Advances in Pure and Applied Mathematics
2026, vol. 17, n° 1, 43-59 pages, DOI: 10.21494/ISTE.OP.2026.1393 ISTE OpenScience

Generalized Pitt's inequality for the Gabor transform

L'inégalité de Pitts généralisée pour la transformation de Gabor

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ABSTRACT. The generalized forms of Pitt's inequality for the L^p-Gabor transform on the groups of the form $\mathbb{R}^n$; $\mathbb{R}^n \times K$, K being a Lie group of type I, in particular, a connected nilpotent Lie group; Heisenberg motion group and diamond Lie groups have been established.
2020 Mathematics Subject Classification. 43A32, 43A30.
KEYWORDS. Diamond Lie groups, Fourier transform, Gabor transform, Hausdorff-Young inequality, Heisenberg motion group, Nilpotent Lie groups, Pitt's inequality

1. Introduction

Pitt's inequality in harmonic analysis provides bounds that relate a function and its Fourier transform. It helps to establish embeddings between weighted L^p-spaces and spaces of Fourier transforms.

Let $1 < p \leq 2$ and q be the conjugate of p, i.e., $\dfrac{1}{p} + \dfrac{1}{q} = 1$. The space of all Schwartz class functions on $\mathbb{R}^n$ will be denoted by $\mathbb{S}(\mathbb{R}^n)$. The well known Pitt's inequality [3, 4] can be stated as follows:

PROPOSITION 1.1. For $f \in \mathbb{S}(\mathbb{R}^n)$ and $0 \leq \beta < n$, we have

$$\int\limits_{\mathbb{R}^n} \|\zeta\|^{-\beta} \, |\widehat{f}(\zeta)|^2 \, d\zeta \leq \mathcal{C}_\beta \int\limits_{\mathbb{R}^n} \|y\|^{\beta} \, |f(y)|^2 \, dy,$$

where $\mathcal{C}_\beta = \pi^\beta \left[\dfrac{\Gamma\left(\dfrac{n-\beta}{4}\right)}{\Gamma\left(\dfrac{n+\beta}{4}\right)} \right]^2$.

The objective here is to consider the generalized Pitt's inequality for the L^p-Gabor transform $G\psi f$ with $f \in \mathbb{S}(G)$ and $\psi \in L^q(G)$ for certain Lie groups G. The Hausdorff-Young inequality [4] on $\mathbb{R}^n$ takes the following form:

$$\left(\int\limits_{\mathbb{R}^n} |\widehat{\varphi}(\zeta)|^q \, d\zeta \right)^{1/q} \leq (A_p)^n \left(\int\limits_{\mathbb{R}^n} |\varphi(y)|^p \, dy \right)^{1/p},$$

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for all $\varphi \in L^p(\mathbb{R}^n)$, where $A_p = p^{\frac{1}{2p}} q^{-\frac{1}{2q}}$ is the Babenko-Beckner constant and $A_p < 1$ when $1 < p < 2$. Proceeding as in the proof of Pitt's inequality [5]*p. 1880 and using the above Hausdorff-Young inequality, the following generalized Pitt's inequality for the Fourier transform on $\mathbb{R}^n$ can be obtained.

THEOREM 1.2. For $f \in \mathbb{S}(\mathbb{R}^n)$, $1 < p \leq 2$, $0 \leq \beta < \dfrac{n}{q}$ and $n > 0$, we have

$$\left(\int_{\mathbb{R}^n} \|\zeta\|^{-\beta q} \, |\widehat{f}(\zeta)|^q \, d\zeta \right)^{1/q} \leq \mathcal{C}_\beta \left(\int_{\mathbb{R}^n} \|y\|^{\beta p} \, |f(y)|^p \, dy \right)^{1/p},$$

where $\mathcal{C}_\beta = \tilde{\mathcal{C}}_\beta \, (A_p)^n$ and $\tilde{\mathcal{C}}_\beta = \pi^{n/2} \left[\dfrac{\Gamma\left(\frac{\beta}{2}\right) \Gamma\left(\frac{n}{2p} - \frac{\beta}{2}\right) \Gamma\left(\frac{n}{2q}\right)}{\Gamma\left(\frac{n}{2} - \frac{\beta}{2}\right) \Gamma\left(\frac{n}{2q} + \frac{\beta}{2}\right) \Gamma\left(\frac{n}{2p}\right)} \right]$.

In [3], Pitt's inequality has been proved for the Fourier transform on several classes of groups.

In Section 2, we shall briefly discuss some important results that shall be used throughout the paper. The generalized Pitt's inequality for the Gabor transform on $\mathbb{R}^n$ is proved in the next section. The Section 4 incorporates the proof of the generalized Pitt's inequality for the Fourier transform and the Gabor transform on the groups of the form $\mathbb{R}^n \times K$, K being a Lie group of type I, in particular, a connected nilpotent Lie group. In the next section, the generalized Pitt's inequality for the Fourier transform and the Gabor transform on Heisenberg motion groups has been established. Finally, in Section 5, we prove the generalized Pitt's inequality for the Fourier transform and the Gabor transform on diamond Lie groups.

2. Preliminaries and Notations

Let G be a separable, unimodular, second-countable locally compact group of type I having left Haar measure μ_G. A continuous unitary representation of G is a pair $(\pi, \mathcal{H}_\pi)$, where $\mathcal{H}_\pi$ is a Hilbert space and π is a homomorphism of G into $\mathcal{U}(\mathcal{H}_\pi)$, the space of unitary operators on $\mathcal{H}_\pi$. We denote by $\widehat{G}$ the set of equivalence classes of irreducible unitary representations of G. The Haar measure μ_G on G is fixed that uniquely determines the Plancherel measure $\mu_{\widehat{G}}$ on $\widehat{G}$.

2.1. *Hausdorff-Young Inequality*

For $\varphi \in L^1(G)$, the Fourier transform of φ is given by

$$\widehat{\varphi}(\pi) = \pi(\varphi) = \int_G \varphi(y) \, \pi(y)^* \, d\mu_G(y).$$

Let $L^\infty(\widehat{G})$ be the space of all measurable fields of bounded operators on $\widehat{G}$. It is easy to see that

$$\|\widehat{\varphi}\|_\infty \leq \|\varphi\|_1.$$

For $\varphi \in L^2(G)$, using Plancherel formula, we have

$$\|\varphi\|_2^2 = \int_{\widehat{G}} \operatorname{tr}\left[\pi(\varphi)^* \pi(\varphi)\right] d\mu_{\widehat{G}}(\pi).$$

The q-th Schatten norm $\|T\|_{c_q}$ of a Hilbert-Schmidt operator T is defined by

$$\|T\|_{c_q}^q = \operatorname{tr}\left[(T^*\, T)^{q/2}\right],$$

Let $L^q(\widehat{G})$ be the space of μ_G-measurable field of bounded operators F on $\widehat{G}$ satisfying

$$\|F\|_q = \left(\int\limits_{\widehat{G}} \|F(\pi)\|_{c_q}^q \, d\mu_{\widehat{G}}(\pi)\right)^{1/q} < \infty.$$

For $\varphi \in L^p(G)$, using Riesz interpolation theorem, we have $\widehat{\varphi} \in L^q(\widehat{G})$ and

$$\|\widehat{\varphi}\|_q \leq C\|\varphi\|_p, \tag{2.1}$$

where the constant $C \leq 1$ and the above inequality is called Hausdorff-Young inequality. The smallest possible value of C for which (2.1) holds is denoted by $H_p(G)$.

2.2. *Gabor Transform*

Let $C_{00}(G)$ denote the set of all continuous complex-valued functions on G having compact support. For $f \in C_{00}(G)$, window function $\psi \in L^2(G)$ and $(x, \pi) \in G \times \widehat{G}$, the Gabor transform of f with respect to ψ is given by

$$G_\psi f(x, \pi) := \int\limits_G f(y)\, \overline{\psi}(x^{-1}y)\, \pi(y)^*\, d\mu_G(y). \tag{2.2}$$

For $x \in G$, consider the function $f_x^\psi : G \to \mathbb{C}$ defined by $f_x^\psi(y) := f(y)\, \overline{\psi}(x^{-1}y)$. For $f \in C_{00}(G)$ and $\psi \in L^2(G)$, it can be observed that $f_x^\psi \in L^1(G) \cap L^2(G)$ and it satisfies

$$\widehat{f_x^\psi}(\pi) = G_\psi f(x, \pi). \tag{2.3}$$

As in [2], the L^p-Gabor transform for $f \in C_{00}(G)$ and $\psi \in L^q(G)$ is given by (2.2) and it satisfies

$$\|G_\psi f\|_q \leq \|\psi\|_q\, \|f\|_p. \tag{2.4}$$

For $\psi \in L^q(G)$, the operator $G_\psi : C_{00}(G) \to L^q(G \times \widehat{G})$ can be extended to a bounded linear operator from $L^p(G)$ into $L^q(G \times \widehat{G})$ which we still denote by G_ψ and it satisfies (2.3) and (2.4) for all $f \in L^p(G)$.

