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Curvature estimates for a class of curvature equation in warped product manifolds

Estimations de courbure pour une classe d'équations de courbure dans les variétés à produits tordus

Jianbo Yang and Yueming Lu*

College of Science, Harbin University of Science and Technology, 52 Xuefu Road, Harbin, 150080, Heilongjiang Province, China
2320800008@stu.hrbust.edu.cn, yueminglu@hrbust.edu.cn

ABSTRACT. In this paper, we establish curvature estimates for a class of curvature equation $\mathcal{F}_p(\kappa) = f(V, \nu)$ for $\frac{n}{2} \leq p \leq n-1$ in the warped product manifolds $\bar{M}$. Additionally, by imposing some constraints on the right-hand side function, we also obtain an existence result for the starshaped hypersurface Σ that satisfies the above equation.
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1. Introduction

Let $\bar{M}^{n+1}$ be a Riemannian manifold and Σ be a closed hypersurface in $\bar{M}^{n+1}$. We use X to denote its position vector. We let $\nu(X)$ be the unit outer-normal vector of Σ at position vector X. The problem of hypersurfaces with prescribed curvature asks to find a closed hypersurfaces Σ in $\bar{M}^{n+1}$ that satisfies the following curvature equation

$$g(\kappa) = f,$$

where g is a smooth symmetric function of n variables, $\kappa = (\kappa_1, \cdots, \kappa_n)$ are the principal curvatures of Σ. The right hand side function $f = f(X)$ or more generally $f = f(X, \nu(X))$.

The above equation is overly broad for our study; therefore, we will impose certain constraints on it. We consider the following prescribed curvature equation with the general right hand side

$$\mathcal{F}_p(\kappa) = \prod_{1\leq i_1<i_2<\cdots<i_p\leq n}(\kappa_{i_1} + \kappa_{i_2} + \cdots + \kappa_{i_p}) = f(X, \nu(X)) \,, X \in \Sigma.$$

We introduce the following elliptic condition by Harvey and Lawson [15].

Definition 1.1. A C^2 regular hypersurface Σ in $\bar{M}^{n+1}$ is called p-convex if $\kappa(X) \in \mathcal{P}_p$ for all $X \in \Sigma$, where $\mathcal{P}_p$ is the p-convex cone

$$\mathcal{P}_p = \{\kappa \in \mathbb{R}^n : \forall 1 \leq i_1 < i_2 < \cdots < i_p \leq n, \kappa_{i_1} + \kappa_{i_2} + \cdots + \kappa_{i_p} > 0\}.$$

When the ambient space is a general Riemannian manifold, difficult-to-handle curvature terms will arise in the prior estimates. Therefore, we consider the warped product manifold with rotational

*Corresponding author.

symmetry properties. Let (M,g') be a compact Riemannian manifold and I be an open interval in $\mathbb{R}$. We consider the warped product manifold $\bar{M}^{n+1} = I \times_\phi M$ endowed with the metric

$$\bar{g} = d\rho^2 + \phi^2(\rho) g'$$

where $\phi : I \to \mathbb{R}^+$ is a smooth positive function. A hypersurface Σ in $\bar{M}^{n+1}$ is called star-shaped if Σ is the graph of a function $\rho = \rho(z)$, $z \in M$, i.e.,

$$\Sigma = \{X = (z, \rho(z)) \in \bar{M}^{n+1} : z \in M\}$$

We also call ρ is p-convex if the star-shaped hypersurface $\Sigma(z, \rho(z))$ is p-convex.

Therefore, based on the above discussion, in this paper, we consider the following prescribed curvature problem

$$\mathcal{F}_p(\kappa) = \prod_{1 \le i_1 < i_2 < \cdots < i_p \le n} (\kappa_{i_1} + \kappa_{i_2} + \cdots + \kappa_{i_p}) = f(X, \nu(X)) \,, X \in \Sigma \tag{1.1}$$

where Σ is a closed star-shaped p-convex hypersurface in the warped product manifold, $V(X)$ is the vector field $V = \phi(\rho)\frac{\partial}{\partial \rho}$ at $X \in \Sigma$. The prescript function f is positive and C^2 on Γ, where Γ is an open neighborhood of $\{(V(X), \nu(X)) \ : \ X \in \Sigma\}$ in $T\bar{M}^{n+1} \times \mathbb{S}^n$. Equation (1.1) is an elliptic partial differential equation of second order of the function ρ due to the p-convexity of Σ.

Let's make a brief review of the current research status of equation (1.1). The equation (1.1) is fully nonlinear for $p \le n-1$. When $p = 1$, equation (1.1) is the Gaussian curvature equation studied by Oliker [20] in $\mathbb{R}^{n+1}$. When $p = n$, it is the mean curvature equation studied by Bakelman-Kantor [1] and Treibergs-Wei [21] in $\mathbb{R}^{n+1}$. In [7], Chu-Jiao considered the (η, k)-convex star-shaped hypersurfaces in $\mathbb{R}^{n+1}$, where $1 \le k \le n$, $\eta_i = \sum_{j \ne i} \kappa_j$. The (η, n) curvature equation there is exactly the equation (1.1) with $p = n-1$. When $\frac{n}{2} \le p \le n-1$, equation (1.1) was firstly proposed by Dong [9] in $\mathbb{R}^{n+1}$ and he obtained the existence theorem by establishing C^0, C^1 and C^2 estimates of equation (1.1). In space forms, Barbosa-Lira-Oliker in [2] studied the curvature estimates for hypersurfaces with prescribed m-th curvature, i.e., $\sigma_m(\kappa) = f$, where the right hand side function f does not depend on $\nu(X)$. Spruck-Xiao in [3] studied the curvature estimates for hypersurfaces with prescribed scalar curvature for general f depending on $\nu(X)$. For equation (1.1), Zhou in [6] studied the curvature estimates for $(n-1)$-convex hypersurfaces. Recently, Lu-Zhong [19] extended the results of Dong to space forms. To obtain the curvature estimates, Lu-Zhong applied another concavity inequality of curvature function $\mathcal{F}_p$ to absorb the bad third order terms in the elliptic space and hyperbolic space. Compared with Dong and Lu-Zhong's work, the important Codazzi property in curvature estimate and gradient estimate for equation (1.1) can't be used here due to the non-constant sectional curvature in the warped product manifold.

For the warped product manifolds, Chen-Li-Wang in [31] derived the curvature estimates for 2-convex and $(n-1)$-convex hypersurfaces, that is, the curvature estimates for hypersurfaces satisfying the curvature equation $\sigma_2(\kappa) = f$ and $\sigma_{n-1}(\kappa) = f$, where the right hand side function f depends on $\nu(X)$. Chen-Tu-Xiang in [32] studied the curvature equation

$$\frac{\sigma_k}{\sigma_l}(\eta) = f(V, \nu)$$

for (η, k)-convex star-shaped hypersurfaces where $1 \le k \le n$, $\eta_i = \sum_{j \ne i} \kappa_i$ and σ_k is the k-th elementary symmetric function with $2 \le k \le n$, $0 \le l \le k-2$. Recently, Wang in [33] extended the results of Chen-Tu-Xiang to $0 \le l < k < n$ by extending the key inequality due to Chen-Dong-Han [25].

If the right hand side function f depending on the gradient term or the normal for Hessian equation or curvature equation, let's give a brief review of the results obtained recently. Guan-Ren-Wang [34] showed that curvature estimate fails for curvature quotient equation

$$\frac{\sigma_k}{\sigma_l}(\kappa(X)) = f(V(X), \nu(X)),\ \forall X \in M,$$

in the Euclidean space for general f, where $0 < l < k \leq n$. However, in the case $l = 0$ and $k \geq 3$, the curvature estimate for *convex* hypersurfaces was obtained by Caffarelli-Nirenberg-Spruck [24] when $k = n$, Guan-Ren-Wang [34] when $3 \leq k \leq n$, and for *non-convex* hypersurfaces by Ren-Wang [27, 28] when $k = n-1$ and $n-2$. Recently, Lu-Tsai in [18] gave a simpler proof of Ren-Wang's results [27]. Assuming that f is convex with respect to the normal ν, Guan [26] obtained the global C^2 estimates for the Dirichlet problem of general Hessian equation on Riemannian manifolds. For the other technical assumptions on f to derive C^2 estimates, we refer the readers to Ivochkina [22, 23], Guan-Guan [11], Guan-Lin-Ma [14], Guan-Li-Li [13] and Guan-Jiao [12] for more details.

