# Functional calculus for sectorial operators via the entire function with the growth regularity

## Calcul fonctionnel pour les opérateurs sectoriels via la fonction entière avec régularité de la croissance

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**ABSTRACT.** In this paper, having analyzed the previously obtained results devoted to the root vectors series expansion in the Abel-Lidskii sense, we come to the conclusion that the concept can be formulated in the classical terms of the spectral theorem. Though, the spectral theorem for a sectorial operator has not been formulated even in the m-sectorial case, we can consider from this point of view a most simplified case related to the sectorial operator with a discrete spectrum. Thus, in accordance with the terms of the spectral theorem, we naturally arrive at the functional calculus for sectorial operators which is the main focus of this paper. Due to the functional calculus methods, we construct the operator class with the asymptotics more subtle then one of the power type.
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## 1. Introduction

### 1.1. *Historical review*

The idea to construct a functional calculus for sectorial operators is not accidental for it appeals to the well-known theory covering self-adjoint and normal operators in which the so called spectral theorem is a concept that gives a comprehensive description of methods allowing to consider a functional calculus. Although we are not ready to formulate a spectral theorem for the sectorial operator with a continuous spectrum the considered analog corresponding to the operator with a discrete spectrum creates a far-reaching prerequisite. The involved auxiliary technique based upon the concept of decomposition of an element of the abstract Hilbert space on the root vectors series. The latter concept lies in the framework of the abstract functional analysis and its appearance arises from elaboration of methods for solving evolution equations investigated in the recent century by Lidskii V.B. [27], Matsaev V.I. [34], Agaranovich M.S. [1], and others [20].

In its simple reduced form, applicably to self-adjoint operators the concept admits the interpretation through the well-known fact that the eigenvectors of the compact self-adjoint operator form a basis in the closure of its range. The question what happens in the case when the operator is non-self-adjoint sectorial is rather complicated and deserves to be considered as a separate part of the spectral theory [24, 25, 26, 30, 31, 28, 29, 32, 33, 43, 44, 2].

We should make a brief digression and explain that a relevance appears just in the case when a senior term of a considered operator is not self-adjoint for there is a number of papers [8, 22, 34, 30, 31, 36, 45] devoted to the perturbed self-adjoint operators. The fact is that most of them deal with a decomposition of the operator on a sum, where the senior term must be either a self-adjoint or normal operator. In other cases, the methods of the papers [13],[12] become relevant and allow us to study spectral properties of operators whether we have the mentioned above representation or not, moreover they have a natural mathematical origin that appears brightly while we are considering abstract constructions expressed in terms of the semigroup theory [14].

Generally, the aim of the mentioned part of the spectral theory are propositions on the convergence of the root vector series in one sense or another to an element belonging to the closure of the operator range, we mean Bari, Riesz, Abel-Lidskii senses of the series convergence [5]. The main condition in terms of which the propositions are mostly described is the asymptotics of the operator singular numbers, here we should note that originally it is formulated in terms of the operator beloning to the Schatten-von Neumann class. However, Agaranovich M.S. made an attempt to express the sufficient conditions of the root vector series basis property, in the mentioned above generalized sense, through the asymptotics of the eigenvalues of the real component [1]. The paper by Markus A.S., Matsaev V.I. [34] can be also considered within the scope since it establishes the relationship between the asymptotics of the operator eigenvalues absolute value and eigenvalues of the real component.

Thus, the interest how to express root vectors series decomposition theorems through the asymptotics of the real component eigenvalues arose previously what along the obvious technical advantage in finding the asymptotics create a prerequisite to investigate the matter properly. We should point out that under the desired relationship between asymptotics we are able to reformulate theorems on the root vectors series expansion in terms of the assumptions related to the real component of the operator. The latter idea is relevant since in many cases the calculation of the real component eigenvalues asymptotics simpler than direct calculation of the singular numbers asymptotics.

If we make a comparison analysis between the methods of root vectors decomposition by Lidskii V.B. [27] and Agaranovich M.S. [1] we will see that the first one formulated the conditions in terms of the singular values but the second one did in terms of the real component eigenvalues. In this regard, we will show that the asymptotics of the real component eigenvalues stronger than the one of the singular numbers, however Agaranovich M.S. [1] imposed the additional condition - spectrum belongs to the domain of the parabolic type. From the latter point of view the results by Lidskii V.B.[27] are more advantageous since the convergence in the Abel-Lidskii sense was established for an operator class wider than the class of sectorial operators. Apparently, a reasonable question that may appear is about minimal conditions that guaranty the desired result what in particular is considered in this paper.

