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Strongly damped wave equations with fractional diffusions: Well-posedness and global attractors

Équations d'ondes fortement amorties avec diffusions fractionnaires

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ABSTRACT. This paper is concerned with the nonlinear strongly damped wave equations involving the fractional Laplacian and regional fractional Laplacian with various boundary conditions. We first prove the existence and uniqueness of weak solutions using the compactness method and weak convergence techniques in Orlicz spaces. Then we study the existence and regularity of global attractors of associated semigroups. The main novelty of the obtained results here is to improve and extend the previous results in [6, 7, A.N. Carvalho and J.W. Cholewa] and [24, J. Shomberg].
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1. Introduction

In this paper, we will consider the strongly damped wave equations involving fractional diffusions with various boundary conditions as follows:

$$\begin{cases} u_{tt} + (-\Delta)^\alpha u_t + (-\Delta)^\beta u + f(u) = g & \text{in} \quad \Omega \times (0,\infty), \\ u = 0 & \text{on} \quad (\mathbb{R}^3 \backslash \Omega) \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \\ u_t(x,0) = u_1(x) & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

and

$$\begin{cases} u_{tt} + \mathcal{A}_\Omega^\alpha u_t + \mathcal{A}_\Omega^\beta u + f(u) = g & \text{in} \quad \Omega \times (0,\infty), \\ u = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \\ u_t(x,0) = u_1(x) & \text{in} \quad \Omega, \end{cases} \tag{1.2}$$

and

$$\begin{cases} u_{tt} + \mathcal{A}_\Omega^\alpha u_t + \mathcal{A}_\Omega^\beta u + f(u) = g & \text{in} \quad \Omega \times (0,\infty), \\ B_{N,\alpha}\mathcal{N}^{2-2\alpha} u + \gamma_\alpha u = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ B_{N,\beta}\mathcal{N}^{2-2\beta} u + \gamma_\beta u = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \\ u_t(x,0) = u_1(x) & \text{in} \quad \Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^3$ is an arbitrary bounded open set with boundary $\partial\Omega$. The nonlinearity f satisfies a dissipativity condition (see Section 3 below). $\mathcal{N}^{2-2\alpha}u$ and $\mathcal{N}^{2-2\beta}u$ are the fractional normal derivatives of the function u. $B_{N,\alpha}$ and $B_{N,\beta}$ are normalized constants (see Theorem 2.2 below). γ_α and γ_β are nonnegative constants. $(-\Delta)^\alpha$ and $(-\Delta)^\beta$ are the fractional Laplacian operators. $\mathcal{A}^\alpha_\Omega$ and $\mathcal{A}^\beta_\Omega$ are the regional fractional Laplacian operators.

The strongly damped wave equations appear in many relevant physical applications (see e.g. [12, 22, 24]). They have been investigated quite extensively by several authors in recent years (see e.g. [3, 6, 7, 12, 13, 14, 21, 22, 24]), with particular regard to their asymptotics.

Recently, the nonlocal equations of fractional order have gained a lot of attention from the partial differential equations research community. To the best of our knowledge, the strongly damped wave equation with the fractional operator $(-\Delta)^s u_t$, $0 < s < 1$ (such powers may be defined through a Fourier series) given by

$$u_{tt} + (-\Delta)^s u_t + (-\Delta)u + f(u) = 0,$$

was studied in [6]. A.N. Carvalho and J.W. Cholewa proved the global well-posedness results for $s \in [\frac{1}{2}, 1]$ and the existence of a compact global attractor for associated semigroup in different cases of nonlinearity. In [7], they studied the local well-posedness of the above equation with $f(u, u_t)$ in place of $f(u)$. Latter, J. Shomberg [24] considered the following semilinear strongly damped wave equations with fractional diffusion operators

$$u_{tt} + (-\Delta)^\alpha u_t + (-\Delta)^\beta u + f(u) = 0,$$

where $(-\Delta)^\alpha$ and $(-\Delta)^\beta$ are the fractional Laplace operators with extended homogeneous Dirichlet boundary condition. The author proved the existence, uniqueness, global well-posedness, regularity of solutions and relation between the solutions. Moreover, the analytic and Gevrey class properties of the semigroup were discussed for certain parameters α and β, and for certain exponents.

Fractional diffusion operators have been used to study anomalous diffusions that cannot be properly described by integer-order partial differential equations. Importantly, fractional diffusion operators appear in the treatment of reaction diffusion equations [16], from which our results are borrowed. In order to make the paper as self-contained as possible, we start by introducing the fractional Laplacian. Let $0 < s < 1$, $\Omega \subset \mathbb{R}^N$ an arbitrary open set and let

$$\mathcal{L}^1(\Omega) := \{u : \Omega \to \mathbb{R} \text{ measurable}, \int\limits_\Omega \frac{|u(x)|}{(1+|x|)^{N+2s}} dx < \infty\}.$$

For $u \in \mathcal{L}^1(\mathbb{R}), x \in \mathbb{R}^N$ and $\varepsilon > 0$, we write

$$(-\Delta)^s_\varepsilon u(x) := C_{N,s} \int\limits_{\{y \in \mathbb{R}^N, |y-x|>\varepsilon\}} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy,$$

with the normalized constant $C_{N,s}$ given by

$$C_{N,s} := \frac{s 2^{2s} \Gamma(\frac{N+2s}{2})}{\pi^{\frac{N}{2}} \Gamma(1-s)}, \tag{1.4}$$

where Γ denotes the usual Gamma function. The fractional Laplacian $(-\Delta)^s u$ of the function u is defined by

$$(-\Delta)^s u(x) := C_{N,s} P.V. \int\limits_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy = \lim_{\varepsilon \downarrow 0} (-\Delta)^s_\varepsilon u(x),\ x \in \mathbb{R}^N,$$

provided that the limit exists.

We notice that in the whole space $\mathbb{R}^N$, using the Fourier transform, $(-\Delta)^s$ can be defined as a pseudo-differential operator with symbol $|\xi|^{2s}$. If one wishes to consider the fractional Laplace operator $(-\Delta)^s$ on open subsets Ω of $\mathbb{R}^N$ it cannot be used on Ω automatically due to its nonlocal character. In order to give a proper definition, we follow [18, 19, 28] in the following fashion. Let Ω be an arbitrary open set in $\mathbb{R}^N$. For $u \in \mathcal{L}^1(\Omega), x \in \Omega$ and $\varepsilon > 0$, we let

$$\mathcal{A}^s_{\Omega,\varepsilon} u(x) := C_{N,s} \int\limits_{\{y \in \Omega, |y-x|>\varepsilon\}} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy,$$

and we define the operator $\mathcal{A}^s_\Omega$ as follows

$$\mathcal{A}^s_\Omega u(x) := C_{N,s} P.V. \int\limits_{\Omega} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy = \lim_{\varepsilon \downarrow 0} \mathcal{A}^s_{\Omega,\varepsilon} u(x),\ x \in \Omega,$$

provided that the limit exists. We call the operator A^s_Ω *the regional fractional Laplacian.*

The fractional Laplacian operator and the regional fractional Laplacian operators have important motivations. In the context of physical motivation, for a point $x \in \Omega$, the fractional Laplacian $(-\Delta)^s$ accounts for the interactions between x and y for all $y \in \mathbb{R}^N \backslash \{x\}$. Whereas, the regional fractional Laplacian $\mathcal{A}^s_\Omega$ accounts for the interactions between x and y for all $y \in \Omega\backslash\{x\}$. They describe a particle jumping from one point $x \in \Omega$ to another point $y \in \Omega$ with intensity proportional to $\frac{1}{|x-y|^{N+2s}}$. In the context of probability, the fractional Laplacian operator $(-\Delta)^s$ represents the infinitesimal generator of a symmetric $2s$-stable Lévy process and the fractional Laplacian $(-\Delta)^s$ on a bounded domain Ω with extended homogeneous Dirichlet boundary condition represents that particles are killed upon leaving the domain Ω. Whereas, the regional fractional Laplacian $\mathcal{A}^s_\Omega$ is considered as the fractional Laplacian $(-\Delta)^s$ by restricting its measure on Ω.

The paper is organized as follows. In Section 2, we recall some intermediate results that will be used to obtain our main results. In Section 3, we prove the existence and uniqueness of weak solutions to the strongly damped wave equations involving fractional diffusions with various boundary conditions (1.1)-(1.3). In Section 4, we prove the existence of global attractors in $\mathbb{H}^\beta_K(\Omega)$ and their regularity.

2. Prelimilaries

In this section, we recall some intermediate results that will be used to obtain our main results.

Let $\Omega \subset \mathbb{R}^N$ be an arbitrary bounded open set with boundary $\partial\Omega$. We denote by $\mathcal{D}(\Omega)$ the space of test functions on Ω. For $s \in (0,1)$, we denote by

$$W^{s,2}(\Omega) := \{u \in L^2(\Omega) : \int\limits_\Omega \int\limits_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy < \infty\}$$

the fractional order Sobolev space endowed with the norm

$$\|u\|_{W^{s,2}(\Omega)} := \left(\int\limits_{\Omega} |u(x)|^2 dx + \frac{C_{N,s}}{2} \int\limits_{\Omega} \int\limits_{\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy \right)^{\frac{1}{2}}.$$

We let

$$W_0^{s,2}(\Omega) := \overline{\mathcal{D}(\Omega)}^{W^{s,2}(\Omega)}.$$

We have the following result taken from [17, Chapter 1] (see also [4, Corollary 2.8 and Remark 2.3] and [28])

Theorem 2.1. *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz continuous boundary. For every $0 < s \leq \frac{1}{2}$, the spaces $W^{s,2}(\Omega)$ and $W_0^{s,2}(\Omega)$ coincide with equivalent norms.*

Remark 2.1. In view of Theorem 2.1, to talk about well-defined traces (not necessarily null) of functions in $W^{s,2}(\Omega)$, it is not a restriction to assume that $\frac{1}{2} < s < 1$.

For advantages of using the method of bilinear Dirichlet forms, our framework is borrowed from [16, 28, 29](see also [26, 27]).

Let $\mathcal{E}_{E,s}$ be the bilinear symmetric closed form with its domain

$$D(\mathcal{E}_{E,s}) = \{u \in W^{s,2}(\mathbb{R}^N), u = 0 \text{ on } \mathbb{R}^N \backslash \Omega\}$$

and defined for $u, v \in D(\mathcal{E}_E)$ by

$$\mathcal{E}_{E,s}(u,v) = \frac{C_{N,s}}{2} \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dxdy.$$

Let A_E^s be the closed linear selfadjoint operator on $L^2(\Omega)$ associated with $\mathcal{E}_{E,s}$ in the sense that

$$\begin{cases} D(A_E^s) := \left\{u \in D(\mathcal{E}_{E,s}), (-\Delta)^s u \in L^2(\Omega)\right\}, \\ A_E^s u = (-\Delta)^s u. \end{cases}$$

We call A_E^s is said to be a realization of the fractional Laplace operator $(-\Delta)^s$ on $L^2(\Omega)$ with the extended homogeneous Dirichlet boundary condition.