In this paper, the curvature estimate for equation (1.1) is established without any technical assumptions on prescript function f in the warped product manifold.

Theorem 1.2. *Let Σ be a closed star-shaped p-convex hypersurface satisfying curvature equation* (1.1) *in the warped product manifold with $\frac{n}{2} \leq p \leq n-1$. Then,*

$$\max_{X \in M, i=1,\cdots,n} |\kappa_i(X)| \leq C, \tag{1.2}$$

where C is a positive constant depending on n, p, $|\Sigma|_{C^1}$, $\inf f$ and $|f|_{C^2}$.

Unlike curvature estimates, both gradient estimates and boundedness estimates require some restrictions on the prescript function f. For the gradient estimate and boundedness estimate, we respectively adopt two assumptions A_1 and A_2 on prescript function f from [1, 21] as follows:

(A_1): for any fixed unit vector ν,

$$\frac{\partial}{\partial \rho}\left(\phi^{C_n^p}(\rho) f(V, \nu)\right) \leq 0, \text{ where } |V| = \phi(\rho);$$

(A_2): there exist two positive constants r_1 and r_2 such that $r_1 < r_2$ and

$$f\left(V, \frac{V}{|V|}\right) \geq p^{C_n^p} \left(\frac{\phi'}{\phi}\right)^{C_n^p}(\rho), \text{for } \rho = r_1, \tag{1.3}$$

$$f\left(V, \frac{V}{|V|}\right) \leq p^{C_n^p} \left(\frac{\phi'}{\phi}\right)^{C_n^p}(\rho), \text{for } \rho = r_2. \tag{1.4}$$

Based on the curvature estimate (C^2 estimate), gradient estimate (C^1 estimate) and boundedness estimate (C^0 estimate), we can prove the existence theorem by the continuity method. Denote by

$$\pi : T\bar{M}^{n+1} \to \bar{M}^{n+1}, X_p \mapsto p, \forall X_p \in T_p\bar{M}^{n+1}$$

the natural projection and ball $B_r = \{(z, \rho) \in \bar{M}^{n+1} : \rho \leq r\}$.

Theorem 1.3. *Let $f \in C^2\left(\pi^{-1}(\overline{B_{r_2}} \setminus B_{r_1}) \times \mathbb{S}^n\right)$ be a positive function satisfying assumptions A_1 and A_2. Then there exists a unique $C^{3,\alpha}$ closed star-shaped p-convex hypersurface $\Sigma(z, \rho(z))$ satisfying equation* (1.1) *such that $r_1 \le \rho(z) \le r_2$ on $\mathbb{S}^n$ for any $\alpha \in (0,1)$ provided $\frac{n}{2} \le p \le n-1$.*

The organization of this paper is as follows. In Section 2, we recall some geometric formulas regarding the warped product manifold and some properties of curvature function $\mathcal{F}_p$ in p-convex cone. In Section 3, we prove the curvature estimate Theorem 1.2. Section 4 is devoted to the gradient estimate. In the last section, we establish boundedness estimate and obtain the existence theorem by the continuity method.

2. Preliminaries

In this section, we recall some geometric formulas in warped product space(see [30, 31]). Let M be a compact Riemannian manifold with the metric g' and I be an open interval in $\mathbb{R}$. The manifold $\bar{M}^{n+1} = I \times_\phi M$ is called warped product if it is endowed with the metric

$$\bar{g} = d\rho^2 + \phi^2(\rho) g' \tag{2.1}$$

where $\phi: I \to \mathbb{R}^+$ is a smooth positive function and $\phi' > 0$.

In this paper, '$\langle\cdot,\cdot\rangle$' is the inner product defined by the metric $\bar{g}$ in the ambient space, $|\cdot|$ is the norm with respect to $\bar{g}$ and $\bar{\nabla}$ denotes the Levi-Civita connection with respect to the metric $\bar{g}$. Let ∇' is the Levi-Civita connection with respect to the metric g'. The curvature tensors in M and $\bar{M}$ are denoted by R and $\bar{R}$.

Let $\{e_1, \cdots, e_n\}$ be an orthonormal frame field in M and let $\{\theta^1, \cdots, \theta^n\}$ be the associated dual frame. An orthonormal frame in $\bar{M}$ may be defined by $\bar{E}_i = \frac{1}{\phi} e_i$, $1 \le i \le n$ and $\bar{E}_{n+1} = \frac{\partial}{\partial \rho}$. Let g be the restriction of $\bar{g}$ on hypersurface Σ and ∇ the Levi-Civita connection with respect to g. Suppose $\{E_1, \cdots, E_n\}$ is a local frame on Σ, we denote $\nabla_i = \nabla_{E_i}$.

A star-shaped compact hypersurface Σ in $\bar{M}$ can be represented as a smooth radial graph

$$\Sigma = \{(z, \rho(z)) : z \in M\},$$

where $\rho : M \to I$ be a smooth function. A tangent vector of Σ in $\bar{M}$ is

$$X_i = \phi \bar{E}_i + \rho_i \bar{E}_{n+1}, \tag{2.2}$$

where ρ_i are the components of the differential $d\rho = \rho_i \theta^i$. The outward unit normal is given by

$$\nu = \frac{1}{\sqrt{\phi^2 + |\nabla' \rho|^2}} \left(\phi \bar{E}_{n+1} - \sum \rho^i \bar{E}_i\right). \tag{2.3}$$

where $|\nabla'\rho|^2 = \rho^i \rho_i$ is the squared norm of $\nabla'\rho = \sum \rho^i e_i$. The components of the induced metric and its inverse in Σ is given by

$$g_{ij} = \langle X_i, X_j \rangle = \phi^2 \delta_{ij} + \rho_i \rho_j, \; g^{ij} = \frac{1}{\phi^2}\left(\delta_{ij} - \frac{\rho_i \rho_j}{\phi^2 + |\nabla'\rho|^2}\right). \tag{2.4}$$

The second fundamental form of Σ with components (h_{ij}) is determined by

$$h_{ij} = \frac{1}{\sqrt{\phi^2 + |\nabla'\rho|^2}} \left(-\phi \rho_{ij} + 2\phi' \rho_i \rho_j + \phi^2 \phi' \delta_{ij}\right), \tag{2.5}$$

where ρ_{ij} are the components of the Hessian $\nabla'^2 z$ of z in M. The Codazzi equation is given by

$$\nabla_k h_{ij} = \nabla_j h_{ik} + \bar{R}_{\nu ijk} \tag{2.6}$$

Lemma 2.1. *[4] On the leaf M_ρ, the curvature satisfies*

$$\bar{R}_{ijk\nu} = 0$$

and the principle curvature is given by

$$\kappa(\rho) = \frac{\phi'(\rho)}{\phi(\rho)}$$

where the upward unit normal $E_{n+1} = \frac{\partial}{\partial\rho}$ is choosen for each leaf M_ρ.

Lemma 2.2. *[4] Let X_0 be a point of Σ and $\{E_1, \cdots, E_n, E_{n+1} = \nu\}$ be an adapted frame field such that E_i is a principal direction and connection forms $\omega_i^k = 0$ at X_0. Let h_{ij} be the second fundamental form of Σ. Then at X_0, we have that*

$$\begin{aligned}\nabla_{ii} h_{ll} =& \nabla_{ll} h_{ii} - \sum h_{lm}(h_{mi}h_{il} - h_{ml}h_{ii}) - \sum h_{mi}(h_{mi}h_{ll} - h_{ml}h_{li}) \\ &+ \nabla_l \bar{R}_{\nu iil} - 2\sum h_{ml}\bar{R}_{miil} + h_{il}\bar{R}_{\nu i\nu l} + h_{ll}\bar{R}_{\nu li\nu} + \nabla_i \bar{R}_{\nu lik} \\ &- 2\sum h_{mi}\bar{R}_{mlil} + h_{ii}\bar{R}_{\nu l\nu l} + h_{li}\bar{R}_{\nu li\nu}\end{aligned} \tag{2.7}$$

The position vector $V = \phi(\rho)\frac{\partial}{\partial\rho}$ is a conformal Killing vector field and the support function on Σ is given by $u = \langle V, \nu\rangle$. We define

$$\Phi(\rho) = \int_0^\rho \phi(r)dr.$$

Let A be a symmetric $n \times n$ matrix, we denote $\kappa(A) = (\kappa_1(A), \kappa_2(A), \cdots, \kappa_n(A))$ where $\kappa_i(A)$ ($i = 1, 2, \cdots, n$) is the eigenvalue of A. The p-convex cone of symmetric $n \times n$ matrices is defined by $P_p = \{A : \kappa(A) \in \mathcal{P}_p\}$. Therefore, the hypersurface Σ in $\bar{M}^{n+1}$ is p-convex if and only if $\sqrt{g^{-1}}\{h_{ij}\}\sqrt{g^{-1}} \in P_p$.

