{x+1\}$ is not an interval of $\cup P$. Since $x-1 \in \cup S$ and $\{x-1, x+1\}$ is an interval of $\cup P$, it follows that $\cup S$ is not an interval of $\cup R$. Therefore R is irreducible, completing the proof. $\square$

6. Proof of Theorem 2.4

We first prove Lemma 2.1.

Proof of Lemma 2.1. We have to prove (2.2). Recall that (2.2) is well-defined (see Remark 2.3). Let $u \in [\![0,4n]\!]_{\text{even}}$. Clearly $T - u = \mathrm{Inv}(\underline{4n+1} - u, R)$, where

$$R = \begin{cases} P \smallsetminus \{\{u, i_P(u)\}\} & \text{if } u \neq \hat{v}_P, \\ P \smallsetminus \{\{u, v_P^-\}, \{u, v_P^+\}\} & \text{if } u = \hat{v}_P. \end{cases} \tag{6.1}$$

Since $|P| = n+1$, it follows that $|R| = n$ or $n-1$. On the other hand, $\underline{4n+1} - u \simeq \underline{4n}$ and hence $\delta(\underline{4n+1} - u) = \delta(\underline{4n}) = n+1$ (see (2.1)). Thus $|R| < \delta(\underline{4n+1} - u)$, and hence $T - u$ is decomposable. Since u was arbitrarily chosen in $[\![0,4n]\!]_{\text{even}}$,

$$\mathrm{supp}(T) \subseteq [\![1,4n-1]\!]_{\text{odd}}. \tag{6.2}$$

Now let $v \in [\![1,4n-1]\!]_{\text{odd}}$. Recall that the tournament $T - v$ is indecomposable if and only if $\widetilde{T-v}$ is. Since $T - v = \mathrm{Inv}(\underline{4n+1} - v, P)$, we have $\widetilde{T-v} = \mathrm{Inv}(\underline{4n}, Q)$, where $Q = \widetilde{P}_{T-v}$ is a partial quasi-pairing of $\underline{4n}$ (see Notation 2.1). Recall that $Q = \{\pi(B) : B \in P\}$, where π is the strictly increasing function from $[\![0,4n]\!] \smallsetminus \{v\}$ onto $[\![0,4n-1]\!]$, and hence $\cup Q = [\![0,v-1]\!]_{\text{even}} \cup [\![v,4n-1]\!]_{\text{odd}}$. In the rest of the proof, we use Theorem 3.3 to check the indecomposability of $\widetilde{T-v}$, and hence that of $T-v$. We therefore consider Conditions (C1)–(C4) of this theorem with $m = 4n$. These conditions on Q are necessary and sufficient for the tournament $\widetilde{T-v}$, and hence for $T-v$, to be indecomposable (see Theorem 3.3). The quasi-pairing Q obviously satisfies Condition (C3) because $\cup Q = [\![0,v-1]\!]_{\text{even}} \cup [\![v,4n-1]\!]_{\text{odd}}$. Since T is indecomposable, then by Theorem 3.3, P is an irreducible quasi-pairing of the transversal $[\![0,4n]\!]_{\text{even}}$ of $\mathrm{mc}(\underline{4n+1})$, which is also a transversal of $\mathrm{mc}(\underline{4n+1} - v)$ (see Fact 4.1(2)). Therefore, P is an irreducible quasi-pairing of a transversal of $\mathrm{mc}(\underline{4n+1} - v)$, and hence Q is an irreducible quasi-pairing of a transversal of $\mathrm{mc}(\underline{4n})$. So Q also satisfies Condition (C1). Thus,

$$Q \text{ satisfies Conditions (C1) and (C3) for every } v \in [\![1,4n-1]\!]_{\text{odd}}. \tag{6.3}$$

Let p and q denote $\min(B_P)$ and $\max(B_P)$, respectively. Since $T - v$ is indecomposable if and only if Q satisfies Conditions (C1)–(C4), then (2.2) follows directly from (6.2), (6.3), and the following three claims.