Definition 2.1. *[29, Definition 2.5(a)] Let $u \in W^{s,2}(\Omega)$. We say that $\mathcal{A}_\Omega^s u \in L^2(\Omega)$ if there exists $w \in L^2(\Omega)$ such that*

$$\frac{C_{N,s}}{2} \int\limits_{\Omega} \int\limits_{\Omega} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dxdy = \int\limits_{\Omega} w(x)v(x)dx$$

for all $v \in \mathcal{D}(\Omega)$, and hence for all $v \in W_0^{s,2}(\Omega)$ by density. In that case, we write $\mathcal{A}_\Omega^s u = w$.

Next, we introduce the bilinear symmetric closed form $\mathcal{E}_{D,s}$ with domain $D(\mathcal{E}_{D,s}) = W_0^{s,2}(\Omega)$ as follows

$$\mathcal{E}_{D,s}(u,v) = \frac{C_{N,s}}{2} \int\limits_{\Omega} \int\limits_{\Omega} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dxdy, \ \forall u, v \in W_0^{s,2}(\Omega).$$

Let A_D^s be the closed linear selfadjoint operator on $L^2(\Omega)$ associated with $\mathcal{E}_{D,s}$ in the sense that

$$\begin{cases} D(A_D^s) := \left\{ u \in W_0^{s,2}(\Omega), \mathcal{A}_\Omega^s u \in L^2(\Omega) \right\}, \\ A_D^s u = \mathcal{A}_\Omega^s u. \end{cases}$$

In that case, A_D^s is said to be a realization of the regional fractional Laplace operator $\mathcal{A}_\Omega^s$ on $L^2(\Omega)$ with the Dirichlet boundary condition.

Before we introduce realizations of the regional fractional Laplace operator with fractional Neumann and fractional Robin boundary conditions, respectively, let us first recall the following integration by parts formula taken from [18, Theorem 3.3].

Theorem 2.2. *Let $\frac{1}{2} < s < 1$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{1,1}$. Setting*

$$C_{2s}^2(\overline{\Omega}) := \left\{ u : \ u(x) = a(x)\rho(x)^{2s-1} + b(x), \ \forall x \in \Omega, \text{ for some } a, b \in C^2(\overline{\Omega}) \right\},$$

where $\rho(x) := dist(x, \partial\Omega), \ x \in \Omega$. For $u \in C_{2s}^2(\overline{\Omega})$ and $z \in \partial\Omega$, we define the fractional normal derivative $\mathcal{N}^{2-2s}u$ of the function u by

$$\mathcal{N}^{2-2s}u(z) = -\lim_{t \downarrow 0} \frac{du(z + \vec{n}(z)t)}{dt} t^{2-2s},$$

where $\vec{n}(z)$ is the inner normal vector of $\partial\Omega$ at the point $z \in \partial\Omega$. Then for every $u \in C_{2s}^2(\overline{\Omega})$ and $v \in W^{s,2}(\Omega)$, one has $\mathcal{A}_\Omega^s u \in L^2(\Omega)$, $\mathcal{N}^{2-2s}u \in L^2(\partial\Omega)$, and

$$\begin{aligned} \int_\Omega v(x)\mathcal{A}_\Omega^s u(x)dx = & \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dxdy \\ & - B_{N,s} \int_{\partial\Omega} v\mathcal{N}^{2-2s}u d\sigma, \end{aligned} \tag{2.1}$$

where the constant $B_{N,s}$ is determined by

$$\frac{C_{1,s}}{C_{N,s}} B_{N,s} := \begin{cases} C_s & \text{if } N = 1, \\ \frac{2\pi^{\frac{N-1}{2}} C_s}{\Gamma(\frac{N-1}{2})} \int_0^{\frac{\pi}{2}} \cos^{2s}(\theta) \sin^{N-2}(\theta) d\theta & \text{if } N \geq 2, \end{cases} \tag{2.2}$$

and

$$C_s := \frac{C_{1,s}}{2s(2s-1)} \int_0^\infty \frac{(\tau - 1)^{1-2s} - \{\max(\tau, 1)\}^{1-2s}}{\tau^{2-2s}} d\tau,$$

and $C_{1,s}$ is given by (1.4) *with $N = 1$.*

Remark 2.2. We see from (2.1) that $B_{N,s}\mathcal{N}^{2-2s}u$ plays the role as normal derivative $\partial_\nu u$ in the classical Green formula for the Laplace operator ($\partial_\nu u := \nabla u \cdot \nu$ is the normal derivative of u in direction of the outer normal vector ν).

Then, we recall the weak formulation on nonsmooth domains of a fractional normal derivative.

Definition 2.2. *[29, Definition 2.5(b)] Let $\frac{1}{2} < s < 1$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz continuous boundary $\partial\Omega$. Let $u \in W^{s,2}(\Omega)$ such that $\mathcal{A}^s_\Omega u \in L^2(\Omega)$. We say that u has a fractional normal derivative in $L^2(\partial\Omega)$ if there exists $\psi \in L^2(\partial\Omega)$ such that*

$$\int_\Omega v(x)\mathcal{A}^s_\Omega u(x)dx = \frac{C_{N,s}}{2}\int_\Omega\int_\Omega \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dxdy - \int_{\partial\Omega}\psi v d\sigma,$$

for all $v \in W^{s,2}(\Omega)\cap C(\overline{\Omega})$, and hence for all $v \in W^{s,2}(\Omega)$ by density and trace theorem. In that case, we write $B_{N,s}\mathcal{N}^{2-2s}u = \psi$ and ψ is called the ***fractional normal derivative*** *of u.*

We have Green's type formula from Definition 2.1 and Definition 2.2 (see [29, Remark 2]) that

$$\begin{aligned}\int_\Omega v(x)\mathcal{A}^s_\Omega u(x)dx =& \frac{C_{N,s}}{2}\int_\Omega\int_\Omega \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dxdy \\ &- B_{N,s}\int_{\partial\Omega} v\mathcal{N}^{2-2s}u d\sigma\end{aligned}$$

holds for all $v \in W^{s,2}(\Omega)$ whenever $u \in W^{s,2}(\Omega)$, $\mathcal{A}^s_\Omega u \in L^2(\Omega)$ and $B_{N,s}\mathcal{N}^{2-2s}u$ exists in $L^2(\partial\Omega)$.

Throughout this section, we assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz continuous boundary $\partial\Omega$ and $\frac{1}{2} < s < 1$. We consider the bilinear symmetric closed form $\mathcal{E}_{\mathcal{N},s}$ on domain $D(\mathcal{E}_{\mathcal{N},s}) = W^{s,2}(\Omega)$ given by

$$\mathcal{E}_{\mathcal{N},s}(u,v) = \frac{C_{N,s}}{2}\int_\Omega\int_\Omega \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dxdy,\ \forall u,v \in W^{s,2}(\Omega).$$

Let $A^s_\mathcal{N}$ is the closed linear self-adjoint operator associated with $\mathcal{E}_{\mathcal{N},s}$ in the sense that

$$\begin{cases} D(A^s_\mathcal{N}) := \{u \in W^{s,2}(\Omega), \mathcal{A}^s_\Omega u \in L^2(\Omega), \mathcal{N}^{2-2s}u \text{ exists in } L^2(\partial\Omega) \\ \qquad\qquad\qquad \text{and } \mathcal{N}^{2-2s}u = 0 \text{ on } \partial\Omega\}, \\ A^s_\mathcal{N} u = \mathcal{A}^s_\Omega u. \end{cases}$$

We call $A^s_\mathcal{N}$ a realization of the regional fractional Laplace operator $\mathcal{A}^s_\Omega$ on $L^2(\Omega)$ with fractional Neumann type boundary conditions.

Finally, we consider the bilinear symmetric closed form $\mathcal{E}_{R,s}$ on domain $D(\mathcal{E}_{R,s}) = W^{s,2}(\Omega)$ and defined for $u, v \in W^{s,2}(\Omega)$ by

$$\mathcal{E}_{R,s}(u,v) = \frac{C_{N,s}}{2}\int_\Omega\int_\Omega \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dxdy + \int_{\partial\Omega}\gamma_s uv d\sigma.$$

Let A^s_R be the closed linear self-adjoint operator associated with $\mathcal{E}_{R,s}$ in the sense that

$$\begin{cases} D(A^s_R) := \{u \in W^{s,2}(\Omega), \mathcal{A}^s_\Omega u \in L^2(\Omega), \mathcal{N}^{2-2s}u \text{ exists in } L^2(\partial\Omega) \\ \qquad\qquad\qquad \text{and } B_{N,s}\mathcal{N}^{2-2s}u + \gamma_s u = 0 \text{ on } \partial\Omega\}, \\ A^s_R u = \mathcal{A}^s_\Omega u, \end{cases}$$

where $\mathcal{N}^{2-2s}u$ is to be understood in the sense of Definition 2.2 and $B_{N,s}$ is the constant given in (2.2). Hence, A^s_R is said to be realization on $L^2(\Omega)$ of the operator $\mathcal{A}^s_\Omega$ with fractional Robin type boundary conditions.

Remark 2.3. The bilinear symmetric closed forms $\mathcal{E}_{D,s}$, $\mathcal{E}_{\mathcal{N},s}$ and $\mathcal{E}_{R,s}$ are continuous and elliptic (see [29, Proposition 2.1]). Since $W^{s,2}(\Omega) = W_0^{s,2}(\Omega)$, if $0 < s \leq \frac{1}{2}$, then we have that $\mathcal{E}_{D,s} = \mathcal{E}_{\mathcal{N},s} = \mathcal{E}_{R,s}$ and hence $A_D^s \equiv A_{\mathcal{N}}^s \equiv A_R^s$.

3. Existence and uniqueness of weak solutions

We rewrite problems (1.1)-(1.3) in the unified form

$$\begin{cases} u_{tt} + A_K^{\alpha} u_t + A_K^{\beta} u + f(u) = g & \text{in} \quad \Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \\ u_t(x,0) = u_1(x) & \text{in} \quad \Omega, \end{cases} \tag{3.1}$$

where A_K^{α} and A_K^{β}, $K \in \{D, E, \mathcal{N}, R\}$, are the linear self-adjoint operators introduced in Section 2.

To study problem (3.1), we suppose that the following assumptions hold:

(D) The domain Ω and s satisfy the following conditions:

(i) If $K = D$, then Ω is an arbitrary bounded open set and $0 < s < 1$,

(ii) If $K \in \{\mathcal{N}, R\}$, then Ω is a bounded set with Lipschitz continuous boundary and $\frac{1}{2} < s < 1$.