2.3. *Minkowski's inequality*

Consider σ-finite measure spaces (X, μ_X) and (Y, μ_Y) and let φ be a $\mu_X \times \mu_Y$-measurable function. Then, for $0 < r \leq s < \infty$, we have

$$\left(\int\limits_X \left\{\int\limits_Y |\varphi(x,y)|^r \, d\mu_Y(y)\right\}^{s/r} d\mu_X(x)\right)^{1/s} \leq \left(\int\limits_Y \left\{\int\limits_X |\varphi(x,y)|^s \, d\mu_X(x)\right\}^{r/s} d\mu_Y(y)\right)^{1/r}. \tag{2.5}$$

For more details, one may refer to [8].

3. Euclidean Group $\mathbb{R}^n$

In this section, we shall discuss the generalized Pitt's inequality for the Gabor transform on $\mathbb{R}^n$. Let $\mathcal{L}(\mathbb{R}^n)$ be the space of slowly growing functions, i.e., it contains the elements $\psi \in C^\infty(\mathbb{R}^n)$ such that for every $\alpha \in \mathbb{N}_0^n$, there exists C_α and $N_\alpha \in I$ such that $|\partial^\alpha f(x)| \leq C_\alpha (1+|x|^2)^{N_\alpha}$. Note that $\mathbb{S}(\mathbb{R}^n) \subseteq \mathcal{L}(\mathbb{R}^n)$.

THEOREM 3.1. For $f \in \mathbb{S}(\mathbb{R}^n)$, $1 < p \leq 2$, $0 \leq \beta < \dfrac{n}{q}$ and $n > 0$, we have

$$\left(\int\limits_{\mathbb{R}^{2n}} \|\zeta\|^{-\beta q} \, |G_\psi f(x,\zeta)|^q \, dx \, d\zeta \right)^{1/q} \leq \mathcal{C}_\beta \, \|\psi\|_q \left(\int\limits_{\mathbb{R}^n} \|y\|^{\beta p} \, |f(y)|^p \, dy \right)^{1/p},$$

where $\mathcal{C}_\beta$ is as in Theorem 1.2 and $\psi \in \mathcal{L}(\mathbb{R}^n)$ or $\psi \in L^q(\mathbb{R}^n)$.

Proof. First assume that $f \in \mathbb{S}(\mathbb{R}^n)$ and $\psi \in \mathcal{L}(\mathbb{R}^n)$. For every $x \in \mathbb{R}^n$, the function $f_x^\psi \in \mathbb{S}(\mathbb{R}^n)$. Using Theorem 1.2, we have for $x \in \mathbb{R}^n$ and $0 \leq \beta < \dfrac{n}{q}$,

$$\begin{aligned}
\int\limits_{\mathbb{R}^n} \|\zeta\|^{-\beta q} \, |G_\psi f(x,\zeta)|^q \, d\zeta &= \int\limits_{\mathbb{R}^n} \|\zeta\|^{-\beta q} \, |\widehat{f_x^\psi}(\zeta)|^q \, d\zeta \\
&\leq \mathcal{C}_\beta^q \left(\int\limits_{\mathbb{R}^n} \|y\|^{\beta p} \, |f_x^\psi(y)|^p \, dy \right)^{q/p} \\
&= \mathcal{C}_\beta^q \left(\int\limits_{\mathbb{R}^n} \|y\|^{\beta p} \, |f(y)|^p \, |\psi(y-x)|^p \, dy \right)^{q/p}. \qquad (3.1)
\end{aligned}$$

Using Fubini's theorem, (3.1) and Minkowski's inequality (2.5), we have

$$\begin{aligned}
\int\limits_{\mathbb{R}^{2n}} \|\zeta\|^{-\beta q} \, |G_\psi f(x,\zeta)|^q \, dx \, d\zeta &= \int\limits_{\mathbb{R}^n} \left(\int\limits_{\mathbb{R}^n} \|\zeta\|^{-\beta q} \, |G_\psi f(x,\zeta)|^q \, d\zeta \right) dx \\
&\leq \mathcal{C}_\beta^q \int\limits_{\mathbb{R}^n} \left(\int\limits_{\mathbb{R}^n} \|y\|^{\beta p} \, |f(y)|^p \, |\psi(y-x)|^p \, dy \right)^{q/p} dx \\
&\leq \mathcal{C}_\beta^q \left(\int\limits_{\mathbb{R}^n} \left(\int\limits_{\mathbb{R}^n} \|y\|^{\beta q} \, |f(y)|^q \, |\psi(y-x)|^q \, dx \right)^{p/q} dy \right)^{q/p} \\
&= \mathcal{C}_\beta^q \left(\int\limits_{\mathbb{R}^n} \|y\|^{\beta p} \, |f(y)|^p \left(\int\limits_{\mathbb{R}^n} |\psi(y-x)|^q \, dx \right)^{p/q} dy \right)^{q/p} \\
&= \mathcal{C}_\beta^q \, \|\psi\|_q^q \left(\int\limits_{\mathbb{R}^n} \|y\|^{\beta p} \, |f(y)|^p \, dy \right)^{q/p} < \infty,
\end{aligned}$$

so the map

$$x \mapsto \int\limits_{\mathbb{R}^n} \|\zeta\|^{-\beta q} \, |G_\psi f(x,\zeta)|^q \, d\zeta$$

is integrable for almost all $x \in \mathbb{R}^n$.

Hence, for all $f \in \mathbb{S}(\mathbb{R}^n)$ and $\psi \in \mathcal{L}(\mathbb{R}^n)$, we have

$$\left(\int\limits_{\mathbb{R}^{2n}} \|\zeta\|^{-\beta q} \, |G_\psi f(x,\zeta)|^q \, dx \, d\zeta \right)^{1/q} \leq \mathcal{C}_\beta \, \|\psi\|_q \left(\int\limits_{\mathbb{R}^n} \|y\|^{\beta p} \, |f(y)|^p \, dy \right)^{1/p}.$$

We now suppose that $f \in \mathbb{S}(\mathbb{R}^n)$ and $\psi \in L^q(\mathbb{R}^n)$. Since $\mathbb{S}(\mathbb{R}^n)$ is dense in $L^q(\mathbb{R}^n)$, there exists $\psi_m \in \mathbb{S}(\mathbb{R}^n)$ such that $\|\psi_m - \psi\|_q \to 0$ as $m \to \infty$. Therefore for all $m \in \mathbb{N}$, we have

$$\left(\int\limits_{\mathbb{R}^{2n}} \|\zeta\|^{-\beta q} \, |G_{\psi_m} f(x,\zeta)|^q \, dx \, d\zeta \right)^{1/q} \leq \mathcal{C}_\beta \, \|\psi_m\|_q \left(\int\limits_{\mathbb{R}^n} \|y\|^{\beta p} \, |f(y)|^p \, dy \right)^{1/p}. \tag{3.2}$$

For each $(x,\zeta) \in \mathbb{R}^{2n}$, we have

$$|G_{\psi_m} f(x,\zeta) - G_\psi f(x,\zeta)| = |G_{\psi_m - \psi} f(x,\zeta)| \leq \|f\|_p \, \|\psi_m - \psi\|_q \to 0 \text{ as } m \to \infty.$$