Denote by

$$\mathcal{F}(\kappa) = \mathcal{F}_p(\kappa)^{\frac{1}{C_n^p}} = \left[\prod\nolimits_{1\leq i_1<i_2<\cdots<i_p\leq n}(\kappa_{i_1} + \kappa_{i_2} + \cdots + \kappa_{i_p})\right]^{\frac{1}{C_n^p}}$$

and

$$F(A) = \mathcal{F}(\kappa(A)), \quad F^{ij}(A) = \frac{\partial F}{\partial A_{ij}}(A), \quad F^{ij,kl}(A) = \frac{\partial^2 F}{\partial A_{ij}\partial A_{kl}}(A).$$

Then equation (1.1) can be rewritten as

$$F(b) = \tilde{f}(V, \nu), \tag{2.8}$$

where $b = \sqrt{g^{-1}}\{h_{ij}\}\sqrt{g^{-1}}$, $\tilde{f} = f^{\frac{1}{C_n^p}}$ and $C_n^p = \frac{n!}{p!(n-p)!}$. Since $\{F^{ij}\}$ is positive on p-convex cone, equation (2.8) is elliptic for p-convex hypersurface Σ. Moreover, on the p-convex cone, F is concave, $\{F^{ij,kl}(b)\}$ is negative, which is extremely important in the curvature estimate.

If $A = \mathrm{diag}(\kappa_1, \kappa_2, \cdots, \kappa_n)$, then $\{F^{ij}(A)\}$ is diagonal and

$$F^{ii}(A) = \frac{\partial \mathcal{F}}{\partial \kappa_i}(\kappa(A)) = \frac{1}{C_n^p}\mathcal{F}(\kappa) \sum_{\substack{i\in\{i_1,i_2,\cdots,i_p\}\\ 1\leq i_1<i_2<\cdots<i_p\leq n}} \frac{1}{\kappa_{i_1}+\kappa_{i_2}+\cdots+\kappa_{i_p}}. \tag{2.9}$$

The following properties of function $\mathcal{F}$ are proved by Dinew [8].

Lemma 2.3. *Let $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_n) \in \mathcal{P}_p$ with $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n$. Then, (1)$\frac{\partial \mathcal{F}}{\partial \kappa_1}(\kappa)\kappa_1 \geq \frac{1}{n}\mathcal{F}(\kappa)$; (2)$\sum \frac{\partial \mathcal{F}}{\partial \kappa_i}(\kappa) \geq p$; (3)$\sum \frac{\partial \mathcal{F}}{\partial \kappa_i}\kappa_i = \mathcal{F}(\kappa)$; (4)there exists a positive constant C=C(n,p) such that $\frac{\partial \mathcal{F}}{\partial \kappa_j}(\kappa) \geq C\sum \frac{\partial \mathcal{F}}{\partial \kappa_i}(\kappa)$ when $j \geq n-p+1$.*

3. Curvature estimate

Let $\kappa_{\max}$ be the largest principal curvature of Σ. Based on the assumption of C^0 and C^1 estimates, we assert that

$$\frac{1}{C} \leq \inf_\Sigma u \leq u \leq \sup_\Sigma u \leq C,$$

where C depends on $\inf_\Sigma \rho$ and $|\rho|_{C^1}$. In the following, we will denote by C a constant depending only on n, p, $|\rho|$, $\inf f$, $|f|_{C^2}$ or some of them. It may change from line to line.

Then we choose $a = \frac{1}{2}\inf_\Sigma u > 0$ and consider the quantity

$$Q = \ln \kappa_{\max} - \ln(u-a) + N\Phi,$$

where N is a large constant to be determined later.

Suppose Q attains its maximum at some point $X_0 \in \Sigma$. Choose an orthonormal frame $E_1, \cdots, E_n$ near X_0 such that at X_0, we have

$$h_{ij} = \kappa_i\delta_{ij}, \quad \kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n.$$

If $\kappa_{\max}$ has multiplicity more than 1, then Q is not smooth at X_0. We consider the new test function $\hat{Q}$ near X_0

$$\hat{Q} = \ln h_{11} - \ln(u-a) + N\Phi. \tag{3.1}$$

Clearly, $\hat{Q}$ also attains its maximum at X_0.

At X_0, we have

$$0 = \frac{\nabla_i h_{11}}{h_{11}} - \frac{\nabla_i u}{u-a} + N\nabla_i\Phi, \tag{3.2}$$

$$0 \geq \frac{\nabla_{ii}h_{11}}{h_{11}} - \left(\frac{\nabla_i h_{11}}{h_{11}}\right)^2 - \frac{\nabla_{ii}u}{u-a} + \left(\frac{\nabla_i u}{u-a}\right)^2 + N\nabla_{ii}\Phi. \tag{3.3}$$

By Lemma 2.2, we have

$$\begin{aligned}\nabla_{ii}h_{11} =& \nabla_{11}h_{ii} + h_{11}^2 h_{ii} - h_{11}h_{ii}^2 + 2(h_{11}-h_{ii})\bar{R}_{i1i1}\\ &- h_{11}\bar{R}_{i\nu i\nu} + h_{ii}\bar{R}_{1\nu 1\nu} - \nabla_1\bar{R}_{i1i\nu} + \nabla_i\bar{R}_{1i1\nu}.\end{aligned}$$

By (2.2) and Gram-Schmidt procedure, we can obtain the orthonormal frame $E_1, \cdots, E_n$, $E_{n+1} = \nu$ from the local frame $X_1, \cdots X_n, \nu$. This implies the bound of the components of $\bar{R}$ and $\bar{\nabla}\bar{R}$ due to the frame $X_1, \cdots, X_n, \nu$ depends only on ρ and $\nabla'\rho$.

Hence, we have

$$
\begin{aligned}
0 \geq & \frac{\nabla_{11} h_{ii}}{h_{11}} - \left(\frac{\nabla_i h_{11}}{h_{11}}\right)^2 - \frac{\nabla_{ii} u}{u-a} + \left(\frac{\nabla_i u}{u-a}\right)^2 + N \nabla_{ii} \Phi \\
& + h_{11} h_{ii} - h_{ii}^2 - C.
\end{aligned} \tag{3.4}
$$

Contracting (3.4) with F^{ii}, we have

$$
\begin{aligned}
0 \geq & \sum_i F^{ii} \frac{\nabla_{11} h_{ii}}{h_{11}} - \sum_i F^{ii} \frac{(\nabla_i h_{11})^2}{h_{11}^2} - \sum_i F^{ii} \frac{\nabla_{ii} u}{u-a} + \sum_i F^{ii} \frac{(\nabla_i u)^2}{(u-a)^2} \\
& + N \sum_i F^{ii} \nabla_{ii} \Phi - \sum_i F^{ii} h_{ii}^2 - C \sum_i F^{ii}.
\end{aligned} \tag{3.5}
$$

we have used the fact $\sum_i F^{ii} h_{ii} = \tilde{f}$ in the above inequality.