Here, we obviously can extend the results devoted to compact operators to operators with the discrete spectrum, for they can be easily reformulated from one realm to another. In this regard, we should make warning that the latter fact does not hold for real components since the real component of the inverse operator does not coincide with the inverse of the operator real component. However, such a complication was diminished due to the results [12],[13], where the asymptotic equivalence between the eigenvalues of the mentioned operators was established.

A couple of words on the applied relevance of the issue. The abstract approach to the Cauchy problem for the fractional evolution equation is classical one [3],[4]. In its framework, the application of results connected with the basis property covers many problems of the theory of evolution equations [27, 13, 14, 15, 16, 17]. In its general statement the problem appeals to many applied ones, we can produce a number of papers dealing with differential equations which can be studied by the abstract methods [35, 38, 46]. Apparently, the main advantage of this paper is a a technique that allows to implement the verification of the abstract conditions of the exitance and uniqueness theorem for the concrete evolution equations. Thereby, we hope that the offered approach is extremely novel in the abstract theory and relevant in applications.

## 1.2. *Preliminaries*

Let $C, C_i,\ i \in \mathbb{N}_0$ be real positive constants. We assume that values of $C$ can be different in formulas but values of $C_i$ are certain. Assume that $\Lambda$ is a subset of the complex plain, we will use a notation $C\Lambda := \left\{ z \in \mathbb{C} :\ z \cdot C^{-1} \in \Lambda \right\}$. Everywhere further, we consider linear densely defined operators acting in a separable complex Hilbert space $\mathfrak{H}$. Denote by $\mathcal{B}(\mathfrak{H})$ the set of linear bounded operators on $\mathfrak{H}$. Denote by $\mathrm{D}(L),\ \mathrm{R}(L),\ \mathrm{N}(L),\ \mathrm{P}(L)$ the *domain of definition*, the *range*, the *kernel*, and the *resolvent set* of the operator $L$ respectively. Denote by $\Sigma(L) := \mathbb{C}\backslash \mathrm{P}(L)$ the spectrum of the operator $L$. Consider a pair of complex Hilbert spaces $\mathfrak{H}, \mathfrak{H}_+$, the notation $\mathfrak{H}_+ \subset\subset \mathfrak{H}$ means that $\mathfrak{H}_+$ is dense in $\mathfrak{H}$ as a set of elements and we have a bounded embedding provided by the inequality

$$\|f\|_{\mathfrak{H}} \leq C\|f\|_{\mathfrak{H}_+},\ f \in \mathfrak{H}_+,$$

moreover any bounded set with respect to the norm $\mathfrak{H}_+$ is compact with respect to the norm $\mathfrak{H}$. Denote by $\mathfrak{Re}L := (L + L^*)/2,\ \mathfrak{Im}L := (L - L^*)/2i$ the real and imaginary components of the operator $L$ respectively. In accordance with the terminology of monograph [7] the set $\Theta(L) := \{z \in \mathbb{C} : z = (Lf, f)_{\mathfrak{H}},\ f \in \mathrm{D}(L),\ \|f\|_{\mathfrak{H}} = 1\}$ is called the *numerical range* of the operator $L$. Define a closed sector in the complex plain $\mathfrak{L}_a(\theta) := \{z \in \mathbb{C} : |\arg(z - a)| \leq \theta < \pi\} \cup \{a\}$, where $a \in \mathbb{C}$ is called by the vertex and $\theta$ is called by the semi-angle of the sector. An operator $L$ is said to be *sectorial* if its numerical range belongs to a sector $\mathfrak{L}_a(\theta),\ \theta < \pi/2$.