Claim 2. *If $v \in [\![1, 4n-1]\!]_{\text{odd}} \setminus \{p+1, q-1\}$, then Q satisfies Conditions (C2) and (C4).*

Claim 3. *If $v = p+1$, then Q satisfies Conditions (C2) and (C4) if and only if $p+2 \notin B_P$.*

Claim 4. *If $v = q-1$, then Q satisfies Conditions (C2) and (C4) if and only if $q-2 \notin B_P$.*

By interchanging $\underline{4n+1}$ and its dual $(\underline{4n+1})^\star$, Claims 3 and 4 are equivalent. So to conclude the proof, it only remains to prove Claims 2 and 3.

For Claim 2, suppose $v \in [\![1, 4n-1]\!]_{\text{odd}} \setminus \{p+1, q-1\}$. Suppose for a contradiction that Condition (C2) is not satisfied, that is, $v_Q^+ = v_Q^- + 1$. Since $\cup Q = [\![0, v-1]\!]_{\text{even}} \cup [\![v, 4n-1]\!]_{\text{odd}}$, we get $v_Q^- = v-1$ and $v_Q^+ = v$. Since $\pi(v_P^-) = v_Q^-$ and $\pi(v_P^+) = v_Q^+$, it follows that $v_P^- = v-1$ and $v_P^+ = v+1$. Thus $v = v_P^- + 1 = v_P^+ - 1$. Since $p = v_P^-$ or $q = v_P^+$ because $|B_P| = 3$ and $v_P^- \le v_P^+$, this contradicts $v \notin \{p+1, q-1\}$. Therefore Condition (C2) is satisfied. Now suppose to the contrary that $\{v-1, v+1\} \in P$. If $\hat{v}_P \notin \{v-1, v+1\}$, then the nontrivial interval $\{v-1, v+1\}$ of $[\![0, 4n]\!]_{\text{even}}$ is a block of P_{part}, which contradicts that the quasi-pairing P of $[\![0, 4n]\!]_{\text{even}}$ is irreducible. Therefore $\hat{v}_P \in \{v-1, v+1\}$, and hence $\{v-1, v+1\} \subseteq B_P$. Since $|B_P| = 3$, it follows that $p = v-1$ or $q = v+1$, which contradicts that $v \neq p+1$ and $v \neq q-1$. Therefore $\{v-1, v+1\} \notin P$. Since $\pi(v-1) = v-1$ and $\pi(v+1) = v$, this implies $\{v-1, v\} \notin Q$. Since $v-1$ and v are the unique consecutive elements of $\cup Q = [\![0, v-1]\!]_{\text{even}} \cup [\![v, 4n-1]\!]_{\text{odd}}$, it follows that Condition (C4) is also satisfied.

For Claim 3, consider the quasi-pairing Q with $v = p+1$. Suppose $p+2 \in B_P$. We will verify that Q does not satisfy both Conditions (C2) and (C4). First suppose $\hat{v}_P \notin \{p, p+2\}$. In this instance, by the choice of p, $v_P^- = p$ and $v_P^+ = p+2$, and hence $v_Q^- = p$ and $v_Q^+ = p+1$. So Condition (C2) is not satisfied. Second suppose $\hat{v}_P \in \{p, p+2\}$. Since $\{p, p+2\} \subseteq B_P$, it follows that $\{p, p+2\} \in P$ and hence $\{p, p+1\} \in Q$. Moreover, since $\hat{v}_Q \in \{p, p+1\}$ because $\hat{v}_P \in \{p, p+2\}$, and since $\cup Q = [\![0, p]\!]_{\text{even}} \cup [\![p+1, 4n-1]\!]_{\text{odd}}$, then $\{\hat{v}_Q - 1, \hat{v}_Q + 1\} \nsubseteq \cup Q$. Thus Condition (C4) is not satisfied. Conversely, suppose that Q does not satisfy both Conditions (C2) and (C4). Since p and $p+1$ are the unique consecutive elements in $\cup Q$, it follows that $v_Q^+ = p+1$ if Condition (C2) is not satisfied, and that $\{p, p+1\} \in Q$ if Condition (C4) is not satisfied. So $v_Q^+ = p+1$ or $\{p, p+1\} \in Q$. In the first instance, $v_P^+ = p+2$ because $\pi(v_P^+) = v_Q^+$ and $\pi(p+2) = p+1$, and hence $p+2 \in B_P$. In the second one, $\{p, p+2\} \in P$ because $\pi(p) = p$ and $\pi(p+2) = p+1$. Since $p \in B_P$, it follows that $p+2 \in B_P$. ☐

We also need the following lemma.