(F) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

(i) There exists a positive constant c_0 such that, for $1 \leq \rho < \frac{3}{3-2\beta}$,

$$|f(s_1) - f(s_2)| \leq c_0 |s_1 - s_2| (1 + |s_1|^{\rho-1} + |s_2|^{\rho-1}), \ \forall s_1, s_2 \in \mathbb{R}. \tag{3.2}$$

(ii) There exists some positive constant $\lambda_{K,\beta}$ (see (3.5) and (3.6)) such that

$$\liminf_{|s|\to\infty} \frac{f(s)}{s} \geq -\lambda_{K,\beta}, \tag{3.3}$$

(iii) If $K = \mathcal{N}$, then $f(s) - \eta s$ satisfies (3.2) and (3.3) for some $\eta \geq |\Omega|$.

(G) $g \in L^2(\Omega)$.

(P) $0 < \frac{\beta}{2} \leq \alpha < \beta < 1$.

We identify $L^2(\Omega)$ with its dual space $L^2(\Omega)^*$, we consider the family of Hilbert spaces $D(A_K^{\frac{s}{2}})$, $0 < s < 1$, whose inner product and norms are given by

$$(u,v)_{D(A_K^{\frac{s}{2}})} = (A_K^{\frac{s}{2}} u, A_K^{\frac{s}{2}} v)_{L^2(\Omega)} \ \text{ and } \ \|u\|_{D(A_K^{\frac{s}{2}})} = \|A_K^{\frac{s}{2}} u\|_{L^2(\Omega)},$$

for $u, v \in D(A_K^{\frac{s}{2}})$ and $K \in \{D, E, \mathcal{N}, R\}$. We also recall the continuous embedding

$$D(A_K^{\frac{s}{2}}) \hookrightarrow L^{\frac{6}{3-2s}}(\Omega), \tag{3.4}$$

and the compact and dense injections

$$D(A_K^{\frac{s}{2}}) \hookrightarrow D(A_K^{\frac{r}{2}}), \ \forall s > r.$$

It follows from [16, Theorem 2.5] that there exists the best Sobolev-Poincaré constant $\lambda_{K,s} := \lambda_{K,s}(s,\Omega) > 0$ such that

$$\lambda_{K,s}\|u\|^2_{L^2(\Omega)} \le \|u\|^2_{D(A_K^{\frac{s}{2}})},\ \forall u \in D(A_K^{\frac{s}{2}}),\ K \in \{D,E,R\}, \tag{3.5}$$

and if $K = \mathcal{N}$, there is $\lambda_{\mathcal{N},s} := \lambda_{\mathcal{N},s}(s,\Omega) > 0$ such that

$$\lambda_{\mathcal{N},s}\|u\|^2_{L^2(\Omega)} \le \|u\|^2_{D(A_{\mathcal{N}}^{\frac{s}{2}})} + \|u\|^2_{L^1(\Omega)},\ \forall u \in D(A_{\mathcal{N}}^{\frac{s}{2}}). \tag{3.6}$$

Setting

$$\mathbb{H}^s_{K,\sigma}(\Omega) := D(A_K^{\frac{(1+\sigma)s}{2}}) \times D(A_K^{\frac{\sigma s}{2}}).$$

The space $\mathbb{H}^s_{K,\sigma}(\Omega)$ is the Hilbert space equipped with the norm as

$$\|\xi\|^2_{\mathbb{H}^s_{K,\sigma}(\Omega)} := \|u\|_{D(A_K^{\frac{(1+\sigma)s}{2}})} + \|v\|^2_{D(A_K^{\frac{\sigma s}{2}})},\ \ \forall \xi = (u,v) \in \mathbb{H}^s_{K,\sigma}(\Omega).$$

and the inner-product in $\mathbb{H}^s_{K,\sigma}(\Omega)$ as

$$\langle \xi_1, \xi_2 \rangle_{\mathbb{H}^s_{K,\sigma}(\Omega)} := (A_K^{\frac{(1+\sigma)s}{2}}u, A_K^{\frac{(1+\sigma)s}{2}}v)_{L^2(\Omega)} + (A_K^{\frac{\sigma s}{2}}v_1, A_K^{\frac{\sigma s}{2}}v_2)_{L^2(\Omega)}.$$

Then

$$\mathbb{H}^s_K(\Omega) := \mathbb{H}^s_{K,0}(\Omega) = D(A_K^{\frac{s}{2}}) \times L^2(\Omega).$$

The space $\mathbb{H}^s_K(\Omega)$ is also the Hilbert space whose square is given by

$$\|\xi\|^2_{\mathbb{H}^s_K(\Omega)} := \|u\|_{D(A_K^{\frac{s}{2}})} + \|v\|^2_{L^2(\Omega)},\ \ \forall \xi = (u,v) \in \mathbb{H}^s_K(\Omega),$$

and its inner-product in $\mathbb{H}^s_K(\Omega)$ is

$$\langle \xi_1, \xi_2 \rangle_{\mathbb{H}^s_K(\Omega)} := (A_K^{\frac{s}{2}}u, A_K^{\frac{s}{2}}v)_{L^2(\Omega)} + (v_1, v_2)_{L^2(\Omega)},$$

for every $\xi_1 = (u_1, v_1)$ and $\xi_2 = (u_2, v_2)$ in $\mathbb{H}^s_K(\Omega)$.

We define the linear unbounded operator $\mathbb{A}_{\beta,\alpha} : D(\mathbb{A}_{\beta,\alpha}) \subset \mathbb{H}^\beta_K(\Omega) \to \mathbb{H}^\beta_K(\Omega)$ given by

$$\mathbb{A}_{\beta,\alpha} := \begin{bmatrix} 0 & -I \\ A_K^\beta & A_K^\alpha \end{bmatrix},$$

and

$$\mathbb{A}_{\beta,\alpha}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & -I \\ A_K^\beta & A_K^\alpha \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -v \\ A_K^\alpha(A_K^{\beta-\alpha}u + v) \end{bmatrix},$$

for all $(u,v) \in D(\mathbb{A}_{\beta,\alpha})$.

The adjoint of $\mathbb{A}_{\beta,\alpha}$ is determined as follows. The proof is calculation similar to [7, Proposition 1] (see also [2, Lemma 3.1]). We omit it here

Proposition 3.1. *The adjoint of* $\mathbb{A}_{\beta,\alpha}$ *is the operator* $\mathbb{A}^*_{\beta,\alpha}$ *defined by, for any* $0 < \alpha, \beta < 1$,

$$\mathbb{A}^*_{\beta,\alpha} := \begin{bmatrix} 0 & I \\ -A_K^\beta & A_K^\alpha \end{bmatrix},$$

Since $\mathbb{A}_{\beta,\alpha} = \mathbb{A}^{**}_{\beta,\alpha}$, then $\mathbb{A}_{\beta,\alpha}$ is closed (see also [24, Proposition 2.3]). Moreover, the operator $-\mathbb{A}_{\beta,\alpha}$ is dissipative on $\mathbb{H}^{\beta}_{K}(\Omega)$ because, for any $\xi = (u, v)$ in $\mathbb{H}^{\beta}_{K}(\Omega)$,

$$\langle -\mathbb{A}_{\beta,\alpha}\xi, \xi\rangle = -\|v\|^2_{D(A_K^{\frac{\alpha}{2}})}. \tag{3.7}$$

We define the functional $\mathcal{F} : \mathbb{H}^{\beta}_{K}(\Omega) \to \mathbb{H}^{\beta}_{K}(\Omega)$ as

$$\mathcal{F}(\xi) = \begin{bmatrix} 0 \\ f(u) \end{bmatrix}, \;\; \forall \xi = (u, v) \in \mathbb{H}^{\beta}_{K}(\Omega).$$

We have the following proposition and its proof is similar to [24, Proposition 2.8]. We skip it here

Proposition 3.2. *Assume the conditions* **(F)** *and* **(P)** *hold. The functional $\mathcal{F}$ is locally Lipschitz continuous.*

We now return to the nonlinearity and give some important estimates. It follows from the condition **(F)** that there exist $\mu \in (0, \lambda_{K,\beta}]$, $c_1 = c_1(f, |\Omega|) \geq 0$ and $c_2 = c_2(f, |\Omega|) \geq 0$ such that, for all $u \in D(A_K^{\frac{\beta}{2}})$,

$$(f(u), u)_{L^2(\Omega)} \geq -(1 - \frac{\mu}{\lambda_{K,\beta}})\|u\|^2_{D(A_K^{\frac{\beta}{2}})} - c_1, \tag{3.8}$$

and

$$\int_\Omega F(u)dx \geq -\frac{1}{2}(1 - \frac{\mu}{\lambda_{K,\beta}})\|u\|^2_{D(A_K^{\frac{\beta}{2}})} - c_2, \tag{3.9}$$

where $F(s) = \int_0^s f(\tau)d\tau$. Moreover, there exists $C > 0$ such that for all $s \in \mathbb{R}$,

$$|f(s)| \leq C(1 + |s|^{\rho}). \tag{3.10}$$

Hence, if $u \in L^{\rho+1}(\Omega)$, there exists $c_3 = c_3(c_0, f, |\Omega|)$ such that

$$\int_\Omega F(u)dx \leq c_3(1 + \|u\|^{\rho+1}_{L^{\rho+1}(\Omega)}), \tag{3.11}$$

and if $u \in L^{2\rho}(\Omega)$, there exists $c_4 = c_4(c_0, f, |\Omega|)$ such that

$$\|f(u)\|^2_{L^2(\Omega)} \leq c_4(1 + \|u\|^{2\rho}_{L^{2\rho}(\Omega)}). \tag{3.12}$$

We define

$$\mathcal{G} := \begin{bmatrix} 0 \\ g \end{bmatrix}, \;\; \forall g \in L^2(\Omega).$$

We can now study the system (3.1) under the abstract form in $\mathbb{H}^{\beta}_{K}(\Omega)$ as

$$\begin{cases} \xi_t + \mathbb{A}_{\beta,\alpha}\xi + \mathcal{F}(\xi) = \mathcal{G} & \text{in} \quad \Omega \times (0, \infty), \\ \xi(x, 0) = \xi_0(x) & \text{in} \quad \Omega, \end{cases} \tag{3.13}$$

where $\xi_0 = (u_0, u_1) \in \mathbb{H}^{\beta}_{K}(\Omega)$.

We now give the definition of weak solutions.

Definition 3.1. *A weak solution to problem* (3.13) *on* $[0,T]$ *is a map* $\xi = (u, u_t) \in L^\infty(0,T;\mathbb{H}^\beta_K(\Omega))$ *satisfying* $\xi(0) = \xi_0$ *almost everywhere and*

$$\langle \frac{d}{dt}\xi(t), \vartheta\rangle_{\mathbb{H}^\beta_K(\Omega)} + \langle \xi(t), \mathbb{A}^*_{\beta,\alpha}\vartheta\rangle_{\mathbb{H}^\beta_K(\Omega)} + \langle \mathcal{F}(\xi), \vartheta\rangle_{\mathbb{H}^\beta_K(\Omega)} = \langle \mathcal{G}, \vartheta\rangle_{\mathbb{H}^\beta_K(\Omega)}, \tag{3.14}$$

holds for all test functions $\vartheta = (\varphi, \psi) \in D(\mathbb{A}^*_{\beta,\alpha})$ *and for a.e.* $t \in (0,T)$.