An operator $L$ is said to be *bounded from below* if the following relation holds $\mathrm{Re}(Lf, f)_{\mathfrak{H}} \geq \gamma_L\|f\|^2_{\mathfrak{H}},\ f \in \mathrm{D}(L),\ \gamma_L \in \mathbb{R}$, where $\gamma_L$ is called by the lower bound of $L$. An operator $L$ is said to be *accretive* if $\gamma_L = 0$. An operator $L$ is said to be *strictly accretive* if $\gamma_L > 0$. An operator $L$ is said to be *m-accretive* if the following relation holds $(L - \zeta I)^{-1} \in \mathcal{B}(\mathfrak{H}),\ \|(L - \zeta)^{-1}\| \leq |\mathrm{Re}\zeta|^{-1},\ \mathrm{Re}\zeta < 0$. An operator $L$ is said to be *m-sectorial* if $L$ is sectorial and $L + \beta$ is m-accretive for some constant $\beta$. An operator $L$ is said to be *symmetric* if one is densely defined and the following relation holds $(Lf, g)_{\mathfrak{H}} = (f, Lg)_{\mathfrak{H}},\ f, g \in \mathrm{D}(L)$. An operator $L$ is called by the *operator with discrete spectrum* if $0 \in \mathrm{P}(L)$ and the inverse operator is compact. Denote by $R_L(\zeta) := (L - \zeta I)^{-1},\ \zeta \in \mathrm{P}(L),\ R_L := R_L(0)$ the resolvent of the operator $L$. The dimension of the root vectors subspace corresponding to a certain eigenvalue of the operator $L$ is called by the *algebraic multiplicity* of the eigenvalue.

Denote by $\mu_j(L),\ j \in \mathbb{N}$ the eigenvalues of the operator $L$, where the numbering is given in accordance with increase or decrease of their absolute value. If the contrary is not stated, we assume that in accordance with the numbering each eigenvalue is counted as many times as its algebraic multiplicity. Denote by $\nu(\mu_j)$ the algebraic multiplicity of the eigenvalue $\mu_j(L)$ and denote by $\nu(L)$ the sum of all algebraic multiplicities of the operator $L$. Suppose $L$ is a compact operator and $|L| := (L^*L)^{1/2}$, then the eigenvalues of the operator $|L|$ are called by the *singular numbers* (*s-numbers*) of the operator $L$

 

and are denoted by $s_j(L),\ j = 1, 2, \dots, \dim \mathrm{R}(|L|)$. If $\dim \mathrm{R}(|L|) < \infty$, then we put by definition $s_j = 0,\ j > \dim \mathrm{R}(|L|)$. Assume that an operator $L$ is compact, the following relation holds

$$\sum_{n=1}^{\infty} s_n^{\sigma}(L) < \infty,\ 0 < \sigma < \infty,$$

then $L$ is said to be in the Schatten-von Neumann class $\mathfrak{S}_{\sigma}(\mathfrak{H})$, i.e. $L \in \mathfrak{S}_{\sigma}(\mathfrak{H})$ in symbol. Denote by $\mathfrak{S}_{\infty}(\mathfrak{H})$ the set of compact operators acting in $\mathfrak{H}$.

Consider a sequence $\{a_j\}_1^{\infty} \subset \mathbb{C}$, define the counting function $n(r, a_j) := \mathrm{card}\{j \in \mathbb{N} : |a_j| \leq r\}$. Assume that an operator $L$ is compact (operator with discrete spectrum), denote by $n(r, L)$ the counting function corresponding to the sequence of the absolute values of the operator characteristic numbers (eigenvalues).

We consider element-functions of the Hilbert space $u : \mathbb{R}_+ \to \mathfrak{H},\ u := u(t),\ t \geq 0$ assuming that if $u$ belongs to $\mathfrak{H}$ then the fact holds for all values of the variable $t$. We understand such operations as differentiation and integration in the generalized sense that is caused by the topology of the Hilbert space $\mathfrak{H}$. The derivative is understood as a limit

$$\frac{u(t + \Delta t) - u(t)}{\Delta t} \xrightarrow{\mathfrak{H}} \frac{du}{dt},\ \Delta t \to 0.$$

Let $t \in J := [a, b],\ 0 < a < b < \infty$. The following integral is understood in the Riemann sense as a limit of partial sums

$$\sum_{i=0}^{n} u(\xi_i)\Delta t_i \xrightarrow{\mathfrak{H}} \int\limits_J u(t)dt,\ \zeta \to 0,$$

where $(a = t_0 < t_1 < \dots < t_n = b)$ is an arbitrary splitting of the segment $J,\ \zeta := \max\limits_i (t_{i+1} - t_i),\ \xi_i$ is an arbitrary point belonging to $[t_i, t_{i+1}]$. The sufficient condition of the integral existence is a continuous property (see[21, p.248]), i.e. $u(t) \xrightarrow{\mathfrak{H}} u(t_0),\ t \to t_0,\ \forall t_0 \in J$. The improper integral is understood as a limit

$$\int\limits_a^b u(t)dt \xrightarrow{\mathfrak{H}} \int\limits_a^c u(t)dt,\ b \to c,\ c \in [0, \infty].$$