Notation 6.1. Given an integer n with $n \ge 2$, we denote by $\mathcal{U}_{\underline{4n}}$ the set of (indecomposable) tournaments $\widetilde{T - x}$, where $T \in \mathcal{T}_{\underline{4n+1}}$ and $x \in \text{supp}(T)$.

Lemma 6.1. *For every integer $n \ge 2$, we have $\mathcal{U}_{\underline{4n}} = \mathcal{T}'_{\underline{4n}} \cup \mathcal{T}''_{\underline{4n}}$.*

Proof. Let $U \in \mathcal{T}'_{\underline{4n}} \cup \mathcal{T}''_{\underline{4n}}$. If $U \in \mathcal{T}''_{\underline{4n}}$, then $U = \widetilde{T - x}$ for some $T \in \mathcal{T}''_{\underline{4n+1}}$ and $x \in \text{supp}(T)$ (see Notation 2.3). Since $\mathcal{T}''_{\underline{4n+1}} \subseteq \mathcal{T}_{\underline{4n+1}}$ (see Theorem 2.3), we obtain $U \in \mathcal{U}_{\underline{4n}}$. Now suppose $U \in \mathcal{T}'_{\underline{4n}}$. In this instance, $U = \widetilde{T - e}$ for some $T \in \mathcal{T}_{\underline{4n+3}}$ and $e \in \binom{[\![1, 4n+1]\!]_{\text{odd}}}{3}$ (see Notation 2.3). Note that $|\overline{e}| = 4n \ge 8$ because $n \ge 2$. It follows from Theorem 2.1 that $T - e$ is indecomposable, and hence U is indecomposable as well. Let x denote $\min(e)$ and let $e' := e \setminus \{x\}$. Consider the tournament $T' := \widetilde{T - e'}$. By construction, $T' \in \mathcal{T}'_{\underline{4n+1}}$ and $U = \widetilde{T' - x}$. By Theorem 2.3, $T' \in \mathcal{T}_{\underline{4n+1}}$. Since U and T' are indecomposable, it follows that $T' - x$ is also indecomposable, that is, $x \in \operatorname{supp}(T')$. Thus $U \in \mathcal{U}_{\underline{4n}}$. So $\mathcal{T}'_{\underline{4n}} \cup \mathcal{T}''_{\underline{4n}} \subseteq \mathcal{U}_{\underline{4n}}$.

Conversely, let $U \in \mathcal{U}_{\underline{4n}}$. By Theorem 2.3, $U = \widetilde{T - x}$ for some $T \in \mathcal{T}'_{\underline{4n+1}} \cup \mathcal{T}''_{\underline{4n+1}}$ and $x \in \operatorname{supp}(T)$. If $T \in \mathcal{T}''_{\underline{4n+1}}$, then $U \in \mathcal{T}''_{\underline{4n}}$ (see Notation 2.3). Now suppose $T \in \mathcal{T}'_{\underline{4n+1}}$. In this instance, $T = \widetilde{\Gamma - \{i,j\}}$ for some $\Gamma \in \mathcal{T}_{\underline{4n+3}}$ and distinct $i, j \in [\![1, 4n+1]\!]_{\mathrm{odd}}$. By construction, we have $U = \widetilde{\Gamma - e}$ where $e = \{i,j,k\}$ for some $k \in [\![0, 4n+2]\!] \setminus \{i,j\}$. (It suffices to take $k = \pi^{-1}_{\Gamma - \{i,j\}}(x)$ (see Notation 2.1).) Recall that $\Gamma = \operatorname{Inv}(\underline{4n+3}, P)$ for some pairing P of $[\![0, 4n+2]\!]_{\mathrm{even}}$ (see Theorem 2.1), and hence $\Gamma - e = \operatorname{Inv}(\underline{4n+3} - e, Q)$ for some $Q \subseteq P$. Clearly $Q = P$ iff k is odd. Since $\Gamma - e$ is indecomposable because U is, $|Q| \geq \delta(\underline{4n+3} - e)$. But $\delta(\underline{4n+3} - e) = \delta(\underline{4n}) = n+1$ (see (2.1)) because $\underline{4n+3} - e \cong \underline{4n}$. Since $Q \subseteq P$ and $|P| = n+1$, it follows that $Q = P$ and hence $k \in [\![1, 4n+1]\!]_{\mathrm{odd}}$. Therefore $U \in \mathcal{T}'_{\underline{4n}}$. This completes the proof. □