It means that $\|\zeta\|^{-\beta q} \, |G_{\psi_m} f(x,\zeta)|^q \to \|\zeta\|^{-\beta q} \, |G_\psi f(x,\zeta)|^q$ pointwise for all $x \in \mathbb{R}^n$ and almost all $\zeta \in \mathbb{R}^n$ as $m \to \infty$. By Fatou's lemma and then using inequality (3.2), we obtain

$$\begin{aligned} \left(\int\limits_{\mathbb{R}^{2n}} \|\zeta\|^{-\beta q} \, |G_\psi f(x,\zeta)|^q \, dx \, d\zeta \right)^{1/q} &\leq \liminf_{m\to\infty} \left(\int\limits_{\mathbb{R}^{2n}} \|\zeta\|^{-\beta q} \, |G_{\psi_m} f(x,\zeta)|^q \, dx \, d\zeta \right)^{1/q} \\ &\leq \mathcal{C}_\beta \, \liminf_{m\to\infty} \|\psi_m\|_q \left(\int\limits_{\mathbb{R}^n} \|y\|^{\beta p} \, |f(y)|^p \, dy \right)^{1/p} \\ &\leq \mathcal{C}_\beta \, \|\psi\|_q \left(\int\limits_{\mathbb{R}^n} \|y\|^{\beta p} \, |f(y)|^p \, dy \right)^{1/p}. \end{aligned}$$

□

4. $\mathbb{R}^n \times K$, K a Lie group of type I

Let K be a Lie group of type I, then the Hausdorff-Young inequality on K takes the following form:

$$\int\limits_{\widehat{K}} \|\pi(\varphi)\|_{c_q}^q \, d\mu_{\widehat{K}}(\pi) \leq \left(\int\limits_K |\varphi(h)|^p \, d\mu_K(h) \right)^{q/p}, \tag{4.1}$$

for all $\varphi \in L^p(K)$. We now prove the generalized Pitt's inequality for the Fourier transform and the Gabor transform on $G = \mathbb{R}^n \times K$, where $n > 0$.

THEOREM 4.1. For $f \in \mathbb{S}(G)$, $1 < p \le 2$, $0 \le \beta < \dfrac{n}{q}$ and $n > 0$, we have

$$\left(\int\limits_{\widehat{G}} \|\zeta\|^{-\beta q}\, \|\widehat{f}(\zeta,\pi)\|_{c_q}^q \, d\zeta \, d\mu_{\widehat{K}}(\pi)\right)^{1/q} \le \mathcal{C}_\beta \left(\int\limits_G \|y\|^{\beta p}\, |f(y,h)|^p \, dy \, d\mu_K(h)\right)^{1/p},$$

where $\mathcal{C}_\beta$ is as in Theorem 1.2.

Proof. For $f \in \mathbb{S}(G)$ and $\zeta \in \mathbb{R}^n$, define $\Phi_\zeta : K \to \mathbb{C}$ by $\Phi_\zeta(h) = \int_{\mathbb{R}^n} f(y,h)\, e^{-2\pi i \zeta\cdot y}\, dy$ for all $h \in K$.

Then $\Phi_\zeta \in L^p(K)$ for all $\zeta \in \mathbb{R}^n$ and by Hausdorff-Young inequality (4.1), we have

$$\begin{aligned}\int\limits_{\widehat{K}} \|\zeta\|^{-\beta q}\, \|\widehat{f}(\zeta,\pi)\|_{c_q}^q \, d\mu_{\widehat{K}}(\pi) = \|\zeta\|^{-\beta q} \int\limits_{\widehat{K}} \|\pi(\Phi_\zeta)\|_{c_q}^q \, d\mu_{\widehat{K}}(\pi) &\le \|\zeta\|^{-\beta q} \left(\int\limits_K |\Phi_\zeta(h)|^p \, d\mu_K(h)\right)^{q/p} \\ &= \left(\int\limits_K \|\zeta\|^{-\beta p}\, |\Phi_\zeta(h)|^p \, d\mu_K(h)\right)^{q/p},\end{aligned}$$

for all $0 \ne \zeta \in \mathbb{R}^n$. Integrating both sides w.r.t. ζ and then using Minkowski's inequality (2.5), we have

$$\begin{aligned}\int\limits_{\widehat{G}} \|\zeta\|^{-\beta q}\, \|\widehat{f}(\zeta,\pi)\|_{c_q}^q \, d\zeta \, d\mu_{\widehat{K}}(\pi) &\le \int\limits_{\mathbb{R}^n} \left(\int\limits_K \|\zeta\|^{-\beta p}\, |\Phi_\zeta(h)|^p \, d\mu_K(h)\right)^{q/p} d\zeta \\ &\le \left(\int\limits_K \left(\int\limits_{\mathbb{R}^n} \|\zeta\|^{-\beta q}\, |\Phi_\zeta(h)|^q \, d\zeta\right)^{p/q} d\mu_K(h)\right)^{q/p} \\ &= \left(\int\limits_K \left(\int\limits_{\mathbb{R}^n} \|\zeta\|^{-\beta q}\, |\mathscr{F}_1 f(\zeta,h)|^q \, d\zeta\right)^{p/q} d\mu_K(h)\right)^{q/p}. \qquad (4.2)\end{aligned}$$

For $h \in K$, define $F_h : \mathbb{R}^n \to \mathbb{C}$ by $F_h(y) = f(y,h)$ for all $y \in \mathbb{R}^n$.

Then $F_h \in \mathbb{S}(\mathbb{R}^n)$ and using Theorem 1.2, we have for $h \in K$ and $0 \le \beta < \dfrac{n}{q}$,

$$\begin{aligned}\left(\int\limits_{\mathbb{R}^n} \|\zeta\|^{-\beta q}\, |\mathscr{F}_1 f(\zeta,h)|^q \, d\zeta\right)^{p/q} = \left(\int\limits_{\mathbb{R}^n} \|\zeta\|^{-\beta q}\, |\widehat{F_h}(\zeta)|^q \, d\zeta\right)^{p/q} &\le \mathcal{C}_\beta^p \int\limits_{\mathbb{R}^n} \|y\|^{\beta p}\, |F_h(y)|^p \, dy \\ &= \mathcal{C}_\beta^p \int\limits_{\mathbb{R}^n} \|y\|^{\beta p}\, |f(y,h)|^p \, dy.\end{aligned}$$

On integrating both sides w.r.t. $d\mu_K(h)$, we have

$$\int\limits_K \left(\int\limits_{\mathbb{R}^n} \|\zeta\|^{-\beta q}\, |\mathscr{F}_1 f(\zeta,h)|^q \, d\zeta\right)^{p/q} d\mu_K(h) \le \mathcal{C}_\beta^p \int\limits_G \|y\|^{\beta p}\, |f(y,h)|^p \, dy \, d\mu_K(h).$$

The inequality (4.2) can be written as

$$\int\limits_{\widehat{G}} \|\zeta\|^{-\beta q} \, \|\widehat{f}(\zeta,\pi)\|_{c_q}^q \, d\zeta \, d\mu_{\widehat{K}}(\pi) \leq \mathcal{C}_\beta^q \left(\int\limits_G \|y\|^{\beta p} \, |f(y,h)|^p \, dy \, d\mu_K(h) \right)^{q/p},$$

which establishes the required inequality. □

THEOREM 4.2. For $f \in \mathbb{S}(G)$, $\psi \in L^q(G)$, $1 < p \leq 2$, $0 \leq \beta < \dfrac{n}{q}$ and $n > 0$, we have

$$\left(\int\limits_{G\times\widehat{G}} \|\zeta\|^{-\beta q} \, \|G_\psi f(x,k,\zeta,\pi)\|_{c_q}^q \, dx \, d\mu_K(k) \, d\zeta \, d\mu_{\widehat{K}}(\pi) \right)^{1/q}$$
$$\leq \mathcal{C}_\beta \, \|\psi\|_q \left(\int\limits_G \|y\|^{\beta p} \, |f(y,h)|^p \, dy \, d\mu_K(h) \right)^{1/p},$$

where $\mathcal{C}_\beta$ is as in Theorem 1.2.

Proof. We first assume that $f, \psi \in \mathbb{S}(G)$. For every $(x,k) \in G$, define $f^\psi_{(x,k)} : G \to \mathbb{C}$ by

$$f^\psi_{(x,k)}(y,h) = f(y,h) \, \overline{\psi}(y-x, k^{-1}h) \text{ for all } (y,h) \in G.$$

Then $f^\psi_{(x,k)} \in \mathbb{S}(G)$ for all $(x,k) \in G$ and $G_\psi f(x,k,\zeta,\pi) = \widehat{f^\psi_{(x,k)}}(\zeta,\pi)$ for all $(x,k) \in G$.