Combining the generalized integro-differential operations, we can consider a fractional differential operator in the Riemann-Liouville sense (see [42]), in the formal form, we have

$$\mathfrak{D}_-^{\alpha} f(t) := \frac{(-1)^n}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int\limits_0^{\infty} f(t + x) x^{n - \alpha - 1} dx,\ \alpha \geq 0,\ n = [\alpha] + 1,$$

here we should remark that

$$\mathfrak{D}_-^{n} f(t) = (-1)^n \frac{d^n f(t)}{dt^n},\ n \in \mathbb{N}_0.$$

We consider a function $\varphi$ of the complex variable that is analytic in the neighborhood of the infinitely distant point. It has a decomposition in the Laurent series

$$\varphi(z)=\sum_{k=-\infty}^{\infty}c_k z^k,\; z\in\mathbb{C},\; c_k:=\frac{1}{2\pi i}\oint\limits_{\Gamma}\frac{\varphi(\xi)}{\xi^{k+1}}\,d\xi,$$

where $\Gamma$ is a closed contour within the region of analyticity.

Throughout the paper, we consider a sectorial operator $W$ with a discrete spectrum $\Theta(W)\subset\mathfrak{L}_0(\theta)$, additionally we assume that $B:=W^{-1}$ belongs to the Schatten-von Neumann class $\mathfrak{S}_\sigma,\; 0<\sigma<\infty$. Everywhere further, unless otherwise stated, we use notations accepted in the literature [5], [7], [9], [10], [42]. The following paragraph is partly devoted to the detailed description of the regular part of the Laurent series as well as various theoretical concepts playing a distinguished role in our narrative.

## 2. Overview of the supplementary results

### Regular part under the condition of the growth regularity

Below, we consider a condition of the growth regularity, i.e a condition in accordance to which the absolute value of an analytic function can be estimated not only from the above but from the below also. In order to show that this class of function is not empty, moreover it was undergone to a detailed study, we represent the following material. To characterize the growth of a function $f(z)$, we introduce the functions

$$M_f(r)=\max_{|z|=r}|f(z)|,\; m_f(r)=\min_{|z|=r}|f(z)|.$$

Assume that $\varrho$ is a finite order of the entire function $f$. A function $\varrho(r)$, satisfying the following conditions

$$\lim_{r\to\infty}\varrho(r)=\varrho;\;\lim_{r\to\infty}r\varrho'(r)\ln r=0,$$

is said to be *proximate order* if the following relation holds

$$\sigma_f=\overline{\lim_{r\to\infty}}\frac{\ln M_f(r)}{r^{\varrho(r)}},\; 0<\sigma_f<\infty,$$

here and further $r:=|z|,\; z\in\mathbb{C}$. In this case the value $\sigma_f$ is said to be the type of the function $f$ under the proximate order $\varrho(r)$.

To guaranty some technical results we need to consider a class of entire functions whose zero distributions have a certain type of regularity. We follow the monograph [23] where the regularity of the distribution of the zeros is characterized by a certain type of density of the set of zeros.

We will say that the set $I$ of the complex plane has an *angular density of index*

$$\xi(r)\to\xi,\; r\to\infty,$$

if for an arbitrary set of values $\phi$ and $\psi$ $(0<\phi<\psi\leq 2\pi)$, maybe except of denumerable sets, there exists the limit

$$\Delta(\phi,\psi)=\lim_{r\to\infty}\frac{n(r,\phi,\psi)}{r^{\xi(r)}},$$

where $n(r,\phi,\psi)$ is the number of points of the set $I$ within the sector $|z|\leq r,\ \phi<\arg z<\psi$. The quantity $\Delta(\phi,\psi)$ will be called the angular density of the set $I$ within the sector $\phi<\arg z<\psi$. A set will be said to be *regularly distributed* relative to $\xi(r)$ if it has an angular density $\xi(r)$ with $\xi$ non-integer.

Consider the following conditions imposed on the sequence of the complex numbers, allowing us to solve technical problems related to estimation of contour integrals.

(C1) There exists a value $d>0$ such that circles of radii

$$r_n = d|a_n|^{1-\frac{\varrho(|a_n|)}{2}}$$

with the centers situated at the points $a_n$ do not intersect each other, where $\{a_n\}_1^\infty \subset \mathbb{C}$.