Proof of Theorem 2.4. Let n be an integer such that $n \geq 2$. We have to prove that $\{\mathcal{T}'_{\underline{4n}}, \mathcal{T}''_{\underline{4n}}\}$ is a partition of $\mathcal{T}_{\underline{4n}}$. Clearly $\mathcal{T}'_{\underline{4n}} \neq \varnothing$ and $\mathcal{T}''_{\underline{4n}} \neq \varnothing$. By the definitions of $\mathcal{T}'_{\underline{4n}}$ and $\mathcal{T}''_{\underline{4n}}$ (see Notation 2.3), a tournament of $\mathcal{T}'_{\underline{4n}}$ (resp. of $\mathcal{T}''_{\underline{4n}}$) is obtained from $\underline{4n}$ by reversing a pairing (resp. a quasi-pairing). Therefore $\mathcal{T}'_{\underline{4n}} \cap \mathcal{T}''_{\underline{4n}} = \varnothing$. By Lemma 6.1, to prove that $\mathcal{T}_{\underline{4n}} = \mathcal{T}'_{\underline{4n}} \cup \mathcal{T}''_{\underline{4n}}$, we may prove instead that $\mathcal{T}_{\underline{4n}} = \mathcal{U}_{\underline{4n}}$.

Let $U \in \mathcal{U}_{\underline{4n}}$. We have $U = \widetilde{T - x}$ for some $T \in \mathcal{T}_{\underline{4n+1}}$ and $x \in \operatorname{supp}(T)$. Let P be the subset of $\binom{[\![0,4n]\!]}{2}$ such that $T = \operatorname{Inv}(\underline{4n+1}, P)$. As $T \in \mathcal{T}_{\underline{4n+1}}$, we have $|P| = n+1$ (see Remark 2.1). Because $T - x$ is indecomposable and $T - x = \operatorname{Inv}(\underline{4n+1} - x, P')$ for some $P' \subseteq P$, we have $\delta(\underline{4n+1} - x) \leq |P'| \leq |P|$. Since $\delta(\underline{4n+1} - x) = \delta(\underline{4n}) = n+1$ (see (2.1)) because $\underline{4n+1} - x \cong \underline{4n}$, and since $|P| = n+1$, it follows that $P' = P$. Since $U = \widetilde{T - x}$ and $T - x = \operatorname{Inv}(\underline{4n+1} - x, P)$, we obtain $U = \operatorname{Inv}(\underline{4n}, \widetilde{P}_{T-x})$ (see Notation 2.1). Because U is indecomposable and $|\widetilde{P}_{T-x}| = |P| = n+1$, it follows that $U \in \mathcal{T}_{\underline{4n}}$ (see Remark 2.1).

Conversely, let $U \in \mathcal{T}_{\underline{4n}}$. To prove that $U \in \mathcal{U}_{\underline{4n}}$, we consider the subset Q of $\binom{V(\underline{4n})}{2}$ such that $U = \operatorname{Inv}(\underline{4n}, Q)$. We have $|Q| = n+1$ (see Remark 2.1) and $|\cup Q| \in \{2n+1, 2n+2\}$ (see Fact 3.1(2)). Thus, Q is a pairing or a quasi-pairing of $\cup Q$. Consider the Δ-decomposition $D := \{\{0\}, \{4n-1\}\} \cup \{\{2i-1, 2i\} : 1 \leq i \leq 2n-1\}$ of $\underline{4n}$ (see Example 3.1). By Fact 3.1(1),

$$\cup Q \text{ is a transversal of } D. \tag{6.4}$$

In particular, $\{0, 4n-1\} \subseteq \cup Q$. So consider the nonempty intersection $X := [\![1, 4n-1]\!]_{\mathrm{odd}} \cap (\cup Q)$. Let p denote $\min(X)$. If $p = 1$, then $\{0, 1\} \subseteq \cup Q$. If $p > 1$, then since $\{p-2, p-1\} \in D$ and $p - 2 \notin \cup Q$ because $p = \min(X)$, it follows from (6.4) that $\{p-1, p\} \subseteq \cup Q$. Thus in all cases