Using Theorem 4.1, we have for $(x,k) \in G$ and $0 \leq \beta < \dfrac{n}{q}$,

$$\int\limits_{\widehat{G}} \|\zeta\|^{-\beta q} \, \|G_\psi f(x,k,\zeta,\pi)\|_{c_q}^q \, d\zeta \, d\mu_{\widehat{K}}(\pi)$$
$$= \int\limits_{\mathbb{R}^n\times\widehat{K}} \|\zeta\|^{-\beta q} \, \|\widehat{f^\psi_{(x,k)}}(\zeta,\pi)\|_{c_q}^q \, d\zeta \, d\mu_{\widehat{K}}(\pi)$$
$$\leq \mathcal{C}_\beta^q \left(\int\limits_G \|y\|^{\beta p} \, |f^\psi_{(x,k)}(y,h)|^p \, dy \, d\mu_K(h) \right)^{q/p}$$
$$= \mathcal{C}_\beta^q \left(\int\limits_G \|y\|^{\beta p} \, |f(y,h)|^p \, |\psi(y-x,k^{-1}h)|^p \, dy \, d\mu_K(h) \right)^{q/p}. \tag{4.3}$$

Using Fubini's theorem, (4.3) and Minkowski's inequality (2.5), we can write

$$\int\limits_{G\times\widehat{G}} \|\zeta\|^{-\beta q} \, \|G_\psi f(x,k,\zeta,\pi)\|_{c_q}^q \, dx \, d\mu_K(k) \, d\zeta \, d\mu_{\widehat{K}}(\pi)$$
$$= \int\limits_G \left(\int\limits_{\widehat{G}} \|\zeta\|^{-\beta q} \, \|G_\psi f(x,k,\zeta,\pi)\|_{c_q}^q \, d\zeta \, d\mu_{\widehat{K}}(\pi) \right) dx \, d\mu_K(k)$$

$$
\leq \mathcal{C}_\beta^q \int_G \left(\int_G \|y\|^{\beta p} \, |f(y,h)|^p \, |\psi(y-x,k^{-1}h)|^p \, dy \, d\mu_K(h) \right)^{q/p} dx \, d\mu_K(k)
$$

$$
\leq \mathcal{C}_\beta^q \left(\int_G \left(\int_G \|y\|^{\beta q} \, |f(y,h)|^q \, |\psi(y-x,k^{-1}h)|^q \, dx \, d\mu_K(k) \right)^{p/q} dy \, d\mu_K(h) \right)^{q/p}
$$

$$
= \mathcal{C}_\beta^q \left(\int_G \|y\|^{\beta p} \, |f(y,h)|^p \left(\int_G |\psi(y-x,k^{-1}h)|^q \, dx \, d\mu_K(k) \right)^{p/q} dy \, d\mu_K(h) \right)^{q/p}
$$

$$
= \mathcal{C}_\beta^q \, \|\psi\|_q^q \left(\int_G \|y\|^{\beta p} \, |f(y,h)|^p \, dy \, d\mu_K(h) \right)^{q/p} < \infty,
$$

so the map

$$
(x,k) \mapsto \int_G \|\zeta\|^{-\beta q} \, |G_\psi f(x,k,\zeta,\pi)\|_{c_q}^q \, d\zeta \, d\mu_{\widehat{K}}(\pi)
$$

is integrable for almost all $(x,k) \in G$ and hence for all $f, \psi \in \mathbb{S}(G)$, we have

$$
\left(\int_{G\times\widehat{G}} \|\zeta\|^{-\beta q} \, \|G_\psi f(x,k,\zeta,\pi)\|_{c_q}^q \, dx \, d\mu_K(k) \, d\zeta \, d\mu_{\widehat{K}}(\pi) \right)^{1/q}
$$

$$
\leq \mathcal{C}_\beta \, \|\psi\|_q \left(\int_G \|y\|^{\beta p} \, |f(y,h)|^p \, dy \, d\mu_K(h) \right)^{1/p}.
$$

Arguments similar to those in Theorem 3.1 give the required inequality. $\square$

In particular, let K be a connected nilpotent Lie group having universal covering group $\widetilde{K}$ whose generic coadjoint orbits have the maximal dimension k. If H is the maximal compact subgroup of G, then by [1]*Theorem 3, we have $H_p(K) = (A_p)^{\frac{2\dim(K/H)-k}{2}}$. The Hausdorff-Young inequality on K takes the following form:

$$
\int_{\widehat{K}} \|\pi(\varphi)\|_{c_q}^q \, d\mu_{\widehat{K}}(\pi) \leq (A_p)^{\frac{q(2\dim(K/H)-k)}{2}} \left(\int_K |\varphi(h)|^p \, d\mu_K(h) \right)^{q/p}, \tag{4.4}
$$

for all $\varphi \in L^p(K)$. The generalized Pitt's inequality for the Fourier transform and the Gabor transform on $G = \mathbb{R}^n \times K$, where $n > 0$, take the following forms:

THEOREM 4.3. For $f \in \mathbb{S}(G)$, $1 < p \leq 2$, $0 \leq \beta < \dfrac{n}{q}$ and $n > 0$, we have

$$
\left(\int_{\widehat{G}} \|\zeta\|^{-\beta q} \, \|\widehat{f}(\zeta,\pi)\|_{c_q}^q \, d\zeta \, d\mu_{\widehat{K}}(\pi) \right)^{1/q} \leq \mathcal{E}_\beta \left(\int_G \|y\|^{\beta p} \, |f(y,h)|^p \, dy \, d\mu_K(h) \right)^{1/p},
$$

where $\mathcal{E}_\beta = \tilde{\mathcal{C}}_\beta \, (A_p)^{\frac{2\dim(K/H)-k+2n}{2}}$ and $\tilde{\mathcal{C}}_\beta$ is as in Theorem 1.2.

THEOREM 4.4. For $f \in \mathbb{S}(G)$, $\psi \in L^q(G)$, $1 < p \leq 2$, $0 \leq \beta < \dfrac{n}{q}$ and $n > 0$, we have

$$\left(\int_{G \times \widehat{G}} \|\zeta\|^{-\beta q} \, \|G_\psi f(x,k,\zeta,\pi)\|_{\mathcal{C}_q}^q \, dx \, d\mu_K(k) \, d\zeta \, d\mu_{\widehat{K}}(\pi) \right)^{1/q}$$

$$\leq \mathcal{E}_\beta \, \|\psi\|_q \left(\int_G \|y\|^{\beta p} \, |f(y,h)|^p \, dy \, d\mu_K(h) \right)^{1/p},$$

where $\mathcal{E}_\beta$ is as in Theorem 4.3.

REMARK 4.5. Another special case can be written when K is a Lie group with only finite-dimensional irreducible representations with K_0, the connected component of K, non-compact. In this case, K contains a normal subgroup N of finite index such that $Z(N)$, the center of N, is cocompact in N. So $H_p(Z(N)) \leq (A_p)^m$, for some m. Thus, the above inequalities can be written with $\mathcal{E}_\beta = \tilde{\mathcal{C}}_\beta \, (A_p)^m$.

REMARK 4.6. Let G be a locally compact abelian group, then there exists a locally compact abelian group H containing a compact open subgroup such that G is topologically isomorphic to $\mathbb{R}^n \times H$. If G has a non-compact connected component of identity, then $n > 0$. Hence, by Theorem 4.1 and Theorem 4.2, the generalized Pitt's inequality can be written for the Fourier transform and the Gabor transform on G for Schwartz functions on G with $\mathcal{E}_\beta = \mathcal{C}_\beta$.

5. Connected Nilpotent Lie Group

In this section, we shall prove the generalized Pitt's inequality for the Fourier transform and the Gabor transform for nilpotent Lie groups which are connected but not necessarily simply connected. Let $\widetilde{G}$ be the simply connected covering group of a connected nilpotent Lie group G such that $\Omega : \widetilde{G} \to G$ be the quotient homomorphism. Let $Z(G)$ be the center of G. Let N be a discrete normal subgroup of G such that $G = \widetilde{G}/N$. Let $\mathfrak{g}$ be the Lie algebra of G and $\widetilde{G}$ having center $\mathfrak{z}(\mathfrak{g})$. Consider a strong Malcev basis $\mathscr{B} = \{Y_1, \ldots, Y_n\}$ of $\mathfrak{g}$ through the ascending central series of $\mathfrak{g}$. Let $\mathscr{B}^* = \{Y_1^*, \ldots, Y_n^*\}$ be the dual basis for $\mathscr{B}$. Let $\mathcal{U}$ denote the Zariski open subset of $\mathfrak{g}^*$ of generic elements under the coadjoint action of $\widetilde{G}$ with respect to $\mathscr{B}$. Let $V_J = \mathbb{R}\text{-}\operatorname{span}\{Y_i^* : i \in J\}$ and $V_I = \mathbb{R}\text{-}\operatorname{span}\{Y_i^* : i \in I\}$, where I is the set of jump indices and J is the complement of I in $\{1, \ldots, n\}$. The Plancherel measure on $\widetilde{G}$ is supported by $\mathcal{W} = \mathcal{U} \cap V_J$ which is a cross-section of the generic orbits.