(C2) The points $a_n$ lie inside angles with a common vertex at the origin but with no other points in common, which are such that if one arranges the points of the set $\{a_n\}_1^\infty$ within any one of these angles in the order of increasing moduli, then for all points which lie inside the same angle the following relation holds

$$|a_{n+1}|-|a_n| > d|a_n|^{1-\varrho(|a_n|)},\ d>0.$$

The circles $|z-a_n|\leq r_n$ in the first case, and $|z-a_n|\leq d|a_n|^{1-\varrho(|a_n|)}$ in the second case, will be called the exceptional circles.

Further, we will call the set of zeros regularly distributed relative to the proximate order $\varrho(r)$ and satisfying one of the conditions $\mathrm{C1},\mathrm{C2}$ by $R$ - set of the proximate order.

**Lemma 1.** *Assume that the entire function $f$ is of the proximate order $\varrho(r),\ \varrho\in(0,1/2]$, its zeros form $R$ - set of the proximate order, there exists $\varsigma>0$ such that the set $\{z\in\mathbb{C}:|\zeta-arg\,z|<\varsigma\},\ \zeta\in[0,2\pi)$ do not contain the zeros with a sufficiently large absolute value. Then, for an arbitrary fixed value $0<\delta<\varsigma$ the following relation holds*

$$\forall\varepsilon>0,\ \exists N(\varepsilon):|f(z)|>e^{(H_0-\varepsilon)r^{\varrho(r)}},\ r>N(\varepsilon),\ |\zeta-arg\,z|<\delta,$$

*where*

$$H_0=\pi\Delta(2\pi,0)\cot\pi\varrho.$$

*Proof.* The proof is represented by the modification of the proofs of Lemmas 1,2 [18]. Using Theorem 13 [23] (Chapter I, § 10) we obtain a representation of the entire function of the finite order due to the canonical product

$$f(z)=z^m\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right),$$

where $a_n$ are non-zero roots of the entire function. Having repeated the reasonings represented in Lemma 2 [18], we prove the fact that the canonical product has the same proximate order $\varrho(r)$, in accordance with Theorem 5 [23] (Chapter II, § 1), we get

$$\forall\varepsilon>0,\ \exists N(\varepsilon):\left|\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)\right|\geq e^{(H(\phi)-\varepsilon)r^{\varrho(r)}},\ r>N(\varepsilon),\ \phi:=\arg z,\ |\zeta-\arg z|<\delta,$$

where in accordance with Lemma 1 [18]

$$H(\phi) := \frac{\pi}{\sin \pi \varrho} \int\limits_0^{2\pi} \cos \varrho(|\phi - \varphi| - \pi) d\Delta(\varphi, 0).$$

Here, we should take into account the fact that the set of circulus of the exponential type does not intersect the set $\{z \in \mathbb{C} : |\zeta - arg\, z| < \delta\}$ for sufficiently large values $r$. It follows easily from conditions $\mathrm{C1}, \mathrm{C2}$ the detailed proof is omitted one can find it in Lemma 2 [18]. Having noticed the fact $\cos \varrho(|\phi - \varphi| - \pi) \geq \cos \varrho\pi,\ \varrho \in (0, 1/2]$, we obtain

$$H(\phi) > \pi\Delta(2\pi, 0) \cot \pi\varrho.$$

The rest part of the proof related to the explicit form of the claimed estimate is omitted since it may not represent any difficulties for the reader. □

## Conditions of the abstract uniform ellipticity

Consider an operator $W$ with a discrete spectrum, note that the class of operators with discrete spectrum is rather wide. As an argument of this statement consider the most important representative of the class the abstract class of uniformly elliptic operators that can be expressed independently on the Hilbert space in which the operator acts due to the following conditions.

(H1) There exists a Hilbert space $\mathfrak{H}_+ \subset\subset \mathfrak{H}$ and a linear manifold $\mathfrak{M}$ that is dense in $\mathfrak{H}_+$. The operator $L$ is a non-sefadjoint closed densely defined operator acting in $\mathfrak{H},\ \mathfrak{M} \subset \mathrm{D}(L)$.