$$\{p-1, p\} \subseteq \cup Q. \tag{6.5}$$

Consider the tournament Γ obtained from U by adding one new vertex $(p - \frac{1}{2})$ in the following manner: $\Gamma = \operatorname{Inv}(\underline{W}, Q)$, where $W := [\![0, 4n-1]\!] \cup \{p - \frac{1}{2}\}$. Note that $U = \Gamma - (p - \frac{1}{2})$. Now consider the tournament $T := \widetilde{\Gamma}$. By construction, we have $U = \widetilde{T - p}$. Since U is indecomposable, so to prove that $U \in \mathcal{U}_{\underline{4n}}$, it remains only to prove that $T \in \mathcal{T}_{\underline{4n+1}}$. Since $\Gamma = \operatorname{Inv}(\underline{W}, Q)$ and $\widetilde{\underline{W}} = \underline{4n+1}$, we have $T = \operatorname{Inv}(\underline{4n+1}, \widetilde{Q}_\Gamma)$ (see Notation 2.1). Since $|\widetilde{Q}_\Gamma| = |Q| = n+1$, so by Remark 2.1, to prove that $T \in \mathcal{T}_{\underline{4n+1}}$, we only have to show that T is indecomposable, or, equivalently, that Γ is indecomposable.

Since $U = \mathrm{Inv}(\underline{4n}, Q)$ is indecomposable, and since Q is a pairing or a quasi-pairing of $\cup Q$, it follows from Theorems 3.1 and 3.2 that Q is irreducible and that $\cup Q$ is a transversal of $\mathrm{mc}(\underline{4n})$. Moreover, because

$$\mathrm{mc}(\underline{W}) = \begin{cases} \mathrm{mc}(\underline{4n}) \cup \{\{p-1, p-\frac{1}{2}\}, \{p-\frac{1}{2}, p\}\} & \text{if } p \in [\![3, 4n-3]\!]_{\mathrm{odd}}, \\ \mathrm{mc}(\underline{4n}) \cup \{\{\frac{1}{2}, 1\}\} & \text{if } p = 1, \\ \mathrm{mc}(\underline{4n}) \cup \{\{4n-2, 4n-\frac{3}{2}\}\} & \text{if } p = 4n-1, \end{cases}$$

and $\cup Q$ is a transversal of $\mathrm{mc}(\underline{4n})$, it follows from (6.5) that $\cup Q$ is also a transversal of $\mathrm{mc}(\underline{W})$. Thus, Q is an irreducible pairing or quasi-pairing of a transversal of $\mathrm{mc}(\underline{W})$. If Q is a pairing, it follows from Theorem 3.1, that the tournament $\mathrm{Inv}(\underline{W}, Q)$, which is Γ, is indecomposable as desired. To finish, suppose that Q is a quasi-pairing. By Theorem 3.2, at least one of the tournaments Γ, $\Gamma - v_Q^-$ or $\Gamma - v_Q^+$ is indecomposable. To show that both $\Gamma - v_Q^-$ and $\Gamma - v_Q^+$ are decomposable, which implies that Γ is again indecomposable, consider $v \in \{v_Q^-, v_Q^+\}$. As $\Gamma = \mathrm{Inv}(\underline{W}, Q)$, we have $\Gamma - v = \mathrm{Inv}(\underline{W} - v, P)$ where $P = Q \setminus \{\{v, \hat{v}_Q\}\}$. Since $|P| = |Q| - 1 = n$, and since $\delta(\underline{W} - v) = \delta(\underline{4n}) = n + 1$ (see (2.1)) because $\underline{W} - v \cong \underline{4n}$, we obtain $|P| < \delta(\underline{W} - v)$ and hence $\Gamma - v$ is decomposable. Thus, both $\Gamma - v_Q^-$ and $\Gamma - v_Q^+$ are decomposable. Therefore T is indecomposable. □

Declarations

Conflicts of interest The authors have no financial or non-financial interests to declare.

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