If $Z(G)$ is non-compact, then there is a closed subgroup in $Z(G)$ which is of the form $\mathbb{T}^k \times \mathbb{R}$ such that $\mathbb{T}^k = G^c$ is the maximal compact subgroup of G. Choose a strong Malcev basis $\mathscr{B} = \{Y_1, \ldots, Y_n\}$ of $\mathfrak{g}$ that satisfies $\mathfrak{b} := \mathbb{R}\text{-}\operatorname{span}\{Y_1, \ldots, Y_{k+1}\} \subset \mathfrak{z}(\mathfrak{g})$ and $\Lambda = \oplus_{i=1}^k \mathbb{Z} Y_i$. Thus $\mathfrak{h} = \mathbb{R}\text{-}\operatorname{span}\{Y_1^*, \ldots, Y_k^*\}$ and $\mathfrak{g}_\Lambda^* = \mathbb{Z}\text{-}\operatorname{span}\{Y_1^*, \ldots, Y_k^*\} + \mathfrak{h}^\perp$.

For $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{Z}^k$, let $\lambda Y^* := \lambda_1 Y_1^* + \ldots + \lambda_k Y_k^*$. Let $\tilde{B} = \exp \mathfrak{b}$. For $\eta \in \mathbb{R}$ and $\lambda \in \mathbb{Z}^k$, consider the unitary character $\chi_{\eta,\lambda} = \chi_\psi$ of $\tilde{B}$, where $\psi = \eta Y_{k+1}^* + \lambda Y^* \in \mathfrak{b}^*$ such that

$$\chi_{\eta,\lambda}(\exp(s_1 Y_1 + \ldots + s_{k+1} Y_{k+1})) = e^{2\pi i \langle \psi, \sum_{j=1}^{k+1} s_j Y_j \rangle} = e^{2\pi i (\eta s_{k+1} + \sum_{j=1}^k \lambda_j s_j)}.$$

Let

$$\widehat{\widetilde{G}}_{\chi_{\eta,\lambda}} = \{\delta \in \widehat{\widetilde{G}} : \delta|_{\tilde{B}} = \chi_{\eta,\lambda} \cdot I\}$$

having the Plancherel measure $d\mu_{\chi_{\eta,\lambda}}$. Thus, for $\varphi \in L^1(\tilde{G}/\tilde{B},\psi) \cap L^2(\tilde{G}/\tilde{B},\psi)$, we have

$$\int\limits_{\tilde{G}/\tilde{B}} |\varphi(\nu)|^2 \, d_{\tilde{G}/\tilde{B}}\,\nu = \int\limits_{\widehat{\tilde{G}}_{\chi_{\eta,\lambda}}} \|\delta(\varphi)\|^2_{\mathrm{HS}} \, d\mu_{\chi_{\eta,\lambda}}(\delta). \tag{5.1}$$

We choose a Borel cross-section Y for the cosets of B in G. We can take $Y = \exp\left(\sum_{k+2}^{n} \mathbb{R}Y_j\right)$. Every element $\dot{\nu}$ of G can be uniquely written as

$$\dot{\nu} = yN\dot{b} \equiv (y,\dot{b}) = (y, \dot{h}(\Omega \circ \exp)(sY_{k+1})),$$

where $y \in Y$, $\dot{h} \in H$ and $s \in \mathbb{R}$. We endow Y with the image of the Haar measure $d_{G/B}\dot{\nu}$ of G/B under the Borel isomorphism $yB \mapsto y$ between G/B and Y. Note that $f \circ \Omega \circ \exp$ is a Schwartz function on $\mathfrak{g}/\Lambda$, since $f \in \mathbb{S}(G)$. For $\psi \in \mathfrak{b}^*$, consider the function $\mathcal{F}_{\mathfrak{b}/\Lambda} f(\psi)(\cdot)$ on G such that

$$\mathcal{F}_{\mathfrak{b}/\Lambda} f(\psi)(\dot{\nu}) = \int\limits_{\mathfrak{b}/\Lambda} f(\dot{\nu}(\Omega \circ \exp)(Y+\Lambda)) \; \chi_\psi(\exp(Y+\Lambda)) \, d\dot{Y}, \;\; \dot{\nu} \in G.$$

For $\eta \in \mathbb{R}$ and $\dot{\nu} \in G$, let

$$\Phi(\eta,\dot{\nu}) := \int\limits_{\mathbb{R}} f(\dot{\nu}(\Omega \circ \exp)(sY_{k+1})) \, e^{-2\pi i s\eta} \, ds.$$

Then $\mathcal{F}_{\mathfrak{h}/\Lambda}\Phi(\eta,\cdot)(\lambda Y^*)(\dot{\nu}) = \mathcal{F}_{\mathfrak{b}/\Lambda} f(\psi)(\dot{\nu})$. For the Torus group $\mathfrak{h}/\Lambda = \mathbb{T}^k$, the Plancherel formula gives

$$\sum_{\lambda \in \mathbb{Z}^k} |\mathcal{F}_{\mathfrak{b}/\Lambda} f(\psi)(\dot{\nu})|^2 = \sum_{\lambda \in \mathbb{Z}^k} |\mathcal{F}_{\mathfrak{h}/\Lambda}\Phi(\eta,\cdot)(\lambda Y^*)(\dot{\nu})|^2 = \int\limits_{H} |\Phi(\eta,\dot{\nu}\dot{h})|^2 \, d\dot{h}.$$

Using Riesz interpolation theorem, we obtain

$$\left(\sum_{\lambda \in \mathbb{Z}^k} |\mathcal{F}_{\mathfrak{b}/\Lambda} f(\psi)(\dot{\nu})|^2\right)^{p/q} = \left(\sum_{\lambda \in \mathbb{Z}^k} |\mathcal{F}_{\mathfrak{h}/\Lambda}\Phi(\eta,\cdot)(\lambda Y^*)(\dot{\nu})|^q\right)^{p/q} \le \int\limits_{H} |\Phi(\eta,\dot{\nu}\dot{h})|^p \, d\dot{h}. \tag{5.2}$$

By localized Plancherel formula (5.1) for $\tilde{G}/\tilde{B}$ and using $\phi(\nu) = \mathcal{F}_{\mathfrak{b}}\widetilde{f}(\psi)(\nu)$, we get

$$\int\limits_{\widehat{\tilde{G}}_{\chi_{\eta,\lambda}}} \|\pi(f)\|^2_{\mathrm{HS}} \, d\mu_{\chi_{\eta,\lambda}}(\delta) = \int\limits_{G/B} |\mathcal{F}_{\mathfrak{b}/\Lambda} f(\psi)(\dot{\nu})|^2 \, d_{G/B}\,\dot{\nu}.$$

Using Riesz interpolation theorem, we have

$$\int\limits_{\widehat{\tilde{G}}_{\chi_{\eta,\lambda}}} \|\pi(f)\|^q_{c_q} \, d\mu_{\chi_{\eta,\lambda}}(\delta) \le \left(\int\limits_{G/B} |\mathcal{F}_{\mathfrak{b}/\Lambda} f(\psi)(\dot{\nu})|^p \, d_{G/B}\,\dot{\nu}\right)^{q/p}. \tag{5.3}$$

THEOREM 5.1. Let G be a connected nilpotent Lie group with non-compact center. Then for any $f \in \mathbb{S}(G)$, $1 < p \le 2$ and $0 \le \beta < \dfrac{1}{q}$, we have

$$\left(\int\limits_{\mathbb{R}} \sum_{\lambda \in \mathbb{Z}^k} \int\limits_{\widehat{\tilde{G}}_{\chi_{\eta,\lambda}}} |\eta|^{-\beta q} \, \|\pi(f)\|^q_{c_q} \, d\eta \, d\mu_{\chi_{\eta,\lambda}}(\delta)\right)^{1/q} \le \mathcal{C}_\beta \left(\int\limits_{G} |s|^{\beta p} \, |f(y,\dot{h}(\Omega \circ \exp)(sY_{k+1}))|^p \, dy \, d\dot{h} \, ds\right)^{1/p},$$

where $\mathcal{C}_\beta$ is as in Theorem 1.2 with $n = 1$.