Differentiating the equation $F(b) = \tilde{f}(V, \nu)$, we have

$$
\sum_i F^{ii} \nabla_k h_{ii} = \overline{\nabla}_k \tilde{f}(V, \nu) = \tilde{f}_V(\overline{\nabla}_k V) + \sum_s h_{ks} \tilde{f}_\nu(E_s), \tag{3.6}
$$

and

$$
\begin{aligned}
\sum_i F^{ii} \nabla_{11} h_{ii} + & \sum_{p,q,r,s} F^{pq,rs} \nabla_1 h_{pq} \nabla_1 h_{rs} \\
& = \tilde{f}_{VV}(\overline{\nabla}_1 V, \overline{\nabla}_1 V) + 2\tilde{f}_{V\nu}(\overline{\nabla}_1 V, \overline{\nabla}_1 \nu) + \tilde{f}_{\nu\nu}(\overline{\nabla}_1 \nu, \overline{\nabla}_1 \nu) \\
& \quad + \tilde{f}_\nu(\overline{\nabla}_{11} \nu) + \tilde{f}_V(\overline{\nabla}_{11} V),
\end{aligned} \tag{3.7}
$$

where $\overline{\nabla}_{ij} = \overline{\nabla}_i \overline{\nabla}_j - \overline{\nabla}_{\nabla_i E_j}$.

Let Y be a smooth vector field in $\bar{M}$, then $Y = \sum_i a_i e_i + b\frac{\partial}{\partial \rho}$ where a_i, b are smooth function on $\bar{M}$. Without loss of generality, we may assume $e_i = \frac{\partial}{\partial z^i}, i = 1, \ldots, n$, is a natural frame on M. Denote by $\bar{g}_{ij} = \bar{g}(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j})$, $\bar{g}_{i\rho} = \bar{g}(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \rho})$ and $\bar{g}_{\rho\rho} = \bar{g}(\frac{\partial}{\partial \rho}, \frac{\partial}{\partial \rho})$. By the definition of Christoffel symbol, we have

$$
\begin{aligned}
\Gamma_{i\rho}^k &= \frac{1}{2} \bar{g}^{kl} \left(\frac{\partial \bar{g}_{\rho l}}{\partial z^i} + \frac{\partial \bar{g}_{il}}{\partial \rho} - \frac{\partial \bar{g}_{i\rho}}{\partial z^l} \right) + \frac{1}{2} \bar{g}^{k\rho} \left(\frac{\partial \bar{g}_{\rho\rho}}{\partial z^i} + \frac{\partial \bar{g}_{i\rho}}{\partial \rho} - \frac{\partial \bar{g}_{i\rho}}{\partial \rho} \right) \\
&= \frac{1}{2} \frac{1}{\phi^2} g'^{kl} (0 + 2\phi\phi' g'_{il} - 0) + 0 \\
&= \frac{\phi'}{\phi} \delta_{ik}.
\end{aligned}
$$

Similarly, we also have $\Gamma_{i\rho}^{\rho}=0$. By direct calculation, we have

$$\begin{aligned}
\overline{\nabla}_Y V = \overline{\nabla}_{\sum_i a_i e_i + b\frac{\partial}{\partial\rho}} V &= \sum_i a_i \overline{\nabla}_i V + b\overline{\nabla}_{\frac{\partial}{\partial\rho}} V \\
&= \sum_i a_i \phi(\rho)\overline{\nabla}_i \frac{\partial}{\partial\rho} + b\phi'(\rho)\frac{\partial}{\partial\rho} \\
&= \sum_{i,k} a_i\phi(\rho)\left(\Gamma_{i\rho}^{k} e_k\right) + b\phi'(\rho)\frac{\partial}{\partial\rho} \\
&= \phi'(\rho)Y.
\end{aligned} \tag{3.8}$$

Then, we have

$$\overline{\nabla}_1 V = \phi'(\rho)E_1, \tag{3.9}$$

and

$$\begin{aligned}
\overline{\nabla}_{11} V &= \overline{\nabla}_1\overline{\nabla}_1 V - \overline{\nabla}_{\nabla_1 E_1} V \\
&= \overline{\nabla}_1(\phi'(\rho)E_1) - \phi'(\rho)\nabla_1 E_1 \\
&= \phi''(\rho)E_1(\rho)E_1 + \phi'(\rho)(\nabla_1 E_1 - h_{11}\nu) - \phi'(\rho)\nabla_1 E_1 \\
&= \phi'' E_1(\rho)E_1 - h_{11}\phi'(\rho)\nu.
\end{aligned} \tag{3.10}$$

By equation (2.6), we have

$$\begin{aligned}
\overline{\nabla}_{ij}\nu = \overline{\nabla}_i\overline{\nabla}_j\nu - \overline{\nabla}_{\nabla_i E_j}\nu &= \overline{\nabla}_i\left(\sum_k h_{jk}E_k\right) - \sum_k h(\nabla_i E_j, E_k)E_k \\
&= \sum_k \left[\overline{\nabla}_i(h_{jk})E_k + h_{jk}\overline{\nabla}_i E_k - h(\nabla_i E_j, E_k)E_k\right] \\
&= \sum_k \left[(\nabla_i h_{jk} + h(\nabla_i E_j, E_k) + h(E_j, \nabla_i E_k))E_k\right] \\
&\quad + \sum_k \left[h_{jk}\overline{\nabla}_i E_k - h(\nabla_i E_j, E_k)E_k\right] \\
&= \sum_k \left(\nabla_k h_{ij}E_k - \bar{R}_{\nu jik}E_k - h_{ki}h_{kj}\nu\right).
\end{aligned}$$

At X_0, it is

$$\overline{\nabla}_{11}\nu = \sum_k \left(\nabla_k h_{11} - \bar{R}_{\nu 11k}\right)E_k - h_{11}^2\nu. \tag{3.11}$$

By critical equation (3.2), we have

$$\begin{aligned}
|\nabla_k h_{11}| &= \left|\frac{h_{11}\nabla_i u}{u-a} - Nh_{11}\nabla_i\Phi\right| \\
&= \left|\frac{h_{11}}{u-a}\sum_i h_{ii}\langle V, E_i\rangle - Nh_{11}\langle V, E_i\rangle\right| \\
&\leq C(h_{11}^2 + Nh_{11}).
\end{aligned} \tag{3.12}$$

Hence

$$|\overline{\nabla}_{11}\nu| \leq C(h_{11}^2 + Nh_{11}). \tag{3.13}$$

Inserting equations (3.9), (3.10), (3.13) and Weingarten formula $\overline{\nabla}_1\nu = \sum_l h_{1l}E_l$ into equation (3.7), we have

$$\sum_i F^{ii}\nabla_{11}h_{ii} + \sum_{p,q,r,s} F^{pq,rs}\nabla_1 h_{pq}\nabla_1 h_{rs} \geq -C(1 + h_{11}^2 + Nh_{11}) \tag{3.14}$$

where C depends on $|\rho|_{C^1}$ and $|f|_{C^2}$.

Plugging (3.14) into equation (3.5), we have

$$\begin{aligned} 0 \geq & -C(h_{11} + N) - \frac{1}{h_{11}}\sum_{p,q,r,s} F^{pq.rs}\nabla_1 h_{pq}\nabla_1 h_{rs} - \sum_i F^{ii}\frac{(\nabla_i h_{11})^2}{h_{11}^2} \\ & - \sum_i F^{ii}\frac{\nabla_{ii}u}{u-a} + \sum_i F^{ii}\frac{(\nabla_i u)^2}{(u-a)^2} + N\sum_i F^{ii}\nabla_{ii}\Phi \\ & - \sum_i F^{ii}h_{ii}^2 - C\sum_i F^{ii}. \end{aligned} \tag{3.15}$$

Recalling that $\nabla_i\Phi = \langle V, E_i\rangle$, by direct calculation, we have

$$\begin{aligned} \nabla_{ij}\Phi &= \nabla_i\nabla_j\Phi - \nabla_{\nabla_i E_j}\Phi \\ &= \nabla_i\langle V, E_j\rangle - \nabla_{\nabla_i E_j}\Phi \\ &= \langle\overline{\nabla}_i V, E_j\rangle + \langle V, \overline{\nabla}_i E_j\rangle - \nabla_{\nabla_i E_j}\Phi \\ &= \phi'(\rho)\delta_{ij} - uh_{ij}. \end{aligned} \tag{3.16}$$

At X_0, $F^{ii}h_{ii} = \tilde{f}$ imply that

$$N\sum_i F^{ii}\nabla_{ii}\Phi = N\phi'\sum_i F^{ii} - Nu\sum_i F^{ii}h_{ii} \geq N\phi'\sum_i F^{ii} - CN. \tag{3.17}$$