The map $\xi = (u, u_t)$ *is a global weak solution if it is a weak solution on on* $[0,T]$*, for every* $T > 0$.

Theorem 3.1. *Let* $\xi_0 = (u_0, u_1) \in \mathbb{H}^\beta_K(\Omega)$ *be given. Assume* **(D) (F), (G)***, and* **(P)** *hold. Then problem* (3.13) *has a unique global weak solution* $\xi = (u, u_t)$ *in sense of Definition 3.1. Moreover, for any* $T > 0$*, the following regularity satisfies*

$$u_t \in L^2(0,T;D(A_K^{\frac{\alpha}{2}})). \tag{3.15}$$

Proof. **i) Existence.** Since the operators A^α_K and A^β_K are positive and self-adjoint, the operator $\mathbb{A}_{\beta,\alpha}$ possesses compact resolvent on $\mathbb{H}^\beta_K(\Omega)$, hence admitting an eigenbasis in $\mathbb{H}^\beta_K(\Omega)$ ([20, Theorem 3.1(a)]). Let $\{\Psi\}_{i=1}^\infty \subset \mathbb{H}^\beta_K(\Omega)$ be eigenfunctions satisfying

$$\mathbb{A}_{\beta,\alpha}\Psi_i = \Lambda_i\Psi, \quad i = 1, 2, \cdots.$$

We know by spectral theory that the eigenvalues Λ_i are nonnegative. Moreover, 0 is an eigenvalue of $\mathbb{A}_{\beta,\alpha}$ for $K = \mathcal{N}$ and is not an eigenvalue of $\mathbb{A}_{\beta,\alpha}$ for $K \in \{D, E, R\}$. Thanks to [16, Theorem 2.5], the components of every $\Psi_i = (\varrho_i, \phi_i)$, $i = 1, 2, \cdots$, satisfy

$$(\varrho_i, \phi_i) \in (D(A^\beta_K) \cap L^\infty(\Omega)) \times (D(A^\alpha_K) \cap L^\infty(\Omega)).$$

Moreover, $\{\varrho_i\}_{i=1}^\infty$ can be an orthogonal basis in $W^{\beta,2}_K(\Omega)$ and $\{\phi_i\}_{i=1}^\infty$ can be an orthonormal basis in $L^2(\Omega)$. Define the subspaces

$$\mathbb{X}_n := \text{span}\{\Psi_1, \Psi_2, \cdots, \Psi_n\} \ \text{ and } \ \mathbb{X}_\infty := \bigcup_{n=1}^\infty \mathbb{X}_n.$$

By construction, $\mathbb{X}_\infty$ is dense in $D(\mathbb{A}_{\beta,\alpha})$. For each integer $n \geq 1$, we consider the approximate solution to (3.13) in the form

$$\xi_n(t) = \sum_{j=1}^n A_j(t)\Psi_j = (\sum_{j=1}^n A_j(t)\varrho_i, \sum_{j=1}^n A_j'(t)\phi_j),$$

where $A_j(t)$ are gotten from solving the following problem

$$\begin{cases} \langle \frac{d}{dt}\xi_n(t), \Psi_j\rangle_{\mathbb{H}^\beta_K(\Omega)} + \langle \xi_n(t), \mathbb{A}^*_{\beta,\alpha}\Psi_j\rangle_{\mathbb{H}^\beta_K(\Omega)} + \langle P_n\mathcal{F}(\xi_n), \Psi_j\rangle_{\mathbb{H}^\beta_K(\Omega)} = \langle \mathcal{G}, \Psi_j\rangle_{\mathbb{H}^\beta_K(\Omega)}, \\ \langle \xi_n(0), \Psi_j\rangle_{\mathbb{H}^\beta_K(\Omega)} = \langle \xi_{0n}, \Psi_j\rangle_{\mathbb{H}^\beta_K(\Omega)}, \end{cases} \tag{3.16}$$

for $j = 1, 2, \cdots, n$, and where $\xi_{0n} = P_n\xi_0$; P_n is the n-dimensional projection of $\mathbb{L}^2(\Omega)$ onto $\mathbb{X}_n$. Since f is the continuous function, we apply Cauchy-Peano theorem for ODEs (3.16) to find that there exists $T_n > 0$ such that the unique solutions $A_j(t)$ of (3.16) belong to $C^2([0,T_n])$ in the classical sense for $j = 1, 2, \cdots, n$ and all $t \in [0, T_n]$. We now prove that the approximate solutions are global for every $n \geq 1$. To do this, multiplying by $A_j(t)$ in (3.16) and summing from $j = 1$ to n, we get

$$\frac{1}{2}\frac{d}{dt}\|\xi_n\|^2_{\mathbb{H}^\beta_K(\Omega)} + \langle \xi_n, \mathbb{A}^*_{\beta,\alpha}\xi_n\rangle_{\mathbb{H}^\beta_K(\Omega)} + \langle P_n\mathcal{F}(\xi_n), \xi_n\rangle_{\mathbb{H}^\beta_K(\Omega)} = \langle \mathcal{G}, \xi_n\rangle_{\mathbb{H}^\beta_K(\Omega)}. \tag{3.17}$$

It follows form (3.7) that

$$\langle \xi_n, \mathbb{A}^*_{\beta,\alpha}\xi_n\rangle_{\mathbb{H}^\beta_K(\Omega)} = \langle \mathbb{A}_{\beta,\alpha}\xi_n, \xi_n\rangle_{\mathbb{H}^\beta_K(\Omega)} = \|u_{nt}\|^2_{D(A_K^{\frac{\alpha}{2}})}, \tag{3.18}$$

where $u_{nt} = \frac{du_n}{dt}$.

We have

$$\begin{aligned}\langle P_n\mathcal{F}(\xi_n), \xi_n\rangle_{\mathbb{H}^\beta_K(\Omega)} &= \langle \mathcal{F}(\xi_n), P_n\xi_n\rangle_{\mathbb{H}^\beta_K(\Omega)} = \langle \mathcal{F}(\xi_n), \xi_n\rangle_{\mathbb{H}^\beta_K(\Omega)}\\ &= (f(u_n), u_{nt})_{L^2(\Omega)}\\ &= \frac{d}{dt}(F(u_n), 1)_{L^2(\Omega)}.\end{aligned} \tag{3.19}$$

We also have

$$\langle \mathcal{G}, \xi_n\rangle_{\mathbb{H}^\beta_K(\Omega)} = (g, u_{nt})_{L^2(\Omega)} = \frac{d}{dt}(g, u_n)_{L^2(\Omega)}. \tag{3.20}$$

Combining (3.17)-(3.20) yields the differential identity

$$\frac{d}{dt}\{\|\xi_n\|^2_{\mathbb{H}^\beta_K(\Omega)} + 2(F(u_n), 1)_{L^2(\Omega)} - 2(g, u_n)_{L^2(\Omega)}\} + 2\|u_{nt}\|^2_{D(A_K^{\frac{\alpha}{2}})} = 0. \tag{3.21}$$

Integrating (3.21) on the interval $[0, t]$, where $t < T_n$, leads

$$\begin{aligned}&\|\xi_n(t)\|^2_{\mathbb{H}^\beta_K(\Omega)} + 2(F(u_n(t)), 1)_{L^2(\Omega)} - 2(g, u_n(t))_{L^2(\Omega)} + 2\int_0^t \|u_{nt}(\tau)\|^2_{D(A_K^{\frac{\alpha}{2}})}d\tau\\ &\qquad = \|\xi_n(0)\|^2_{\mathbb{H}^\beta_K(\Omega)} + 2(F(u_n(0)), 1)_{L^2(\Omega)} - 2(g, u_n(0))_{L^2(\Omega)}.\end{aligned} \tag{3.22}$$

It follows from the Cauchy inequality, (3.5), (3.6), (3.9), (3.11) and (3.22) that

$$\begin{aligned}\frac{\mu(1-\varepsilon)}{\lambda_{K,\beta}}\|\xi_n(t)\|^2_{\mathbb{H}^\beta_K(\Omega)} &\le 2\lambda_{K,\alpha}\int_0^t \|\xi_n(\tau)\|^2_{\mathbb{H}^\beta_K(\Omega)}d\tau + 2\|\xi_n(0)\|^2_{\mathbb{H}^\beta_K(\Omega)} + 2c_2\\ &\quad + 2c_3(1 + \|u_n(0)\|^{\rho+1}_{L^{\rho+1}(\Omega)}) + C(\varepsilon, \mu, \lambda_{K,\beta})\|g\|^2_{L^2(\Omega)}\\ &\le 2\lambda_{K,\alpha}\int_0^t \|\xi_n(\tau)\|^2_{\mathbb{H}^\beta_K(\Omega)}d\tau + 2\|\xi_0\|^2_{\mathbb{H}^\beta_K(\Omega)} + 2c_2\\ &\quad + 2c_3(1 + \|u_0\|^{\rho+1}_{D(A_K^{\frac{\beta}{2}})}) + C(\varepsilon, \mu, \lambda_{K,\beta})\|g\|^2_{L^2(\Omega)},\end{aligned} \tag{3.23}$$

where the last inequality follows from the embedding $D(A_K^{\frac{\beta}{2}}) \hookrightarrow L^{\rho+1}(\Omega)$ when $1 \le \rho < \frac{3}{3-2\beta}$ and ε is small enough.

Using the Gronwall integral inequality, we deduce from (3.23) that

$$\|\xi_n(t)\|^2_{\mathbb{H}^\beta_K(\Omega)} \le C(\|\xi_0\|_{\mathbb{H}^\beta_K(\Omega)}, c_2, c_3)e^{\frac{2\lambda_{K,\beta}\lambda_{K,\alpha}T}{\mu(1-\varepsilon)}}. \tag{3.24}$$

Since the right hand side of (3.24) is independent of n and t, we deduce $T_n = +\infty$, for every $n \ge 1$, i.e., the approximate solutions are unique global in time. Furthermore, we obtain the following estimates for any given $0 < T < +\infty$,

$$\{\xi_n\} \text{ is uniformly bounded in } L^\infty(0, T; \mathbb{H}^\beta_K(\Omega)). \tag{3.25}$$

$$\{u_n\} \text{ is uniformly bounded in } L^\infty(0,T;D(A_K^{\frac{\beta}{2}})). \tag{3.26}$$

$$\{u_{nt}\} \text{ is uniformly bounded in } L^\infty(0,T;L^2(\Omega)). \tag{3.27}$$

Since $D(A_K^{\frac{\beta}{2}}) \hookrightarrow L^{2\rho}(\Omega)$ when $1 \le \rho < \frac{3}{3-2\beta}$, it follows from (3.12) and (3.20) that

$$\{f(u_n)\} \text{ is uniformly bounded in } L^q((0,T)\times\Omega),$$

for some $q \ge 1$. Moreover, we deduce from (3.10) that

$$\begin{aligned}\operatorname*{ess\,sup}_{t\in(0,T)} \|f(u_n(t))\|_{L^1(\Omega)} &\le \operatorname*{ess\,sup}_{t\in(0,T)} C(|\Omega| + \|u_n\|^\rho_{L^\rho(\Omega)}) \\ &\le C(1 + \operatorname*{ess\,sup}_{t\in(0,T)} \|u_n\|^{2\rho}_{D(A_K^{\frac{\beta}{2}})}).\end{aligned}$$