Proof. For $y \in Y$ and $\dot{h} \in H$, define $f_{(y,\dot{h})} : \mathbb{R} \to \mathbb{C}$ by $f_{(y,\dot{h})}(s) = f(y, \dot{h}(\Omega \circ \exp)(sY_{k+1}))$ for all $s \in \mathbb{R}$.

Then, $f_{(y,\dot{h})} \in \mathbb{S}(\mathbb{R})$ and we have

$$\widehat{f_{(y,\dot{h})}}(\eta) = \int_{\mathbb{R}} f(y, \dot{h}(\Omega \circ \exp)(sY_{k+1}))\, e^{-2\pi i s\eta}\, ds.$$

Then

$$\begin{aligned}\int_Y |\widehat{f_{(y,\dot{h})}}(\eta)|^p\, dy &= \int_Y \left| \int_{\mathbb{R}} f(y, \dot{h}(\Omega \circ \exp)(sY_{k+1}))\, e^{-2\pi i s\eta}\, ds \right|^p dy \\ &= \int_{G/B} \left| \int_{\mathbb{R}} f(\dot{\nu}\dot{h}(\Omega \circ \exp)(sY_{k+1}))\, e^{-2\pi i s\eta}\, ds \right|^p d_{G/B}\, \dot{\nu} \\ &= \int_{G/B} |\Phi(\eta, \dot{\nu}\dot{h})|^p\, d_{G/B}\, \dot{\nu}. \end{aligned} \tag{5.4}$$

Using Theorem 1.2, we have for $y \in Y$, $\dot{h} \in H$ and $0 \le \beta < \dfrac{1}{q}$,

$$\left(\int_{\mathbb{R}} |\eta|^{-\beta q}\, |\widehat{f_{(y,\dot{h})}}(\eta)|^q\, d\eta \right)^{p/q} \le \mathcal{C}_\beta^p \int_{\mathbb{R}} |s|^{\beta p}\, |f_{(y,\dot{h})}(s)|^p\, ds.$$

Integrating both sides with respect to $dy\, d\dot{h}$, we get

$$\int_{Y \times H} \left(\int_{\mathbb{R}} |\eta|^{-\beta q}\, |\widehat{f_{(y,\dot{h})}}(\eta)|^q\, d\eta \right)^{p/q} dy\, d\dot{h} \le \mathcal{C}_\beta^p \int_{Y \times H \times \mathbb{R}} |s|^{\beta p}\, |f_{(y,\dot{h})}(s)|^p\, dy\, d\dot{h}\, ds. \tag{5.5}$$

Thus, using Fubini's theorem, (5.2), (5.3), (5.4), (5.5) and Minkowski's inequality (2.5), we obtain

$$\begin{aligned} &\int_{\mathbb{R}} \sum_{\lambda \in \mathbb{Z}^k} \int_{\widehat{\widetilde{G}}_{\chi_{\eta,\lambda}}} |\eta|^{-\beta q}\, \|\pi(f)\|_{c_q}^q\, d\eta\, d\mu_{\chi_{\eta,\lambda}}(\delta) \\ &\le \int_{\mathbb{R}} |\eta|^{-\beta q} \sum_{\lambda \in \mathbb{Z}^k} \left(\int_{G/B} |\mathcal{F}_{\mathfrak{b}/\Lambda} f(\psi)(\dot{\nu})|^p\, d_{G/B}\, \dot{\nu} \right)^{q/p} d\eta \\ &\le \int_{\mathbb{R}} |\eta|^{-\beta q} \left(\int_{G/B} \left(\sum_{\lambda \in \mathbb{Z}^k} |\mathcal{F}_{\mathfrak{b}/\Lambda} f(\psi)(\dot{\nu})|^q \right)^{p/q} d_{G/B}\, \dot{\nu} \right)^{q/p} d\eta \\ &\le \int_{\mathbb{R}} |\eta|^{-\beta q} \left(\int_{(G/B) \times H} |\Phi(\eta, \dot{\nu}\dot{h})|^p\, d_{G/B}\, \dot{\nu}\, d\dot{h} \right)^{q/p} d\eta \end{aligned}$$

$$= \int_{\mathbb{R}} |\eta|^{-\beta q} \left(\int_{Y\times H} |\widehat{f_{(y,\dot{h})}}(\eta)|^p \, dy\, d\dot{h} \right)^{q/p} d\eta$$

$$\leq \left(\int_{Y\times H} \left(\int_{\mathbb{R}} |\eta|^{-\beta q} \, |\widehat{f_{(y,\dot{h})}}(\eta)|^q \, d\eta \right)^{p/q} dy\, d\dot{h} \right)^{q/p}$$

$$\leq \mathcal{C}_\beta^q \left(\int_{Y\times H\times\mathbb{R}} |s|^{\beta p} \, |f_{(y,\dot{h})}(s)|^p \, dy\, d\dot{h}\, ds \right)^{q/p} = \mathcal{C}_\beta^q \left(\int_G |s|^{\beta p} \, |f(y, \dot{h}(\Omega \circ \exp)(sY_{k+1}))|^p \, dy\, d\dot{h}\, ds \right)^{q/p},$$

which gives the required inequality. □

The following generalized Pitt's inequality for the L^p-Gabor transform can be obtained from Theorem 5.1 using similar techniques as in Theorem 3.1.

THEOREM 5.2. For $f \in \mathbb{S}(G)$, $\psi \in L^q(G)$, $1 < p \leq 2$, $0 \leq \beta < \dfrac{1}{q}$, we have

$$\left(\int_G \int_{\mathbb{R}} \sum_{\lambda\in\mathbb{Z}^k} \int_{\widehat{\widetilde{G}}_{\chi_{\eta,\lambda}}} |\eta|^{-\beta q} \, \|G_\psi f(\dot{z}, \pi)\|_{\mathcal{C}_q}^q \, d\dot{z}\, d\eta\, d\mu_{\chi_{\eta,\lambda}}(\delta) \right)^{1/q}$$

$$\leq \mathcal{C}_\beta \, \|\psi\|_q \left(\int_G |s|^{\beta p} \, |f(y, \dot{h}(\Omega \circ \exp)(sY_{k+1}))|^p \, dy\, d\dot{h}\, ds \right)^{1/p},$$

where $\mathcal{C}_\beta$ is as in Theorem 1.2 with $n = 1$.

6. Heisenberg Motion Group

In this section, we establish the generalized Pitt's inequality for the Fourier transform and the Gabor transform on Heisenberg motion group. Let $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$ be the Heisenberg group with the group law given by

$$(w, r) \cdot (w', r') = \left(w + w', r + r' + \frac{1}{2} \operatorname{Im} (w \cdot \overline{w'}) \right),$$

where w, $w' \in \mathbb{C}^n$ and r, $r' \in \mathbb{R}$. Let $U(n)$ be a maximal compact connected group of automorphisms of $\mathbb{H}_n$. The action of the unitary group $U(n)$ on $\mathbb{H}_n$ is given by $(k, (w, r)) \mapsto (kw, r)$, for all $k \in K$ and $(w, r) \in \mathbb{H}_n$. Let $G = \mathbb{H}_n \rtimes K$, where K is a compact, connected Lie subgroup of $U(n)$ such that $(K, \mathbb{H}_n)$ is a Gelfand pair. G forms a group with the group law given by

$$(w, r, k) \cdot (w', r', k') = ((w, r) \cdot (kw', r'), kk'),$$

where (w, r), $(w', r') \in \mathbb{H}_n$; k, $k' \in K$ and it is called the *Heisenberg motion group*. For more details, refer to [9]. Using [9, Theorem 4.2], we have

$$\sum_{\rho\in\widehat{K}} d_\rho \, \|\widehat{f}(\delta, \rho)\|_{\mathrm{HS}}^2 \, |\delta|^n = \int_{\mathbb{C}^n\times K} |f^\delta(w, k)|^2 \, dw\, dk,$$

where

$$f^\delta(w,k) = \int_{\mathbb{R}} f(w,r,k)\, e^{-2\pi i\delta r}\, dr.$$

Using Riesz interpolation theorem, we get

$$\sum_{\rho\in\widehat{K}} d_\rho \, \|\widehat{f}(\delta,\rho)\|_{c_q}^q \, |\delta|^n \le \left(\int_{\mathbb{C}^n\times K} |f^\delta(w,k)|^p \, dw\, dk \right)^{q/p}, \tag{6.1}$$