By Weingarten formula $\overline{\nabla}_i\nu = \sum_l h_{il}E_l$, we have

$$\begin{aligned} \nabla_{ij}u &= \nabla_i\nabla_j u - \nabla_{\nabla_i E_j}u \\ &= \nabla_i\left(\sum_l h_{jl}\nabla_l\Phi\right) - \sum_l h(\nabla_i E_j, E_l)\nabla_l\Phi \\ &= \sum_l (\nabla_i h_{jl}\nabla_l\Phi + h(\nabla_i E_j, E_l)\nabla_l\Phi + h_{jl}\nabla_{il}\Phi - h(\nabla_i E_j, E_l)\nabla_l\Phi) \\ &= \sum_l [\nabla_i h_{jl}\nabla_l\Phi + h_{jl}(\phi'(\rho)\delta_{il} - uh_{il})] \\ &= \sum_k (\nabla_k h_{ij} + \bar{R}_{\nu jki})\nabla_k\Phi + \phi'(\rho)h_{ij} - \sum_k uh_{ik}h_{jk}. \end{aligned} \tag{3.18}$$

At X_0, we have

$$\begin{aligned}\sum_i \frac{F^{ii}\nabla_{ii}u}{u-a} &= \frac{1}{u-a}\left(\sum_k F^{ii}\nabla_k h_{ii}\nabla_k\Phi + \phi' F^{ii}h_{ii} - u\sum_i F^{ii}h_{ii}^2 + \sum_k F^{ii}\bar{R}_{\nu iki}\nabla_k\Phi\right)\\ &= \frac{1}{u-a}\left(\sum_k \tilde{f}_V(\overline{\nabla}_k V)\nabla_k\Phi + \sum_k h_{kk}\tilde{f}_\nu(E_k) + \phi'\tilde{f} - u\sum_i F^{ii}h_{ii}^2\right)\\ &\quad + \frac{1}{u-a}\sum_k F^{ii}\bar{R}_{\nu iki}\nabla_k\Phi\\ &\leq Ch_{11} + C\sum_i F^{ii} - \frac{u}{u-a}\sum_i F^{ii}h_{ii}^2.\end{aligned} \tag{3.19}$$

Inserting (3.17) and (3.19) into (3.15) yields

$$\begin{aligned}0 \geq &-C(h_{11}+N) - \frac{1}{h_{11}}\sum_{p,q,r,s} F^{pq,rs}\nabla_1 h_{pq}\nabla_1 h_{rs} + \frac{a}{u-a}\sum_i F^{ii}h_{ii}^2\\ &+ (N\phi' - C)\sum_i F^{ii} - \sum_i \frac{F^{ii}(\nabla_i h_{11})^2}{h_{11}^2} + \sum_i \frac{F^{ii}(\nabla_i u)^2}{(u-a)^2}.\end{aligned} \tag{3.20}$$

Now we split the proof into two case as Chou-Wang [16] and Hou-Ma-Wu [29] to eliminate the negative third order terms.

Case A. $h_{nn} \leq -\frac{h_{11}}{n}$. Thanks to the concavity of F, we can drop the positive term $-\frac{1}{h_{11}}\sum_{p,q,r,s} F^{pq,rs}\nabla_1 h_{pq}\nabla_1 h_{rs}$ in (3.20). Using critical equation (3.2), we have

$$\begin{aligned}\sum_i \frac{F^{ii}(\nabla_i h_{11})^2}{h_{11}^2} &\leq (1+\epsilon)\sum_i \frac{F^{ii}(\nabla_i u)^2}{(u-a)^2} + (1+\frac{1}{\epsilon})N^2\sum_i F^{ii}(\nabla_i\Phi)^2\\ &\leq \sum_i \frac{F^{ii}(\nabla_i u)^2}{(u-a)^2} + \epsilon\frac{\sum_i F^{ii}h_{ii}(\nabla_i\Phi)^2}{(u-a)^2} + \frac{1+\epsilon}{\epsilon}N^2\sum_i F^{ii}(\nabla_i\Phi)^2\end{aligned} \tag{3.21}$$

for any $\epsilon > 0$. Inserting (3.21) into (3.20) and choose ϵ sufficiently small yields

$$0 \geq -C(h_{11}+N) + (N\phi' - C - CN^2)\sum_i F^{ii} + \frac{a}{u-a}\sum_i F^{ii}h_{ii}^2. \tag{3.22}$$

Since $F^{11} \leq F^{22} \leq \cdots \leq F^{nn}$ and $h_{nn} \leq -\frac{h_{11}}{n}$, we get

$$\sum_i F^{ii}h_{ii}^2 > F^{nn}h_{nn}^2 \geq \frac{1}{n^3}h_{11}^2\sum_i F^{ii}. \tag{3.23}$$

Hence,

$$0 \geq -C(h_{11}+N) + (N\phi' - C - CN^2 + Ch_{11}^2)\sum_i F^{ii}. \tag{3.24}$$

Since $\sum_i F^{ii} \geq p$, inequality (3.23) implies $h_{11} \leq CN$ at X_0.

Case B. $h_{nn} > -\frac{h_{11}}{n}$. Let

$$I = \{i | F^{ii} \leq n^2 F^{11}\}.$$

By (3.21), we have

$$\begin{aligned}\sum_{i\in I}\frac{F^{ii}(\nabla_i h_{11})^2}{h_{11}^2} &\le \sum_i \frac{F^{ii}(\nabla_i u)^2}{(u-a)^2}+\epsilon\frac{\sum_i F^{ii}h_{ii}(\nabla_i\Phi)^2}{(u-a)^2}+\frac{1+\epsilon}{\epsilon}N^2\sum_i F^{ii}(\nabla_i\Phi)^2\\ &\le \sum_i \frac{F^{ii}(\nabla_i u)^2}{(u-a)^2}+\epsilon\frac{\sum_i F^{ii}h_{ii}(\nabla_i\Phi)^2}{(u-a)^2}+C\frac{1+\epsilon}{\epsilon}N^2F^{11}.\end{aligned} \tag{3.25}$$

Inserting (3.25) into (3.20) and choosing ϵ sufficiently small yield

$$\begin{aligned}0\ge& -C(h_{11}+N)-\frac{1}{h_{11}}\sum_{p,q,r,s}F^{pq,rs}\nabla_1 h_{pq}\nabla_1 h_{rs}+C\sum_i F^{ii}h_{ii}^2\\ &+(N\phi'-C)\sum_i F^{ii}-\sum_{i\notin I}\frac{F^{ii}(\nabla_i h_{11})^2}{h_{11}^2}-CN^2F^{11}.\end{aligned} \tag{3.26}$$

By Lemma 1.1 in Gerhardt [5] and the concavity of F, we have

$$\begin{aligned}-\sum_{p,q,r,s}F^{pq,rs}\nabla_1 h_{pq}\nabla_1 h_{rs} &= -\sum_{p\neq q}\frac{\partial^2 F}{\partial h_{pp}\partial h_{qq}}\nabla_1 h_{pp}\nabla_1 h_{qq}+\sum_{p\neq q}F^{pp,qq}(\nabla_1 h_{pq})^2\\ &= -\sum_{p\neq q}\frac{\partial^2 F}{\partial h_{pp}\partial h_{qq}}\nabla_1 h_{pp}\nabla_1 h_{qq}-\sum_{p\neq q}\frac{F^{pp}-F^{qq}}{h_{pp}-h_{qq}}(\nabla_1 h_{pq})^2\\ &\ge \sum_{p\neq q}\frac{F^{pp}-F^{qq}}{h_{qq}-h_{pp}}(\nabla_1 h_{pq})^2.\end{aligned} \tag{3.27}$$

Plugging into (3.26), we have

$$\begin{aligned}0\ge& -C(h_{11}+N)+\frac{1}{h_{11}}\sum_{p\neq q}\frac{F^{pp}-F^{qq}}{h_{qq}-h_{pp}}(\nabla_1 h_{pq})^2+C\sum_i F^{ii}h_{ii}^2\\ &+(N\phi'-C)\sum_i F^{ii}-\sum_{i\notin I}\frac{F^{ii}(\nabla_i h_{11})^2}{h_{11}^2}-CN^2F^{11}.\end{aligned} \tag{3.28}$$