Hence,

$$\{f(u_n)\} \text{ is uniformly bounded in } L^\infty(0,T;L^1(\Omega)). \tag{3.28}$$

From the uniform bounds (3.25), (3.26), (3.27) and using Alaoglu's Theorem (cf. e.g., [23, Theorem 4.18]), there exist subsequence of $\{\xi_n\}$ (we label the same) and functions

$$\xi \in L^\infty(0,T;\mathbb{H}_K^\beta(\Omega)), \tag{3.29}$$

$$u \in L^\infty(0,T;D(A_K^{\frac{\beta}{2}})), \tag{3.30}$$

$$u_t \in L^\infty(0,T;L^2(\Omega)), \tag{3.31}$$

such that

$$\xi_n \rightharpoonup^* \xi \text{ in } L^\infty(0,T;\mathbb{H}_K^\beta(\Omega)),$$

$$u_n \rightharpoonup^* u \text{ in } L^\infty(0,T;D(A_K^{\frac{\beta}{2}})),$$

$$u_{nt} \rightharpoonup^* u_t \text{ in } L^\infty(0,T;L^2(\Omega)).$$

Since $D(A_K^{\frac{\beta}{2}}) \hookrightarrow L^2(\Omega)$ is compact for any $\beta \in (0,1)$, we use the Aubin-Lions-Simon compactness lemma (see e.g. [5]) to deduce the following embedding is compact

$$\{u \in L^2(0,T;D(A_K^{\frac{\beta}{2}})) : u_t \in L^2((0,T)\times\Omega)\} \hookrightarrow L^2((0,T)\times\Omega).$$

Thus,

$$u_n \to u \text{ strongly in } L^2((0,T)\times\Omega), \tag{3.32}$$

and deduce that $u_n \to u$ a.e. in Ω_T. Because of the continuity of f, we infer from (3.32) that, up to subsequences, (by repeating the standard argument as in [24, Theorem 3.5, p.1329])

$$f(u_n) \rightharpoonup f(u) \text{ in } L^2((0,T)\times\Omega).$$

Hence,

$$P_n\mathcal{F}(u_n) \rightharpoonup \mathcal{F}(u) \text{ in } L^2((0,T)\times\Omega).$$

Using convergence properties above allow us to pass the limit in (3.16) in order to recover (3.14) via standard density arguments. Moreover, repeating the standard argument as in [23, p. 225], we get $\xi(0) = \xi_0$ and this implies that ξ is a weak solution to problem (3.13).

ii) Additional regularity. In (3.14), we take $\vartheta = (u_{nt}, u_{nt})$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\{\|u_{nt}\|^2_{L^2(\Omega)} + \|u_n\|^2_{D(A_K^{\frac{\beta}{2}})} + 2(F(u_n),1)_{L^2(\Omega)} - 2(g,u_n)_{L^2(\Omega)}\}\\ +\|u_{nt}\|^2_{D(A_K^{\frac{\alpha}{2}})} + \|u_{nt}\|^2_{D(A_K^{\frac{\beta}{2}})} = 0.\end{aligned} \tag{3.33}$$

Integrating (3.33) over $(0,t)$ leads to the identity

$$\begin{aligned}&\|u_{nt}(t)\|^2_{L^2(\Omega)} + \|u_n(t)\|^2_{D(A_K^{\frac{\beta}{2}})} + 2(F(u_n(t)),1)_{L^2(\Omega)} - 2(g,u_n(t))_{L^2(\Omega)}\\ &\qquad + 2\int_0^t \|u_{nt}(s)\|^2_{D(A_K^{\frac{\alpha}{2}})}ds + 2\int_0^t \|u_{nt}(s)\|^2_{D(A_K^{\frac{\beta}{2}})}ds\\ &\qquad = \|u_{nt}(0)\|^2_{L^2(\Omega)} + \|u_n(0)\|^2_{D(A_K^{\frac{\beta}{2}})}\\ &\qquad\quad + 2(F(u_n(0)),1)_{L^2(\Omega)} - 2(g,u_n(0))_{L^2(\Omega)}.\end{aligned} \tag{3.34}$$

Since $D(A_K^{\frac{\beta}{2}}) \hookrightarrow D(A_K^{\frac{\alpha}{2}})$, it follows from (3.9), (3.11) and (3.34) that

$$\{u_{nt}\} \text{ is uniformly bounded in } L^2(0,T;D(A_K^{\frac{\alpha}{2}})).$$

Thus, the additional regularity property (3.15) follows.

iii) Uniqueness and continuous dependence on the initial data. As usual, we consider any two weak solutions $\xi_1 = (u_1, u_{1t})$ and $\xi_2 = (u_2, u_{2t})$ of (3.13) with initial data $\xi_{01}, \xi_{02} \in \mathbb{H}^\beta_K(\Omega)$, respectively. We infer that $\xi(t) = \xi_1(t) - \xi_2(t)$ satisfies

$$\begin{cases}\xi_t + \mathbb{A}_{\beta,\alpha}\xi + \mathcal{F}(\xi_1) - \mathcal{F}(\xi_2) = 0 & \text{in} \quad \Omega\times(0,\infty),\\ \xi(x,0) = \xi_{01}(x) - \xi_{02}(x) & \text{in} \quad \Omega.\end{cases} \tag{3.35}$$

It follows from Proposition 3.2 that

$$|\langle\mathcal{F}(\xi_1) - \mathcal{F}(\xi_2), \xi\rangle_{\mathbb{H}^\beta_K(\Omega)}| \le C(\|\xi\|_{\mathbb{H}^\beta_K(\Omega)}, T)\|\xi\|^2_{\mathbb{H}^\beta_K(\Omega)}. \tag{3.36}$$

Taking inner-product (3.35) with ξ, and using (3.7), (3.36) yields

$$\frac{d}{dt}\|\xi\|^2_{\mathbb{H}^\beta_K(\Omega)} \le C(t)\|\xi\|^2_{\mathbb{H}^\beta_K(\Omega)}.$$

Applying the Gronwall inequality, we infer from the last inequality that the solution is uniqueness and continuous dependence with respect to the initial data. □

4. Global attractors and their regularity

4.1. *Existence of global attractors*

Thanks to Theorem 3.1, for each $K \in \{D, E, \mathcal{N}, R\}$, we can define a strongly continuous (nonlinear) semigroup $S_K(t) : \mathbb{H}^\beta_K(\Omega) \to \mathbb{H}^\beta_K(\Omega)$ associated to problem (3.1) as follows:

$$S_K(t)\xi_0 := \xi(t) = (u(t), u_t(t)),$$

where $\xi(t)$ is the unique global weak solution of (3.1) with the initial datum ξ_0.

We will see that in case of $K = \mathcal{N}$, the analysis may be more complicated, and we may need stronger conditions to obtain a result as in Lemma 4.2. Hence, we only consider $K \in \{D, E, R\}$ in the sequel. We will prove that the semigroup $S_K(t)$ has a global attractor $\mathcal{A}_K$ on $\mathbb{H}^\beta_K(\Omega)$. We also use C to denote various positive constants whose values may change with each appearance.

We first recall the following technical lemma. Its proof can be found, for instance, in [3, Lemma 2.7]

Lemma 4.1. *Let X ba a Banach space, and let $U \subset C(\mathbb{R}^+, X)$. Let $E :\ X \to \mathbb{R}$ bbe a function such that*

$$\sup_{t\in\mathbb{R}^+} E(\xi(t)) \geq -m \ \text{ and } \ E(\xi(0)) \leq M,$$

for some $m, M \geq 0$ and every $\xi \in U$. In addition, assume that for every $\xi \in U$ the function $E(\xi(t))$ is continuous differentiable, and satisfies the differential inequality

$$\frac{d}{dt}E(\xi(t)) + \delta\|\xi(t)\|^2_X \leq \ell,$$

for some $\delta > 0$ and $\ell > 0$, both independent of $\xi \in U$. Then

$$E(\xi(t)) \leq \sup_{z\in X}\left\{E(z) : \delta\|z\|^2_X \leq 2\ell\right\},\ \forall t \geq \frac{m+M}{\ell}.$$

We now show that $\{S_K(t)\}_{t\geq 0}$ has a bounded absorbing set $\mathcal{B}_0 \subset \mathbb{H}^\beta_K(\Omega)$.

Lemma 4.2. *Assume that $\|\xi_0\|_{\mathbb{H}^\beta_K(\Omega)} \leq R$ for any given $R \geq 0$. There exist a constant $R_0 > 0$ and $t_0 = t_0(R)$ such that*

$$\|S_K(t)\xi_0\|_{\mathbb{H}^\beta_K(\Omega)} \leq R_0,\ \ \forall t \geq t_0.$$

Consequently, the set

$$\mathcal{B}_0 = \{z \in \mathbb{H}^\beta_K(\Omega) : \|z\|_{\mathbb{H}^\beta_K(\Omega)} \leq R_0\}$$

is a bounded absorbing set for $\{S_K(t)\}_{t\geq 0}$ on $\mathbb{H}^\beta_K(\Omega)$.