Also, the Plancherel formula for G takes the form

$$\int_{\mathbb{R}^*} \sum_{\rho\in\widehat{K}} d_\rho \, \|\widehat{f}(\delta,\rho)\|_{\mathrm{HS}}^2 \, |\delta|^n \, d\delta = \int_G |f(w,r,k)|^2 \, dw\, dr\, dk.$$

Again using Riesz interpolation theorem, we obtain

$$\left(\int_{\mathbb{R}^*} \sum_{\rho\in\widehat{K}} d_\rho \, \|\widehat{f}(\delta,\rho)\|_{c_q}^q \, |\delta|^n \, d\delta \right)^{1/q} \le \left(\int_G |f(w,r,k)|^p \, dw\, dr\, dk \right)^{1/p}.$$

We now prove the generalized Pitt's inequality for the Fourier transform on Heisenberg motion group.

THEOREM 6.1. For $f\in\mathbb{S}(G)$, $1<p\le 2$ and $0\le\beta<\dfrac{1}{q}$, we have

$$\left(\int_{\mathbb{R}^*} \sum_{\rho\in\widehat{K}} d_\rho \, |\delta|^{-\beta q} \, \|\widehat{f}(\delta,\rho)\|_{c_q}^q \, |\delta|^n \, d\delta \right)^{1/q} \le \mathcal{C}_\beta \left(\int_G \|(w,r)\|^{\beta p} \, |f(w,r,k)|^p \, dw\, dr\, dk \right)^{1/p},$$

where $\mathcal{C}_\beta$ is as in Theorem 1.2 with $n=1$.

Proof. For $w\in\mathbb{C}^n$ and $k\in K$, define $f_{w,k}:\mathbb{R}\to\mathbb{C}$ such that $f_{w,k}(r)=f(w,r,k)$.

Then, $f_{w,k}\in\mathbb{S}(\mathbb{R})$ and using Theorem 1.2, we have for $(w,k)\in\mathbb{C}^n\times K$ and $0\le\beta<\dfrac{1}{q}$,

$$\begin{aligned}\left(\int_{\mathbb{R}^*} |\delta|^{-\beta q} \, |f^\delta(w,k)|^q \, d\delta \right)^{p/q} = \left(\int_{\mathbb{R}} |\delta|^{-\beta q} \, |\widehat{f_{w,k}}(\delta)|^q \, d\delta \right)^{p/q} &\le \mathcal{C}_\beta^p \int_{\mathbb{R}} |r|^{\beta p} \, |f_{w,k}(r)|^p \, dr \\ &= \mathcal{C}_\beta^p \int_{\mathbb{R}} |r|^{\beta p} \, |f(w,r,k)|^p \, dr.\end{aligned}$$

Integrating both sides with respect to $dw\,dk$, we get

$$\int_{\mathbb{C}^n\times K} \left(\int_{\mathbb{R}^*} |\delta|^{-\beta q} \, |f^\delta(w,k)|^q \, d\delta \right)^{p/q} dw\, dk \le \mathcal{C}_\beta^p \int_G |r|^{\beta p} \, |f(w,r,k)|^p \, dw\, dr\, dk. \tag{6.2}$$

Using (6.1), (6.2) and Minkowski's inequality (2.5), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^*} \sum_{\rho\in\widehat{K}} d_\rho \, |\delta|^{-\beta q} \, \|\widehat{f}(\delta,\rho)\|_{\mathcal{C}_q}^q \, |\delta|^n \, d\delta &= \int_{\mathbb{R}^*} |\delta|^{-\beta q} \left(\sum_{\rho\in\widehat{K}} d_\rho \, \|\widehat{f}(\delta,\rho)\|_{\mathcal{C}_q}^q \, |\delta|^n \right) d\delta \\
&= \int_{\mathbb{R}^*} |\delta|^{-\beta q} \left(\int_{\mathbb{C}^n\times K} |f^\delta(w,k)|^p \, dw \, dk \right)^{q/p} d\delta \\
&\leq \left(\int_{\mathbb{C}^n\times K} \left(\int_{\mathbb{R}^*} |\delta|^{-\beta q} \, |f^\delta(w,k)|^q \, d\delta \right)^{p/q} dw \, dk \right)^{q/p} \\
&\leq \mathcal{C}_\beta^q \left(\int_G |r|^{\beta p} \, |f(w,r,k)|^p \, dw \, dr \, dk \right)^{q/p} \\
&\leq \mathcal{C}_\beta^q \left(\int_G \|(w,r)\|^{\beta p} \, |f(w,r,k)|^p \, dw \, dr \, dk \right)^{q/p},
\end{aligned}$$

which can be reduced to the required inequality. □

For $f, \psi \in \mathbb{S}(G)$ and $(v,s,h) \in G$, let $f^\psi_{(v,s,h)} : G \to \mathbb{C}$ be defined by

$$f^\psi_{(v,s,h)}(w,r,k) = f(w,r,k) \, \overline{\psi}((v,s,h)^{-1}(w,r,k))$$

for all $(w,r,k) \in G$. The generalized Pitt's inequality for the L^p-Gabor transform on Heisenberg motion group can be proved in a similar manner as in Theorem 3.1 using Theorem 6.1.

THEOREM 6.2. For $f \in \mathbb{S}(G)$, $\psi \in L^q(G)$, $1 < p \leq 2$, $0 \leq \beta < \dfrac{1}{q}$, we have

$$\begin{aligned}
&\left(\int_{\mathbb{R}^*} \sum_{\rho\in\widehat{K}} d_\rho \, |\delta|^{-\beta q} \, \|G_\psi f(v,s,h,\delta,\rho)\|_{\mathcal{C}_q}^q \, |\delta|^n \, d\delta \right)^{1/q} \\
&\leq \mathcal{C}_\beta \, \|\psi\|_q \left(\int_G \|(w,r)\|^{\beta p} \, |f(w,r,k)|^p \, dw \, dr \, dk \right)^{1/p},
\end{aligned}$$

where $\mathcal{C}_\beta$ is as in Theorem 1.2 with $n = 1$.

7. Diamond Lie Groups

In this section, we shall establish the generalized Pitt's inequality for the Fourier transform and the Gabor transform on diamond Lie groups. Let $\mathfrak{h}_{2n+1}$ be $(2n+1)$-dimensioanl Heisenberg Lie algebra having a basis $\{X_1, Y_1, \ldots, X_n, Y_n, Z\}$ with the non-trivial brackets such that $[X_i, Y_i] = Z, i = 1, \ldots, n$. Consider the simply connected Lie group associated to the $\mathfrak{h}_{2n+1}$ and denote it by $\mathbb{H}$. The diamond Lie groups are of the form $G_n = \mathbb{R}^n \ltimes \mathbb{H}$, where $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ acts on $x \in \mathbb{H}$ by

$$(\xi_1, \ldots, \xi_n) \cdot x = ((r(\xi_1)x_1)^T, \ldots, (r(\xi_n)x_n)^T, \eta),$$

where $x = (x_1, \ldots, x_n, \eta)^T$, $x_i = (a_i, b_i)^T \in \mathbb{R}^2$, $\eta \in \mathbb{R}$ and for $\theta \in \mathbb{R}$ we have

$$r(\theta) = \begin{bmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ \sin 2\pi\theta & \cos 2\pi\theta \end{bmatrix}.$$

G_n is non-exponential, connected, 3-step solvable, simply connected Lie group of type I. In particular, G_1 is the oscillator group. See [7, 10] for more details. Also the representations of the form $\delta_{\nu,\eta} \otimes \Lambda$, where Λ is an arbitrary character of $\mathbb{T}^n$, are the elements of $\widehat{G_n}$. Let Λ_τ be the associated character of $\mathbb{T}^n$, where $\tau \in \mathbb{Z}^n$. Then $\widehat{G_n} \setminus \widehat{G_n/Z(\mathbb{H})} = \{\delta_{\nu,\eta} \otimes \Lambda_\tau : \tau \in \mathbb{Z}^n, \nu \in \mathbb{R}^*, \eta \in \mathbb{T}^n\}$.