Now

$$\sum_{p\neq q}\frac{F^{pp}-F^{qq}}{h_{qq}-h_{pp}}(\nabla_1 h_{pq})^2\ge 2\sum_{p\notin I}\frac{F^{11}-F^{pp}}{h_{pp}-h_{11}}(\nabla_1 h_{1p})^2\ge 2\frac{(n-1)}{nh_{11}}\sum_{i\notin I}F^{ii}(\nabla_1 h_{1i})^2. \tag{3.29}$$

By Codazzi equation (2.6) and inequality $(a+b)^2\ge \frac{n}{2(n-1)}a^2-\frac{n}{n-2}b^2$, we have

$$\sum_{i\notin I}F^{ii}(\nabla_1 h_{1i})^2=\sum_{i\notin I}F^{ii}(\nabla_i h_{11}+\bar R_{\nu 1i1})^2\ge \sum_{i\notin I}F^{ii}(\nabla_i h_{11})^2-C\sum_{i\notin I}F^{ii}. \tag{3.30}$$

Combining (3.28), (3.29) and (3.30) gives

$$\begin{aligned}0&\ge -C(h_{11}+N)+C\sum_i F^{ii}h_{ii}^2+(N\phi'-C)\sum_i F^{ii}-CN^2F^{11}\\ &\ge -C(h_{11}+N)+(N\phi'-C)\sum_i F^{ii}+CF^{11}(h_{11}^2-CN^2).\end{aligned} \tag{3.31}$$

Denote $A = \sum_{i=n-p+1}^{n} h_{ii}$. By equation (2.9), we have

$$F^{nn} = \frac{1}{C_n^p}\mathcal{F}(\lambda) \sum_{\substack{n\in\{i_1,i_2,\cdots,i_p\}\\ 1\le i_1<i_2<\cdots<i_p\le n}} \frac{1}{\lambda_{i_1}+\lambda_{i_2}+\cdots+\lambda_{i_p}}$$
$$\ge \frac{1}{C_n^p}\frac{F(b)}{A} = \frac{1}{C_n^p}\frac{\tilde{f}}{A} \ge \frac{C}{A}, \tag{3.32}$$

and

$$F^{nn} \ge \frac{1}{C_n^p}\frac{F(b)}{A} = \frac{1}{C_n^p F(b)^{C_n^p-1}A}F(b)^{C_n^p}$$
$$= \frac{1}{C_n^p \tilde{f}^{C_n^p-1}A}\prod\nolimits_{2\le i_1<i_2<\cdots<i_{p-1}\le n}(h_{11}+\sum_{k=1}^{p-1}h_{i_ki_k}) \times \prod\nolimits_{2\le i_1<i_2<\cdots<i_p\le n}(\sum_{k=1}^{p}h_{i_ki_k})$$
$$\ge C\left(\frac{n-p+1}{n}h_{11}\right)^{C_{n-1}^{p-1}} \times A^{C_{n-1}^p-1}, \tag{3.33}$$

when $\frac{n}{2} \le p \le n-1$. Hence

$$\sum_i F^{ii} \ge F^{nn} \ge Ch_{11}. \tag{3.34}$$

Plugging (3.34) into (3.31), we have

$$0 \ge (N\phi' - C)h_{11} - CN + CF^{11}(h_{11}^2 - CN^2). \tag{3.35}$$

If we choose N sufficiently large and assume that $h_{11} > CN$, then

$$0 \ge CN^2\phi' - CN \ge CN^2 - CN.$$

This leads to a contradiction when we take sufficiently large N. Hence $h_{11} < CN$ at X_0.

4. Gradient estimate

In this section, we establish the C^1 estimate for equation (1.1) by the method of Guan-Lin-Ma in [14].

Theorem 4.1. *Let M be a closed star-shaped hypersurface in $\bar{M}$ satisfying equation* (1.1) *and assumption* (A_1) *hold. If ρ has positive upper and lower bounds, then there is a constant $C = C(\inf_M \rho, \sup_M \rho)$, such that $|\nabla'\rho| \le C$.*

Proof. Since

$$u = \frac{\phi^2(\rho)}{\sqrt{\phi^2(\rho)+|\nabla'\rho|^2}},$$

it suffices to obtain a positive lower bound of u on Σ. We consider the quantity

$$H = -\log u + \gamma(\Phi(\rho)),$$

where γ is a single-variable function to be determined later. Suppose that the maximum of H is achieved at $X_0 \in \Sigma$, where $X_0 = X(z_0)$, $z_0 \in M$.

Let θ be the angle between $V(X_0)$ and $\nu(X_0)$. By equation (2.3), $\langle V(X), \nu(X)\rangle = \frac{\phi}{\sqrt{1+\frac{|\nabla'\rho|^2}{\phi^2}}} > 0$. Hence, $\theta \in (0, \frac{\pi}{2}]$. Now we split the proof into two cases.

Case A. $\theta \in (0, \theta_0]$ for some $\theta_0 \in (0, \frac{\pi}{2})$. In this case,

$$H(X) \leq H(X_0) = -\log u(X_0) + \gamma(\Phi(\rho(z_0))) \leq -\log(|V(X_0)| \cos\theta_0) + \gamma(\rho(z_0)),\ \forall X \in \Sigma,$$

which implies a uniform lower bound of u on Σ.

Case B. $\theta \in [\theta_0, \frac{\pi}{2}]$. We can choose a local orthonormal frame $\{E_1, E_2, \cdots, E_n\}$ on Σ around X_0 such that $\langle V, E_1\rangle \neq 0$, and $\langle V, E_i\rangle = 0$, $i \geq 2$. Differentiating H twice at X_0 yields

$$0 = -\frac{\nabla_i u}{u} + \gamma'\nabla_i\Phi, \tag{4.1}$$

and

$$0 \geq \nabla_{ii}H = -\frac{\nabla_{ii}u}{u} + \left(\frac{\nabla_i u}{u}\right)^2 + \gamma'\nabla_{ii}\Phi + \gamma''(\nabla_i\Phi)^2. \tag{4.2}$$

By critical equation (4.1), we have

$$u\gamma'\nabla_1\Phi = \nabla_1 u = \sum_k h_{1k}\nabla_1\Phi = h_{11}\langle V, E_1\rangle.$$

At X_0, we have $h_{11} = u\gamma'$. For $i \geq 2$, we get

$$u\gamma'\nabla_i\Phi = \nabla_i u = \sum_k h_{ik}\nabla_k\Phi = h_{i1}\langle V, E_1\rangle.$$

By $\nabla_i\Phi = \langle V, E_i\rangle = 0,\ i \geq 2$, we have $h_{1i} = 0,\ i \geq 2$ at X_0. Hence we can rotate $\{E_2, E_3, \cdots, E_n\}$ so that $\{h_{ij}\}$ is diagonal at X_0 and hence,

$$\{F^{ij}(b)\} = \mathrm{diag}(\mathcal{F}_1(\kappa), \mathcal{F}_2(\kappa), \cdots, \mathcal{F}_n(\kappa)).$$

By equation (4.1), (3.16) and (3.18), (4.2) implies that

$$\begin{aligned} 0 \geq &-\frac{1}{u}\left(\nabla_1 h_{ii}\nabla_1\Phi + \phi' h_{ii} - u h_{ii}^2 + \bar{R}_{\nu i1i}\nabla_1\Phi\right) + [(\gamma')^2 + \gamma''](\delta_{i1}\nabla_1\Phi)^2 \\ &+ \gamma'(\phi' g_{ii} - u h_{ii}). \end{aligned} \tag{4.3}$$

Contracting (4.3) with F^{ii}, we have

$$\begin{aligned} 0 \geq &-\frac{1}{u}\left(F^{ii}\nabla_1 h_{ii}\nabla_1\Phi + \phi' F^{ii}h_{ii} - u\sum F^{ii}h_{ii}^2 + \sum F^{ii}\bar{R}_{\nu i1i}\nabla_1\Phi\right) \\ &+ [(\gamma')^2 + \gamma'']F^{11}(\nabla_1\Phi)^2 + \gamma'\left(\phi'\sum F^{ii} - u\sum F^{ii}h_{ii}\right). \end{aligned} \tag{4.4}$$