Proof. We consider the functional

$$\Im(t) = \Im(u(t)) = 2\int_\Omega \int_0^{u(x,t)} f(\tau)d\tau dx.$$

Define

$$\theta(t) = u_t(t) + \varepsilon u(t)$$

for some $\varepsilon \in [0, \varepsilon_0] \subset [0, 1]$. Taking inner product (3.1) with θ yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}E(t) + \|A_K^{\frac{\alpha}{2}}\theta\|^2_{L^2(\Omega)} + \varepsilon\|A_K^{\frac{\beta}{2}}u\|^2_{L^2(\Omega)} - \varepsilon^2\|A_K^{\frac{\alpha}{2}}u\|^2_{L^2(\Omega)} \\
&= \varepsilon\|\theta\|^2_{L^2(\Omega)} - \varepsilon^2(u, \theta)_{L^2(\Omega)} + \varepsilon(g, u)_{L^2(\Omega)} - \varepsilon(f(u), u)_{L^2(\Omega)},
\end{aligned} \tag{4.1}$$

where the energy functional E is defined as

$$\begin{aligned}
E(t) = E(\xi(t)) = \|\theta(t)\|^2_{L^2(\Omega)} + \|A_K^{\frac{\beta}{2}}u\|^2_{L^2(\Omega)} - \varepsilon\|A_K^{\frac{\alpha}{2}}u\|^2_{L^2(\Omega)} \\
+\Im(t) - 2(g, u(t))_{L^2(\Omega)}.
\end{aligned}$$

Using (3.11), we find the bound of E from above

$$E(t) \leq c_5(1 + \|\xi\|^6_{\mathbb{H}_K^\beta(\Omega)}). \tag{4.2}$$

On the other hand, using (3.9), we get the bound of E from below

$$E(t) \geq c_6\|\xi\|^2_{\mathbb{H}_K^\beta(\Omega)} - c_7, \tag{4.3}$$

provided that ε_0 is small enough and for some $c_6 > 0$ (possibly very small). We now estimate the right hand side of (4.1). Using (3.5), we obtain

$$\|A_K^{\frac{\alpha}{2}}\theta\|^2_{L^2(\Omega)} \geq \lambda_{K,\alpha}\|\theta\|^2_{L^2(\Omega)}, \tag{4.4}$$

$$-\varepsilon^2(u, \theta)_{L^2(\Omega)} \leq \frac{\varepsilon^3}{4\lambda_{K,\beta}}\|A_K^{\frac{\beta}{2}}u\|^2_{L^2(\Omega)} + \varepsilon\|\theta\|^2_{L^2(\Omega)}, \tag{4.5}$$

$$\varepsilon(g, u)_{L^2(\Omega)} \leq \frac{\varepsilon\mu}{2\lambda_{K,\beta}}\|A_K^{\frac{\beta}{2}}u\|^2_{L^2(\Omega)} + c_8\varepsilon, \tag{4.6}$$

where $c_8 = c_8(\mu, \lambda_{K,\beta}, \|g\|_{L^2(\Omega)})$. We infer from (3.8), (4.1), (4.4), (4.5), (4.6) that

$$\begin{aligned}
\frac{d}{dt}E(t) + 2(\lambda_{K,\alpha} - 2\varepsilon)\|\theta\|^2_{L^2(\Omega)} + [\frac{\mu\varepsilon}{\lambda_{K,\beta}} - \frac{\varepsilon^2}{2\lambda_{K,\beta}}]\|A_K^{\frac{\beta}{2}}u\|^2_{L^2(\Omega)} \\
-2\varepsilon^2\|A_K^{\frac{\alpha}{2}}u\|^2_{L^2(\Omega)} \leq 2(c_1 + c_8)\varepsilon,
\end{aligned}$$

which, for ε_0 small enough, becomes

$$\frac{d}{dt}E(t) + 2(\lambda_{K,\alpha} - 2\varepsilon)\|\theta\|^2_{L^2(\Omega)} + \frac{\mu\varepsilon}{2\lambda_{K,\beta}}\|A_K^{\frac{\beta}{2}}u\|^2_{L^2(\Omega)} \leq c_9\varepsilon, \tag{4.7}$$

where $c_9 = 2(c_1 + c_8)$. It is easy to see that $\lambda_{K,\alpha} - 2\varepsilon \geq \frac{\mu\varepsilon}{4\lambda_{K,\beta}}$ for ε_0 small enough. We deduce from (4.7) that for ε_0 sufficiently small

$$\frac{d}{dt}E(t) + \frac{\mu\varepsilon}{2\lambda_{K,\beta}}\|\xi\|^2_{\mathbb{H}_K^\beta(\Omega)} \leq c_9\varepsilon.$$

Applying Lemma 4.1, it follows from (4.2) and (4.3) that there exists

$$t_0 = \frac{c_7 + c_5(1 + R^6)}{c_9\varepsilon}$$

such that

$$E(\xi(t)) \le \sup_{z\in\mathbb{H}^{\beta}_{K}(\Omega)} \left\{E(z) : \mu\|z\|^2_{\mathbb{H}^{\beta}_{K}(\Omega)} \le 4c_9\lambda_{K,\beta}\right\},\ \forall t \ge t_0.$$

On account of (4.2) and (4.3), this implies the conclusion of Lemma 4.2. □

Corollary 4.1. *Given any $R \ge 0$, there exists $M_1 = M_1(R)$ and $M_2 = M_2(R)$ such that if $\|\xi_0\|_{\mathbb{H}^{\beta}_{K}(\Omega)} \le R$ then*

$$\|S_K(t)\xi_0\|_{\mathbb{H}^{\beta}_{K}(\Omega)} \le M_1,\ \ \forall t \in \mathbb{R}^+,$$
$$\int_0^{\infty} \|A_K^{\frac{\alpha}{2}} u_t(\tau)\|^2_{L^2(\Omega)} d\tau \le M_2.$$

Proof. Let $\varepsilon = 0$ and integrating (4.1). We infer the results from (4.2), (4.3) and Lemma 4.2. □

Remark 4.1. It follows from Lemma 4.2 and Corollary 4.1 that the set $\{\xi_0 \in \mathbb{H}^{\beta}_{K}(\Omega) : \|S_K(t)\xi_0\|_{\mathbb{H}^{\beta}_{K}(\Omega)} \le M_1(R_0), \forall t \in \mathbb{R}^+\}$ is a bounded absorbing set for $\{S_K(t)\}_{t\ge 0}$. Moreover, it is invariant.

In light of [21, Remark 1, Remark 2] (see also [1, Lemma 1.2], [12, Lemma 3.2]), it follows from the condition **(F)** that f admits the following decomposition

$$f = f_0 + f_1,$$

where f_0 and f_1 are continuous functions satisfying

$$|f_0(s)| \le C(|s| + |s|^{\rho}),\ \ \forall s \in \mathbb{R}, \tag{4.8}$$
$$f_0(s)s \ge 0,\ \ \forall s \in \mathbb{R}, \tag{4.9}$$
$$|f_1(s)| \le C(1 + |s|^{\gamma}),\ \ \gamma < \rho,\ \forall s \in \mathbb{R}, \tag{4.10}$$
$$\liminf_{|s|\to\infty} \frac{f_1(s)}{s} \ge -\lambda_{K,\beta}. \tag{4.11}$$

We decompose the solution of (3.1) into the sum

$$u(t) = v(t) + w(t),$$

where v and w are the solutions to the following problems:

$$\begin{cases} v_{tt} + A_K^{\alpha} v_t + A_K^{\beta} v + f_0(v) = 0 & \text{in} \quad \Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \\ u_t(x,0) = u_1(x) & \text{in} \quad \Omega, \end{cases} \tag{4.12}$$

and

$$\begin{cases} w_{tt} + A_K^{\alpha} w_t + A_K^{\beta} w + f(u) - f_0(v) = g & \text{in} \quad \Omega \times (0,\infty), \\ u(x,0) = 0 & \text{in} \quad \Omega, \\ u_t(x,0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{4.13}$$

Denote

$$\xi(t) = (u(t), u_t(t)), \quad \xi_d(t) = (v(t), v_t(t)), \quad \xi_c(t) = (w(t), w_t(t)).$$

We have the following important lemmas.

Lemma 4.3. *Given any $R \geq 0$. Assume that $\|\xi_0\|_{\mathbb{H}^\beta_K(\Omega)} \leq R$. There exist $M_3 = M_3(R) \geq 0$ and $\nu = \nu(R) > 0$ such that*

$$\|\xi_d(t)\|_{\mathbb{H}^\beta_K(\Omega)} \leq M_3 e^{-\nu t}, \quad \forall t \in \mathbb{R}^+.$$

Proof. We argue as in the proof of Lemma 4.2, we get the differential inequality

$$\frac{d}{dt}E_d(t) + 2(\lambda_{K,\alpha} - 2\varepsilon)\|\theta_d\|^2_{L^2(\Omega)} + \varepsilon\|A_K^{\frac{\beta}{2}}v\|^2_{L^2(\Omega)} \leq 0, \tag{4.14}$$

for some ε_0 small enough and

$$E_d(t) = \|\theta_d(t)\|^2_{L^2(\Omega)} + \|A_K^{\frac{\beta}{2}}v\|^2_{L^2(\Omega)} - \varepsilon\|A_K^{\frac{\alpha}{2}}v\|^2_{L^2(\Omega)} + \Im_d(t), \tag{4.15}$$

$$E_d(t) \leq C(1 + \|\xi_d\|^6_{\mathbb{H}^\beta_K(\Omega)}), \tag{4.16}$$

$$E_d(t) \geq C\|\xi_d\|^2_{\mathbb{H}^\beta_K(\Omega)} - C, \tag{4.17}$$

$$\theta_d(t) = v_t(t) + \varepsilon v(t),$$

$$\Im_d(t) = \Im_d(v(t)) = 2\int_\Omega \int_0^{v(x,t)} f_0(\tau)d\tau dx.$$

Integrating (4.14) for $\varepsilon = 0$ and using (4.16), (4.17) give

$$\sup_{\|\xi_0\|_{\mathbb{H}^\beta_K(\Omega)} \leq R} \sup_{t\in\mathbb{R}^+} \xi_d(t)\|_{\mathbb{H}^\beta_K(\Omega)} < \infty.$$

Hence, we get the uniform estimate

$$\Im_d(t) \leq C(\|v(t)\|^2_{L^2(\Omega)} + \|v(t)\|^{\rho+1}_{L^{\rho+1}(\Omega)}) \leq \kappa\|A_K^{\frac{\beta}{2}}v\|^2_{L^2(\Omega)},$$

for some $\kappa = \kappa(R) \geq 1$ and all $t \in \mathbb{R}^+$.

On the other hand, we have

$$\begin{aligned}\frac{1}{2\kappa}\|A_K^{\frac{\beta}{2}}v\|^2_{L^2(\Omega)} - \frac{\varepsilon}{2\kappa}\|A_K^{\frac{\alpha}{2}}v\|^2_{L^2(\Omega)} + \frac{1}{2\kappa}\Im_d(t) &= \frac{1}{2\kappa}E_d(t) - \frac{1}{2\kappa}\|\theta_d(t)\|^2_{L^2(\Omega)} \\ &\leq \|A_K^{\frac{\beta}{2}}v\|^2_{L^2(\Omega)}.\end{aligned}$$

Put this into (4.14) yields

$$\frac{d}{dt}E_d(t) + \frac{\varepsilon}{2\kappa}E_d(t) + 2(\lambda_{K,\alpha} - 2\varepsilon - \frac{\varepsilon}{4\kappa})\|\theta_d\|^2_{L^2(\Omega)} \leq 0.$$

Thus, we infer that

$$\frac{d}{dt}E_d(t) + \nu E_d(t) \leq 0,$$

where $\nu = \frac{\varepsilon}{2\kappa}$ satisfying $\lambda_{K,\alpha} - 2\varepsilon - \frac{\varepsilon}{4\kappa} > 0$ for ε_0 small enough. By application of the Gronwall inequality and using (4.16), (4.17), the proof is completed. □

Corollary 4.2. *If $f_1 \equiv 0$ and $g \equiv 0$, then $\{S_K(t)\}_{t\geq 0}$ decay to zero. Thus the set $\{0\} \subset \mathbb{H}^\beta_K(\Omega)$ is the global attractor for $\{S_K(t)\}_{t\geq 0}$ on $\mathbb{H}^\beta_K(\Omega)$.*