Also, one can write

$$\int\limits_{\mathbb{T}^n} \sum_{\tau \in \mathbb{Z}^n} \|\delta_{\nu,\eta} \otimes \Lambda_\tau(f)\|_{\mathrm{HS}}^2 \, |\nu|^n \, d\eta = \|f^\nu\|_{L^2(G_n/\mathbb{R}Z)}^2,$$

where $f^\nu = \mathscr{F}_1 f(\nu, \cdot)$. Using Riesz interpolation theorem, we have

$$\int\limits_{\mathbb{T}^n} \sum_{\tau \in \mathbb{Z}^n} \|\delta_{\nu,\eta} \otimes \Lambda_\tau(f)\|_{c_q}^q \, |\nu|^n \, d\eta \leq \|f^\nu\|_{L^p(G_n/\mathbb{R}Z)}^q, \tag{7.1}$$

Let $\Sigma = \{(r, (x, y, 0)) : r, x, y \in \mathbb{R}^n\}$ be a cross-section for the cosets of $\mathbb{R} = Z(\mathbb{H})$ in G.

Following is the generalized Pitt's inequality for the Fourier transform on diamond Lie groups.

THEOREM 7.1. For $f \in \mathbb{S}(G)$, $1 < p \leq 2$ and $0 \leq \beta < \dfrac{1}{q}$, we have

$$\left(\int\limits_{\mathbb{R}^*} \int\limits_{\mathbb{T}^n} \sum_{\tau \in \mathbb{Z}^n} (|\nu|^2 + \|\tau\|^2)^{-\beta q/2} \, \|\delta_{\nu,\eta} \otimes \Lambda_\tau(f)\|_{c_q}^q \, |\nu|^n \, d\nu \, d\eta\right)^{1/q}$$
$$\leq \mathcal{C}_\beta \left(\int\limits_{\mathbb{R}} \int\limits_{\Sigma} \|(t, \theta)\|^{\beta p} \, |f(t, \theta)|^p \, dt \, d\theta\right)^{1/p},$$

where $\mathcal{C}_\beta$ is as in Theorem 1.2 with $n = 1$.

Proof. For $\theta \in \Sigma$, define $f_\theta : \mathbb{R} \to \mathbb{C}$ by $f_\theta(t) = f(t, \theta)$, for all $t \in \mathbb{R}$.

Clearly, $f_\theta \in \mathbb{S}(\mathbb{R})$ and using Theorem 1.2, we have for $\theta \in \Sigma$ and $0 \leq \beta < \dfrac{1}{q}$,

$$\left(\int\limits_{\mathbb{R}^*} |\nu|^{-\beta q} \, |\widehat{f_\theta}(\nu)|^q \, d\nu\right)^{p/q} \leq \mathcal{C}_\beta^p \int\limits_{\mathbb{R}} |t|^{\beta p} \, |f_\theta(t)|^p \, dt = \mathcal{C}_\beta^p \int\limits_{\mathbb{R}} |t|^{\beta p} \, |f(t, \theta)|^p \, dt.$$

Integrating both sides over Σ and then applying Fubini's theorem, we get

$$\int\limits_{\Sigma} \left(\int\limits_{\mathbb{R}^*} |\nu|^{-\beta q} \, |\widehat{f_\theta}(\nu)|^q \, d\nu\right)^{p/q} d\theta \leq \mathcal{C}_\beta^p \, (A_p)^p \int\limits_{\mathbb{R}} \int\limits_{\Sigma} |t|^{\beta p} \, |f(t, \theta)|^p \, dt \, d\theta. \tag{7.2}$$

Using (7.1), we have

$$\int_{\mathbb{T}^n} \sum_{\tau \in \mathbb{Z}^n} \|\delta_{\nu,\eta} \otimes \Lambda_\tau(f)\|_{c_q}^q \, |\nu|^n \, d\eta \leq \|f^\nu\|_{L^p(G_n/\mathbb{R}Z)}^q = \left(\int_{\Sigma} |\widehat{f}_\theta(\nu)|^p \, d\theta \right)^{q/p}. \tag{7.3}$$

Using Fubini's theorem, (7.3), (7.2) and Minkowski's inequality (2.5), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^*} \int_{\mathbb{T}^n} \sum_{\tau \in \mathbb{Z}^n} (|\nu|^2 + \|\tau\|^2)^{-\beta q/2} \, \|\delta_{\nu,\eta} \otimes \Lambda_\tau(f)\|_{c_q}^q \, |\nu|^n \, d\nu \, d\eta \\
&\leq \int_{\mathbb{R}^*} \int_{\mathbb{T}^n} \sum_{\tau \in \mathbb{Z}^n} |\nu|^{-\beta q} \, \|\delta_{\nu,\eta} \otimes \Lambda_\tau(f)\|_{c_q}^q \, |\nu|^n \, d\nu \, d\eta \\
&= \int_{\mathbb{R}^*} |\nu|^{-\beta q} \left(\int_{\mathbb{T}^n} \sum_{\tau \in \mathbb{Z}^n} \|\delta_{\nu,\eta} \otimes \Lambda_\tau(f)\|_{c_q}^q \, |\nu|^n \, d\eta \right) d\nu \\
&\leq \int_{\mathbb{R}^*} |\nu|^{-\beta q} \left(\int_{\Sigma} |\widehat{f}_\theta(\nu)|^p \, d\theta \right)^{q/p} d\nu \\
&\leq \left(\int_{\Sigma} \left(\int_{\mathbb{R}^*} |\nu|^{-\beta q} \, |\widehat{f}_\theta(\nu)|^q \, d\nu \right)^{p/q} d\theta \right)^{q/p} \\
&\leq \mathcal{C}_\beta^q \left(\int_{\mathbb{R}} \int_{\Sigma} |t|^{\beta p} \, |f(t,\theta)|^p \, dt \, d\theta \right)^{q/p} \leq \mathcal{C}_\beta^q \left(\int_{\mathbb{R}} \int_{\Sigma} \|(t,\theta)\|^{\beta p} \, |f(t,\theta)|^p \, dt \, d\theta \right)^{q/p},
\end{aligned}$$

which can be reduced to the required inequality. □

Consider $f^\psi_{(u,v)} : G \to \mathbb{C}$ defined by $f^\psi_{(u,v)}(\theta,t) = f(\theta,t) \, \overline{\psi}((u,v)^{-1}(\theta,t))$, for $f, \psi \in \mathbb{S}(G)$, $(\theta,t),(u,v) \in G$. Note that

$$G_\psi f(u,v,\delta_{\nu,\eta} \otimes \Lambda_\tau) = \delta_{\nu,\eta} \otimes \Lambda_\tau(f^\psi_{(u,v)}).$$

The following generalized Pitt's inequality for the L^p-Gabor transform on diamond Lie groups can be proved using Theorem 7.1 and the techniques similar to those used in Theorem 3.1.

THEOREM 7.2. For $f \in \mathbb{S}(G)$, $\psi \in L^q(G)$, $1 < p \leq 2$, $0 \leq \beta < \dfrac{1}{q}$, we have

$$\begin{aligned}
&\left(\int_{\Sigma} \int_{\mathbb{R}} \int_{\mathbb{R}^*} \int_{\mathbb{T}^n} \sum_{\tau \in \mathbb{Z}^n} (|\nu|^2 + \|\tau\|^2)^{-\beta q/2} \, \|G_\psi f(u,v,\delta_{\nu,\eta} \otimes \Lambda_\tau)\|_{c_q}^q \, |\nu|^n \, du \, dv \, d\nu \, d\eta \right)^{1/q} \\
&\leq \mathcal{C}_\beta \, \|\psi\|_q \left(\int_{\mathbb{R}} \int_{\Sigma} \|(t,\theta)\|^{\beta p} \, |f(t,\theta)|^p \, dt \, d\theta \right)^{1/p},
\end{aligned}$$

where $\mathcal{C}_\beta$ is as in Theorem 1.2 with $n = 1$.

Acknowledgement

The second author acknowledges the support from National Academy of Sciences, India.

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