Differentiating equation $F(b) = \tilde{f}$ at X_0 yields

$$F^{ii}\nabla_1 h_{ii} = \tilde{f}_V(\overline{\nabla}_1 V) + h_{11}\tilde{f}_\nu(E_1). \tag{4.5}$$

Inserting (4.5) and $F^{ii}h_{ii} = \tilde{f}$ into (4.4) yields

$$\begin{aligned} 0 \geq &-\frac{1}{u}\left(\langle V, e_1\rangle \tilde{f}_V(\overline{\nabla}_1 V) + \phi'\tilde{f}\right) - \gamma'\langle V, E_1\rangle \tilde{f}_\nu(E_1) + \sum F^{ii}h_{ii}^2 \\ &-\frac{1}{u}\sum F^{ii}\bar{R}_{\nu i1i}\nabla_1\Phi + [(\gamma')^2 + \gamma'']F^{11}\langle V, E_1\rangle^2 + \gamma'\phi'\sum F^{ii} - \gamma' u\tilde{f}. \end{aligned} \tag{4.6}$$

Assumption (A_1) is equivalent to

$$\phi'\tilde{f} + \tilde{f}_V(\overline{\nabla}_V V) \leq 0. \tag{4.7}$$

Since at X_0, $V = \langle V, E_1\rangle E_1 + \langle V, \nu\rangle\nu$, we have

$$\tilde{f}_V(\overline{\nabla}_V V) = \langle V, E_1\rangle \tilde{f}_V(\overline{\nabla}_1 V) + u\tilde{f}_V(\overline{\nabla}_\nu V). \tag{4.8}$$

It follows from (3.8) that

$$u\tilde{f}_V(\overline{\nabla}_\nu V) = u\phi'\tilde{f}_V(\nu). \tag{4.9}$$

Combing (4.6)-(4.9), we obtain

$$\begin{aligned} 0 \geq\ &\phi'\tilde{f}_V(\nu) - \gamma'\langle V, E_1\rangle \tilde{f}_\nu(E_1) + \sum F^{ii}h_{ii}^2 + [(\gamma')^2 + \gamma'']F^{11}\langle V, E_1\rangle^2 \\ &+ \gamma'\phi'\sum F^{ii} - \gamma' u\tilde{f} - \frac{1}{u}\sum F^{ii}\bar{R}_{\nu i1i}\nabla_1\Phi. \end{aligned} \tag{4.10}$$

Now we choose

$$\gamma(t) = \frac{\alpha}{t}, \tag{4.11}$$

where α is sufficiently large. Recalling that $h_{11} = u\gamma'$, we have $h_{11} < 0$. Since $\{h_{ij}\} \in P_p$, h_{11} is in the p smallest ones of $\{h_{11}, h_{22}, \cdots, h_{nn}\}$. By Lemma 2.3, we have

$$F^{11} \geq \theta\sum F^{ii} \geq p\theta. \tag{4.12}$$

Therefore, by (4.10) and (4.11), we have

$$\begin{aligned} 0 \geq\ &F^{11}h_{11}^2 + \left(\frac{\alpha^2}{t^4} + \frac{2\alpha}{t^3}\right)F^{11}\langle V, E_1\rangle^2 - \frac{\alpha}{t^2}\phi'\sum F^{ii} - \frac{1}{u}\sum F^{ii}\bar{R}_{\nu i1i}\nabla_1\Phi \\ &+ \frac{\alpha}{t^2}\langle V, E_1\rangle \tilde{f}_\nu(E_1) + \frac{\alpha}{t^2}u\tilde{f} + \phi'\tilde{f}_V(\nu). \end{aligned} \tag{4.13}$$

Now we extract factor u from the curvature term $\bar{R}_{\nu i1i}$ above by the method as Andrade-Barbosa-Lira [30] and Chen-Li-Wang [4]. Let $\{e_1', \cdots, e_n'\}$ a local orthonormal frame of M, then we can define an orthonormal frame on $\bar{M}^{n+1}$ by $\bar{E}_i = \frac{1}{\phi}e_i'$, $1 \leq i \leq n$, $\bar{E}_0 = \frac{\partial}{\partial\rho}$, and vice versa. Since $V = \phi\bar{E}_0$ and $V \perp E_i, i \geq 2$ at X_0, we may choose $\bar{E}_2 = E_2, \cdots, \bar{E}_n = E_n$ around X_0. Then $V = \phi\bar{E}_0$ and $\bar{E}_1$ lie in the plane perpendicular to Span$\{\bar{E}_2, \cdots, \bar{E}_n\}$ in $T_{X_0}\bar{M}^{n+1}$. Similarly, since E_1 and ν are perpendicular to Span$\{E_2, \cdots, E_n\}$, they lie in the same plane as $V(X_0)$ and $\bar{E}_1$ in $T_{X_0}\bar{M}^{n+1}$.

By $V = \langle V, E_1\rangle E_1 + \langle V, \nu\rangle\nu$ and $\langle V, \bar{E}_1\rangle = \phi\langle\bar{E}_0, \bar{E}_1\rangle = 0$, we obtain

$$0 = \langle V, \bar{E}_1\rangle = \langle V, E_1\rangle\langle E_1, \bar{E}_1\rangle + u\langle\nu, \bar{E}_1\rangle \Rightarrow \langle E_1, \bar{E}_1\rangle = -\frac{u\langle\nu, \bar{E}_1\rangle}{\langle V, E_1\rangle}. \tag{4.14}$$

In the orthonormal frame $\{\bar{E}_1, \cdots, \bar{E}_n, \bar{E}_0\}$, the vectors ν and E_1 can be decomposed into

$$\nu = \langle \nu, \bar{E}_0\rangle \bar{E}_0 + \langle \nu, \bar{E}_1\rangle \bar{E}_1 = \frac{u}{\phi}\bar{E}_0 + \langle \nu, \bar{E}_1\rangle \bar{E}_1, \tag{4.15}$$

$$E_1 = \langle E_1, \bar{E}_0\rangle \bar{E}_0 + \langle E_1, \bar{E}_1\rangle \bar{E}_1. \tag{4.16}$$

By Lemma 2.1 and (4.14)-(4.16), we have $\bar{R}_{\nu 111} = 0$ and for $i \geq 2$,

$$\begin{aligned}\bar{R}_{\nu i1i} &= \bar{R}(\nu, E_i, E_1, E_i)\\ &= \frac{u}{\phi}\bar{R}(\bar{E}_0, \bar{E}_i, E_1, \bar{E}_i) + \langle \nu, \bar{E}_1\rangle \bar{R}(\bar{E}_1, \bar{E}_i, E_1, \bar{E}_i)\\ &= \frac{u}{\phi}\langle E_1, \bar{E}_0\rangle \bar{R}(\bar{E}_0, \bar{E}_i, \bar{E}_0, \bar{E}_i) + \langle \nu, \bar{E}_1\rangle\langle E_1, \bar{E}_1\rangle \bar{R}(\bar{E}_1, \bar{E}_i, \bar{E}_1, \bar{E}_i)\\ &= u\left(\frac{1}{\phi}\langle E_1, \bar{E}_0\rangle \bar{R}(\bar{E}_0, \bar{E}_i, \bar{E}_0, \bar{E}_i) - \frac{\langle \nu, \bar{E}_1\rangle^2}{\langle V, E_1\rangle}\bar{R}(\bar{E}_1, \bar{E}_i, \bar{E}_1, \bar{E}_i)\right)\end{aligned} \tag{4.17}$$