Proof. This result is a straightforward consequence of Lemma 4.3. ☐

Lemma 4.4. *Given any $T \in \mathbb{R}^+$, there exists a compact set $\mathcal{K}_{T,\beta} \subset \mathbb{H}^\beta_K(\Omega)$ such that*

$$\bigcup_{\xi_0 \in \mathcal{B}_0} \xi_c(t) \in \mathcal{K}_{T,\beta}, \quad \forall t \in [0,T].$$

Proof. We deduce from Corollary 4.1, Remark 4.1 and Lemma 4.3 that

$$\|A_K^{\frac{\beta}{2}} u\|^2_{L^2(\Omega)} + \|A_K^{\frac{\beta}{2}} v\|^2_{L^2(\Omega)} \leq C, \ \forall t \in \mathbb{R}^+. \tag{4.18}$$

Choosing

$$\sigma \in (0, \min\{\frac{1}{2}, \frac{3\rho - 3\gamma + 2\alpha\rho}{2\beta\rho}\}).$$

Taking the inner product (4.13) with $A_K^{\sigma\beta} w_t$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\xi_c\|^2_{\mathbb{H}^\beta_{K,\sigma}(\Omega)} + \|A_K^{\frac{\alpha+\sigma\beta}{2}} w_t\|_{L^2(\Omega)} &= -(f(u) - f(v), A_K^{\sigma\beta} w_t)_{L^2(\Omega)} \\ &\quad -(f_1(v), A_K^{\sigma\beta} w_t)_{L^2(\Omega)} + (g, A_K^{\sigma\beta} w_t)_{L^2(\Omega)}. \end{aligned} \tag{4.19}$$

We now estimate the terms in the right hand side of (4.19)

$$\begin{aligned} (g, A_K^{\sigma\beta} w_t)_{L^2(\Omega)} &\leq \|A_K^{\frac{\sigma\beta-\alpha}{2}} g\|_{L^2(\Omega)} \|A_K^{\frac{\alpha+\sigma\beta}{2}} w_t\|_{L^2(\Omega)} \\ &\leq C + \frac{1}{3}\|A_K^{\frac{\alpha+\sigma\beta}{2}} w_t\|^2_{L^2(\Omega)}, \end{aligned}$$

and it follows from (3.4), (4.10) and (4.18) that

$$\begin{aligned} -(f_1(v), A_K^{\sigma\beta} w_t)_{L^2(\Omega)} &\leq C(1 + \|v\|^\gamma_{L^{\frac{6\gamma}{3+2\alpha-2\sigma\beta}}(\Omega)})\|A_K^{\sigma\beta} w_t\|_{L^{\frac{6}{3-2\alpha+2\sigma\beta}}(\Omega)} \\ &\leq C(1 + \|A_K^{\frac{\beta}{2}} v\|^\gamma_{L^2(\Omega)})\|A_K^{\frac{\alpha+\sigma\beta}{2}} w_t\|_{L^2(\Omega)} \\ &\leq C + \frac{1}{3}\|A_K^{\frac{\alpha+\sigma\beta}{2}} w_t\|^2_{L^2(\Omega)}. \end{aligned}$$

Finally, we infer from (3.2), (3.4) and (4.18) that

$$\begin{aligned} &-(f(u) - f(v), A_K^{\sigma\beta} w_t)_{L^2(\Omega)} \\ &\leq c_0\big(1 + \|u\|^{\rho-1}_{L^{\frac{3(\rho-1)}{\alpha+\beta}}(\Omega)} + \|v\|^{\rho-1}_{L^{\frac{3(\rho-1)}{\alpha+\beta}}(\Omega)}\big)\|w\|_{L^{\frac{6}{3-2(1+\sigma)\beta}}(\Omega)} \|A_K^{\sigma\beta} w_t\|_{L^{\frac{6}{3-2\alpha+2\sigma\beta}}(\Omega)} \\ &\leq C\big(1 + \|A_K^{\frac{\beta}{2}} u\|^{\rho-1}_{L^2(\Omega)} + \|A_K^{\frac{\beta}{2}} v\|^{\rho-1}_{L^2(\Omega)}\big)\|A_K^{\frac{(1+\sigma)\beta}{2}} w\|_{L^2(\Omega)} \|A_K^{\frac{\alpha+\sigma\beta}{2}} w_t\|_{L^2(\Omega)} \\ &\leq C\|\xi_c\|^2_{\mathbb{H}^\beta_{K,\sigma}(\Omega)} + \frac{1}{3}\|A_K^{\frac{\alpha+\sigma\beta}{2}} w_t\|^2_{L^2(\Omega)}. \end{aligned}$$

Combining the above estimates, we deduce from (4.19) that

$$\frac{d}{dt}\|\xi_c\|^2_{\mathbb{H}^\beta_{K,\sigma}(\Omega)} \leq C\|\xi_c\|^2_{\mathbb{H}^\beta_{K,\sigma}(\Omega)} + C.$$

Applying the Gronwall inequality for the last inequality and using $\mathbb{H}^\beta_{K,\sigma}(\Omega) \hookrightarrow \mathbb{H}^\beta_K(\Omega)$, we complete the proof. ☐

From Lemma 4.2, Lemma 4.3, Lemma 4.4 and [25, Chapter 1, Theorem 1.1], we get the main result.

Theorem 4.1. *The semigroup $\{S_K(t)\}_{t\geq 0}$ possesses a connected global attractor $\mathcal{A}_K$ on $\mathbb{H}^\beta_K(\Omega)$.*

4.2. *Regularity*

In this subsection, we want to persue the smoothness of the global attractors when the nonlinearity is more regular. More precisely, we require stronger assumptions as follows:

(Fbis) The function $f \in C^1(\mathbb{R})$, with $f(0) = 0$, satisfies the condition **(F)**.

We see that the condition **(Fbis)** implies that

$$|f'(s)| \leq C(1 + |s|^{\rho-1}), \tag{4.20}$$
$$f'(s) \geq -\ell, \tag{4.21}$$

for some $\ell \geq \lambda_{K,\beta}$ and all $s \in \mathbb{R}$.

We now establish some estimates for the time derivatives of u.

Lemma 4.5. *For every $t > 0$, we have the following inequality*

$$\min\{t,1\}\|A_K^{\frac{\alpha}{2}}u_t\|^2_{L^2(\Omega)} + \int_0^t \min\{s,1\}\|A_K^{\frac{\alpha-\beta}{2}}u_{tt}(s)\|^2_{L^2(\Omega)}ds \leq C(R_0),$$

where R_0 is determined in Lemma 4.2.

Proof. Multiply (3.1) by $A_K^{\frac{\alpha-\beta}{2}}u_{tt}$ in $L^2(\Omega)$ to yield

$$\frac{d}{dt}\Lambda_1 + 2\|A_K^{\frac{\alpha-\beta}{2}}u_{tt}\|^2_{L^2(\Omega)} = 2\|A_K^{\frac{\alpha}{2}}u_t\|^2_{L^2(\Omega)} + 2(f'(u)u_t, A_K^{\alpha-\beta}u_t)_{L^2(\Omega)}, \tag{4.22}$$

where

$$\begin{aligned}\Lambda_1 &= \|A_K^{\alpha-\frac{\beta}{2}}u_t\|^2_{L^2(\Omega)} + 2(A_K^{\frac{\alpha}{2}}u, A_K^{\frac{\alpha}{2}}u_t)_{L^2(\Omega)}\\ &\quad + 2(f(u), A_K^{\alpha-\beta}u_t)_{L^2(\Omega)} - 2(g, A_K^{\alpha-\beta}u_t)_{L^2(\Omega)}.\end{aligned}$$

Since $\alpha - \frac{\beta}{2} < \frac{\alpha}{2} < \frac{\beta}{2}$ and $\rho < \frac{3}{3-2\beta}$, in view of Lemma 4.2, we infer that

$$\frac{1}{2}\|A_K^{\frac{\alpha}{2}}u_t\|^2_{L^2(\Omega)} - C(R_0) \leq \Lambda_1 \leq C(R_0)\|A_K^{\frac{\alpha}{2}}u_t\|^2_{L^2(\Omega)}. \tag{4.23}$$

We have

$$\begin{aligned}2(f'(u)u_t, A_K^{\alpha-\beta}u_t)_{L^2(\Omega)} &\leq 2\|f'(u)\|_{L^{\frac{3}{2\beta}}(\Omega)}\|u_t\|_{L^{\frac{6}{3-2\alpha}}(\Omega)}\|A_K^{\alpha-\beta}u_t\|_{L^{\frac{6}{3+2\alpha-4\beta}}(\Omega)}\\ &\leq C(1 + \|u\|^{\rho-1}_{L^{\frac{3(\rho-1)}{2\beta}}(\Omega)})\|A_K^{\frac{\alpha}{2}}u_t\|^2_{L^2(\Omega)}\\ &\leq C(1 + \|A_K^{\frac{\beta}{2}}u\|^{\rho-1}_{L^2(\Omega)})\|A_K^{\frac{\alpha}{2}}u_t\|^2_{L^2(\Omega)}\\ &\leq C(R_0)\|A_K^{\frac{\alpha}{2}}u_t\|^2_{L^2(\Omega)}.\end{aligned}$$

Inserting this inequality into (4.22), we get

$$\frac{d}{dt}\Lambda_1 + 2\|A_K^{\frac{\alpha-\beta}{2}} u_{tt}\|^2_{L^2(\Omega)} \leq C(R_0)\|A_K^{\frac{\alpha}{2}} u_t\|^2_{L^2(\Omega)}. \tag{4.24}$$

Assume first that $t \in (0, 1]$. Multiplying (4.24) by t and integrating over $[0, t]$, we have

$$t\Lambda_1(t) + 2\int_0^t s\|A_K^{\frac{\alpha-\beta}{2}} u_{tt}(s)\|^2_{L^2(\Omega)}ds \leq C(R_0)\int_0^t \|A_K^{\frac{\alpha}{2}} u_t(s)\|^2_{L^2(\Omega)}ds + \int_0^t \Lambda_1(s)ds.$$

Using Lemma 4.2, Corollary 4.1 and (4.23), we deduce that

$$\frac{t}{2}\|A_K^{\frac{\alpha}{2}} u_t\|^2_{L^2(\Omega)} + 2\int_0^t s\|A_K^{\frac{\alpha-\beta}{2}} u_{tt}(s)\|^2_{L^2(\Omega)}ds \leq C(R_0). \tag{4.25}$$

In case of $t > 1$, we integrate (4.24) over $[1, t]$

$$\Lambda_1(t) + 2\int_0^t \|A_K^{\frac{\alpha-\beta}{2}} u_{tt}(s)\|^2_{L^2(\Omega)}ds \leq C(R_0)\int_0^t \|A_K^{\frac{\alpha}{2}} u_t(s)\|^2_{L^2(\Omega)}ds + \Lambda_1(1).$$

Using Lemma 4.2, Corollary 4.1 and (4.23) again, we deduce that

$$\frac{1}{2}\|A_K^{\frac{\alpha}{2}} u_t\|^2_{L^2(\Omega)} + 2\int_0^t \|A_K^{\frac{\alpha-\beta}{2}} u_{tt}(s)\|^2_{L^2(\Omega)}ds \leq C(R_0). \tag{4.26}$$

The conclusion follows from (4.25) and (4.26). □

Lemma 4.6. *For every $t > 0$, we have the following inequality*

$$\min\{t^2, 1\}\|A_K^{\frac{\alpha-\beta}{2}} u_{tt}\|^2_{L^2(\Omega)} \leq C(R_0).$$

Proof. Set $q = u_t$ and differentiate (3.1) with respect to time, we get

$$q_{tt} + A_K^{\alpha} q_t + A_K^{\beta} q + f'(u)u_t = 0.$$

Then, multiplying the above equation by $A_K^{\frac{\alpha-\beta}{2}} q_t$, we obtain

$$\frac{d}{dt}\Lambda_2 + 2\|A_K^{\alpha-\frac{\beta}{2}} q_t\|^2_{L^2(\Omega)} = -2(f'(u)u_t, A_K^{\frac{\alpha-\beta}{2}} q_t)_{L^2(\Omega)}, \tag{4.27}$$

where

$$\Lambda_2 = \|A_K^{\frac{\alpha}{2}} q\|^2_{L^2(\Omega)} + \|A_K^{\frac{\alpha-\beta}{2}} q_t\|^2_{L^2(\Omega)}.$$

Using Lemma 4.2, Corollary 4.1 and Lemma 4.5, we have

$$\int_0^t \min\{s, 1\}\Lambda_2(s)ds \leq C(R_0). \tag{4.28}$$

In addition, we estimate the right hand side as

$$\begin{aligned}-2(f'(u)u_t, A_K^{\frac{\alpha-\beta}{2}}q_t)_{L^2(\Omega)} &\leq C\|f'(u)\|_{L^{\frac{6}{2\alpha+2\beta}}(\Omega)}\|u_t\|_{L^{\frac{6}{3-2\alpha}}(\Omega)}\|A_K^{\frac{\alpha-\beta}{2}}q_t\|_{L^{\frac{6}{3-2\beta}}(\Omega)}\\ &\leq C(1+\|u\|^{\rho-1}_{L^{\frac{6(\rho-1)}{2\alpha+2\beta}}(\Omega)})\|A_K^{\frac{\alpha}{2}}u_t\|_{L^2(\Omega)}\|A_K^{\alpha-\frac{\beta}{2}}q_t\|_{L^2(\Omega)}\\ &\leq C(R_0)\|A_K^{\frac{\alpha}{2}}u_t\|^2_{L^2(\Omega)}+2\|A_K^{\alpha-\frac{\beta}{2}}q_t\|^2_{L^2(\Omega)}.\end{aligned}$$

Inserting this inequality into (4.27) yields the differential inequality

$$\frac{d}{dt}\Lambda_2 \leq C(R_0)\|A_K^{\frac{\alpha}{2}}u_t\|^2_{L^2(\Omega)}. \tag{4.29}$$

If $t \in (0,1]$, we multiply (4.29) by t^2 and integrating over $[0,t]$. On account of Lemma 4.2, Corollary 4.1 and (4.28), we obtain

$$t^2\|A_K^{\frac{\alpha-\beta}{2}}q_t\|^2_{L^2(\Omega)} \leq C(R_0). \tag{4.30}$$

If $t > 1$, we integrate (4.29) over $[1,t]$ to get

$$\Lambda_2(t) \leq C(R_0)\int_1^t \|A_K^{\frac{\alpha}{2}}u_t(s)\|^2_{L^2(\Omega)}ds + \Lambda_2(1) \leq C(R_0),$$

which in turn gives

$$\|A_K^{\frac{\alpha-\beta}{2}}q_t\|^2_{L^2(\Omega)} \leq C(R_0). \tag{4.31}$$

Putting together (4.30) and (4.31), the proof is finished. □

We now rewrite (3.1) as

$$A_K^{\alpha}u_t + A_K^{\beta}u + f(u) = h, \tag{4.32}$$

where $h = g - u_{tt}$. In view of Lemma 4.6,

$$\sup_{t>0}\|A_K^{\frac{\alpha-\beta}{2}}h\|_{L^2(\Omega)} \leq C(R_0). \tag{4.33}$$

Remark 4.2. We can establish existence and continuous dependence results for (4.32) : For every $u_0 \in D(A_K^{\frac{\beta}{2}})$ and $h \in L^\infty([0,\infty); D(A_K^{\frac{\alpha-\beta}{2}}))$, (4.32) possesses a unique solution $u \in C([0,\infty); D(A_K^{\frac{\beta}{2}}))$ such that $u(0) = u_0$. Indeed, rewriting (4.32) as

$$u_t + A_K^{\beta-\alpha}u + A_K^{-\alpha}f(u) = A_K^{-\alpha}h.$$

Thanks to the works by C.G. Gal and M. Warma [16], we get the results.

We define

$$\widehat{f}(s) = f(s) + \ell s,$$

with ℓ as in (4.21). Thus $\widehat{f}'(s) \geq 0$ for every $s \in \mathbb{R}$. Thus, we decompose $u = v + w$, where v and w now are the solutions to the following equations

$$\begin{cases} A_K^{\alpha}v_t + A_K^{\beta}v + \widehat{f}(u) - \widehat{f}(w) = 0, \\ v(x,0) = u_0(x), \end{cases} \tag{4.34}$$

and

$$\begin{cases} A_K^{\alpha} w_t + A_K^{\beta} w + \widehat{f}(w) = \ell u + h, \\ w(x,0) = 0. \end{cases} \tag{4.35}$$

We have the following important result.

Lemma 4.7. *The solution u to* (4.32) *can be decomposed as $u = v + w$ which satisfies*

$$\|A_K^{\frac{\beta}{2}} v\|_{L^2(\Omega)} \le \|A_K^{\frac{\beta}{2}} u_0\|_{L^2(\Omega)} e^{-Ct},$$

and

$$\|A_K^{\alpha} w\|_{L^2(\Omega)} \le C(R_0, \ell).$$

Proof. Multiplying (4.34) by $A_K^{\beta-\alpha} u$ and from the monotonicity of $\widehat{f}$ we get

$$\frac{d}{dt}\|A_K^{\frac{\beta}{2}} v\|^2_{L^2(\Omega)} + 2\|A_K^{\beta-\frac{\alpha}{2}} v\|^2_{L^2(\Omega)} \le 0.$$

Since $\beta - \frac{\alpha}{2} > \frac{\beta}{2}$ and applying the Gronwall inequality we deduce from the above inequality that

$$\|A_K^{\frac{\beta}{2}} v\|_{L^2(\Omega)} \le \|A_K^{\frac{\beta}{2}} u_0\|_{L^2(\Omega)} e^{-Ct}.$$

Next, multiplying (4.35) by $A_K^{\alpha} w$ and from the nondecreasing of $\widehat{f}$ we get

$$\begin{aligned} &\frac{d}{dt}\|A_K^{\alpha} w\|^2_{L^2(\Omega)} + 2\|A_K^{\frac{\alpha+\beta}{2}} w\|^2_{L^2(\Omega)} \\ &\quad \le 2\ell(u, A_K^{\alpha} w)_{L^2(\Omega)} + 2(h, A_K^{\alpha} w)_{L^2(\Omega)} \\ &\quad \le 2\ell\|A_K^{\frac{\alpha}{2}} u\|_{L^2(\Omega)}\|A_K^{\frac{\alpha}{2}} w\|_{L^2(\Omega)} + 2\|A_K^{\frac{\alpha-\beta}{2}} h\|_{L^2(\Omega)}\|A_K^{\frac{\alpha+\beta}{2}} w\|_{L^2(\Omega)} \\ &\quad \le C(\ell^2\|A_K^{\frac{\alpha}{2}} u\|^2_{L^2(\Omega)} + \|A_K^{\frac{\alpha-\beta}{2}} h\|^2_{L^2(\Omega)}) + \|A_K^{\frac{\alpha+\beta}{2}} w\|^2_{L^2(\Omega)}. \end{aligned}$$

Based on Lemma 4.2, Corollary 4.1, and (4.33), we have

$$\frac{d}{dt}\|A_K^{\alpha} w\|^2_{L^2(\Omega)} + \|A_K^{\frac{\alpha+\beta}{2}} w\|^2_{L^2(\Omega)} \le C(R_0, \ell).$$

Since $\frac{\alpha+\beta}{2} > \alpha$, we can infer from the above inequality and the Gronwall inequality that

$$\|A_K^{\alpha} w\|_{L^2(\Omega)} \le C(R_0, \ell).$$

□

Now, it follows from Lemma 4.2, Lemma 4.5 and Lemma 4.7 that if all initial data $\xi_0 \in \mathcal{B}_0$, we have that

$$\|A_K^{\frac{\alpha}{2}} u_t\|_{L^2(\Omega)} \le C(R_0, \ell), \ \ \|A_K^{\frac{\beta}{2}} v\|_{L^2(\Omega)} \le C(R_0, \ell)e^{-Ct}, \ \ \|A_K^{\alpha} w\|_{L^2(\Omega)} \le C(R_0, \ell),$$

for t large enough. Let R_1 be the best constant $C(R_0, \ell)$ in the above inequality. Setting $\mathcal{B}_1$ as the ball of $D(A_K^{\alpha}) \times D(A_K^{\frac{\alpha}{2}})$ with radius $R_1\sqrt{2}$, since $D(A_K^{\alpha}) \times D(A_K^{\frac{\alpha}{2}})$ is a subspace of $\mathbb{H}_K^{\beta}(\Omega)$. We infer that

$$\mathrm{dist}_{\mathbb{H}_K^{\beta}(\Omega)}(S(t)\mathcal{B}_0, \mathcal{B}_1) \le R_1 e^{-Ct},$$

for t large enough and here $\mathrm{dist}_{\mathbb{H}_K^{\beta}(\Omega)}$ denotes the usual Hausdorff semidistance in $\mathbb{H}_K^{\beta}(\Omega)$. In other words, $\mathcal{B}_1$ is a compact absorbing set, and so by standard arguments, the semigroup $\{S_K(t)\}_{t\ge 0}$ possesses a compact global attractor $\mathcal{A}_K \subset \mathcal{B}_1$. In brief, we have the following theorem.

Theorem 4.2. *The attractor $\mathcal{A}_K$ of the semigroup $\{S_K(t)\}_{t\geq 0}$ on $\mathbb{H}_K^\beta(\Omega)$ is a bounded subset of $D(A_K^\alpha)\times D(A_K^{\frac{\alpha}{2}})$.*

Remark 4.3. In fact, we believe that Theorem 4.1 and Theorem 4.2 still hold if $\alpha, \beta \in (0,1)$; $\alpha = \beta$ and $K \in \{D, E, R\}$ by following the same approach as the works by Pata et. al. [21, 22]. In case of non-uniqueness, the theory of the evolutionary system recently developed by Cheskidov, Foias, and Lu in [8, 9, 10, 11, 15] seems suitable for studying these issues.

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