Inserting $h_{11} = u\gamma'$, $h_{1i} = 0,\ i \geq 2$, (4.12) and (4.17) into (4.13), we obtain

$$\begin{aligned}0 \geq\ & \alpha^2 u^2 F^{11} + \left(\frac{\alpha^2}{t^4} + \frac{2\alpha}{t^3}\right) F^{11}\langle V, E_1\rangle^2 - C_1\alpha F^{11} - C_2\alpha|\langle V, E_1\rangle||\tilde{f}_\nu(E_1)|\\ & - C_3|\tilde{f}_V(\nu)| - C_4\alpha,\end{aligned} \tag{4.18}$$

where $C_1, \cdots, C_4$ are large positive constants depending on $n, p, |\rho|_{C^0}$. Based on the C^0 estimate, we have $0 < r_1 \leq \rho \leq r_2$. This implies that $t = \Phi(\rho)$ has positive lower and upper bounds. Note that $|\langle V, E_1\rangle| \geq |V|\sin\theta_0 = \phi^2\sin\theta_0$ has a positive lower bound at X_0, and therefore we have

$$\left(\frac{\alpha^2}{t^4} + \frac{2\alpha}{t^3}\right) F^{11}\langle V, E_1\rangle^2 - C_1\alpha F^{11} \geq 2c_0\alpha^2 F^{11} - C_1\alpha F^{11},$$

where c_0 is a small positive constant depending on $|\rho|_{C^0}$ and θ_0. By taking α sufficiently large such that $c_0\alpha > C_1$, we have

$$\left(\frac{\alpha^2}{t^4} + \frac{2\alpha}{t^3}\right) F^{11}\langle V, E_1\rangle^2 - C_1\alpha F^{11} \geq c_0\alpha^2 F^{11} \geq c_0 p\theta\alpha^2 \tag{4.19}$$

where we have used (4.12) in the last inequality. Therefore, by dropping the positive term $\alpha^2 u^2 F^{11}$ in (4.18) and inserting (4.19) into (4.18), we derive

$$0 \geq c_0\alpha^2 - C_2\alpha|\langle V, E_1\rangle||\tilde{f}_\nu(E_1)| - C_3|\tilde{f}_V(\nu)| - C_4\alpha.$$

Thus, we get a contradiction if α is large enough. Therefore Case B cannot take place for large α. The proof is completed. □

5. C^0 estimate and existence result

In this section, we establish C^0 estimate of the solution for equation (1.1), and use the method of continuity [10] to obtain the existence of a closed p-convex star-shaped hypersurface Σ satisfying equation (1.1) with $\frac{n}{2} \leq p \leq n-1$.

For $t \in [0,1]$, we define

$$f_t(V(X),\nu(X)) = tf(V(X),\nu(X)) + (1-t)\varphi(\rho)\mathcal{F}_p(\kappa(\rho)),$$

where $\kappa(\rho) = \frac{\phi'}{\phi}$ and φ is a positive function defined on I satisfying (a) $\varphi > 0$; (b) $\varphi(\rho) > 1$ for $\rho \le r_1$; (c) $\varphi(\rho) < 1$ for $\rho \ge r_2$ and (d) $\varphi'(\rho) < 0$.

Now we embed equation (1.1) into the following equations for star-shaped hypersurfaces Σ_t:

$$F(b) = \tilde{f}_t(V(X_t),\nu_t(X_t)), \tag{5.1}$$

where $\tilde{f}_t = f_t^{\frac{1}{C_n^p}}$, $X_t = (z,\rho_t(z))$ $(z \in M)$ is the position vector of Σ_t, ν_t is the unit normal vector of Σ_t. Obviously, $f_t(V(X_t),\nu_t(X_t))$ is strictly positive for $|V(X_t)| \in [r_1,r_2]$. There exists a unique point $r_0 \in (r_1,r_2)$ such that $\varphi(r_0) = 1$, then the geodesic sphere with radius r_0 satisfying equation (5.1) when $t = 0$. By Evans-Krylov theorem, we need to establish the C^0, C^1 and C^2 estimates of solutions for equation (5.1) to obtain the existence result.

Theorem 5.1. *Let $f \in C^2\left(\pi^{-1}(\overline{B_{r_2}} \setminus B_{r_1}) \times \mathbb{S}^n\right)$ be a positive function. Suppose assumptions (A_1) and (A_2) hold. Then there is a unique closed star-shaped p-convex hypersurface Σ_t in $\bar{M}^{n+1}$ satisfying* (5.1) *with $\frac{n}{2} \le p \le n-1$. Moreover, for $0 \le t \le 1$, $r_1 \le \rho_t \le r_2$.*

Proof. For the C^0 estimates of ρ_t, we need the crucial assumption (A_2). By the assumption (b), (c) of φ, we can verify that, for $t \in [0,1)$

$$f_t(V,\frac{V}{|V|}) > p^{C_n^p}\left(\frac{\phi'}{\phi}\right)^{C_n^p}(\rho), \text{for } \rho = r_1, \tag{5.2}$$

$$f_t(V,\frac{V}{|V|}) < p^{C_n^p}\left(\frac{\phi'}{\phi}\right)^{C_n^p}(\rho), \text{for } \rho = r_2. \tag{5.3}$$

We claim that for any $t \in [0,\delta)$, $r_1 < \rho_t < r_2$ on M. Due to the continuous dependence of ρ_t on $t \in [0,\delta)$, we can assume that $\max \rho_{t_0} = \rho_{t_0}(z_0) = r_2$ for some $t_0 \in (0,\delta)$ on the contrary. By (2.3)-(2.5), we have, at $X_{t_0}(z_0)$

$$\sqrt{(g_{t_0})^{-1}}\{(h_{t_0})_{ij}\}\sqrt{(g_{t_0})^{-1}} = \frac{1}{\phi^2}\delta_{ij}^2\left(-\nabla'_{ij}(\rho_{t_0}) + \phi\phi'\delta_{ij}\right) \ge \frac{\phi'}{\phi}\delta_{ij}. \tag{5.4}$$

By (5.3), (5.4) and equation (1.1), we have, at X_0,

$$f_{t_0}(V(X_{t_0}(z_0)),\frac{V(X_{t_0}(z_0))}{|V(X_{t_0}(z_0))|}) < p^{C_n^p}\left(\frac{\phi'}{\phi}\right)^{C_n^p}(\rho_{t_0}(z_0)),\ \mathcal{F}_p(\kappa) \ge p^{C_n^p}\left(\frac{\phi'}{\phi}\right)^{C_n^p}(\rho_{t_0}(z_0)),$$

which contradicts with equation (5.1). Similarly, we can show $\rho_t(z) > r_1$ on M for $t \in [0,\delta)$.

For the C^1 estimates of ρ_t, we need the crucial assumption (A_1). For any given ν,

$$\begin{aligned}\frac{\partial}{\partial\rho}\left(\phi^{C_n^p}f_t(V,\nu)\right) &= C_n^p\phi^{C_n^p-1}\phi' f_t(V,\nu) + \phi^{C_n^p}\frac{\partial}{\partial\rho}\left(f_t(V,\nu)\right)\\ &= C_n^p\phi^{C_n^p}\left[\kappa(\rho)f_t(V,\nu) + \frac{\partial}{\partial\rho}\left(f_t(V,\nu)\right)\right].\end{aligned}$$

By the Lemma 4 in [30], we can choose φ such that

$$\kappa(\rho)f_t(V,\nu)+\frac{\partial}{\partial\rho}\left(f_t(V,\nu)\right)<0.$$

So we have, for any given ν,

$$\frac{\partial}{\partial\rho}[\phi^{C_n^p}f_t(V,\nu)]\leq 0, \text{ where } |V|=\phi(\rho). \tag{5.5}$$

The C^2 estimates for ρ_t has already been established in Section 2. Based on the above prior estimates, we use the continuity methods to obtain the existence results.

Let $S=\{s\in[0,1]$: equation (5.1) has a p-convex solution when $t=s\}$. Recalling that $\rho_0(z)=r_0$ is the unique p-convex solution of equation (5.1). Therefore, $0\in S$. Since equation (5.1) is strictly elliptic, by implicit function theorem that there exists a small constant $\delta\in(0,1)$ such that for any $t\in[0,\delta)$, equation (5.1) admits a unique p-convex solution ρ_t which continuously depends on t.

By Evans-Krylov theorem, for $\alpha\in(0,1)$, we have

$$\|\rho_t\|_{2,\alpha}\leq C, \text{ when } t\in[0,\delta).$$

for some constant $C>0$. Hence, equation (5.1) with $t=\delta$ obtain a unique solution $\rho_\delta=\lim_{t\to\delta^-}\rho_t$,

By implicit function theorem, equation (5.1) admits a unique p-convex solution ρ_t for $t\in[\delta,2\delta)$. We also can show $r_1<\rho_t<r_2$ for $t\in[\delta,2\delta)\cap[0,1)$ as above. Repeating the process, we have $S=[0,1]$. It implies that there is a unique solution of equation (5.1) at $t=1$. The theorem is now proved. $\square$

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