(H2) $|(Lf, f)_{\mathfrak{H}}| \leq C_1 \|f\|^2_{\mathfrak{H}_+},\ \mathrm{Re}(Lf, f)_{\mathfrak{H}} \geq C_2 \|f\|^2_{\mathfrak{H}_+},\ f \in \mathfrak{M}.$

It was shown in the paper [11] that conditions H1,H2 guarantee the existence of the compact inverse operator, i.e. the operator $L$ has a discrete spectrum. We should note, it follows from the conditions, that the numerical range of the operator belongs to a sector $\mathfrak{L}_0(\theta),\ \theta < \pi/2$ with the vertex situated at the point zero and the semi-angle $\theta$.

Apparently, we obtain the operator characteristics such as location of the numerical range of values and discreetness of the spectrum through the abstract conditions applicable to exotic Hilbert spaces.

Below, we represent the main concept of the functional calculus for a suitable operator class, here we mainly mention the classes of selfandjoint and normal operators for which the spectral theorem theory is constructed.

## Spectral theorem and its generalization via the Abel-Lidskii method

**1.** Denote by $\lambda_n,\ n \in \mathbb{N}$ the characteristic numbers of the operator $B := W^{-1}$ i.e. $\lambda_n = 1/\mu_n,\ n \in \mathbb{N}$, where $\mu_n := \mu_n(B)$. In this paragraph for the simplicity of the form of writing, we use the numbering of the operator $B$ eigenvalues in order of their absolute value decreasing. Thus, in accordance with this numbering there is a one to one correspondence between a subset of the complex plane and the set of the eigenvalues. In accordance with the Hilbert theorem (see [40], [5, p.32]) the spectrum of an arbitrary compact operator $B$ consists of the so-called normal eigenvalues, it gives us an opportunity to consider a decomposition to a direct sum of the subspaces

$$\mathfrak{H} = \mathfrak{N}_q \oplus \mathfrak{M}_q, \tag{1}$$

where both summands are invariant subspaces of the operator $B$, the first one is a finite dimensional root subspace corresponding to the eigenvalue $\mu_q$, $\dim \mathfrak{N}_q = \nu(\mu_q)$, $q \in \mathbb{N}$ and the second one is a subspace wherein the operator $B - \mu_q I$ is invertible. We can choose a Jordan basis in $\mathfrak{N}_q$ that consists of Jordan chains of eigenvectors and root vectors of the operator $B$ defined by the following formula

$$Be_{q_\xi} = \mu_q e_{q_\xi},\ Be_{q_\xi+1} = \mu_q e_{q_\xi+1} + e_{q_\xi}, ..., Be_{q_\xi+k} = \mu_q e_{q_\xi+k} + e_{q_\xi+k-1}, \tag{2}$$

where $\xi = 1, 2, ..., m(q)$, the latter symbol denotes the geometric multiplicity of the of the eigenvalue $\mu_q$ and $k := k(q_\xi) \in \mathbb{N}$ indicates a number of elements in the $q_\xi$-th Jordan chain. It is clear that

$$\nu(\mu_q) = \sum_{\xi=1}^{m(q)} \{k(q_\xi) + 1\}.$$

Considering the sequence $\{\mu_q\}_1^\infty$ of the eigenvalues of the operator $B$ and choosing a Jordan basis in each corresponding subspace $\mathfrak{N}_q$, we can arrange a system of vectors $\{e_i\}_1^\infty$ which we will call a system of the root vectors or following Lidskii a system of the major vectors of the operator $B$. It is proved in the paper [27] (in addition see [15]) that there exists a biorthogonal system $\{g_n\}_1^\infty$ with respect to the system of the major vectors of the operator $B$ having the following property

$$B^* g_{q_\xi+k} = \bar{\mu}_q g_{q_\xi+k},\ B^* g_{q_\xi+k-1} = \bar{\mu}_q g_{q_\xi+k-1} + g_{q_\xi+k}, ..., B^* g_{q_\xi} = \bar{\mu}_q g_{q_\xi} + g_{q_\xi+1}.$$

Define

$$c_{q_\xi+i} = \frac{(f, g_{q_\xi+k-i})}{(e_{q_\xi+i}, g_{q_\xi+k-i})},\ 0 \le i \le k.$$

Since the main result of the paper relates to the concept of the operator function, we need state main principles of the spectral theorem in the primitive form. Consider a unit decomposition formula

$$I = \int_0^\infty dP_\lambda$$

corresponding to a self-adjoint operator having a zero kernel. Here $P_\lambda,\ 0 < \lambda < \infty$ is a family of the orthogonal projectors corresponding to the operator. The latter relation is understood in the following sense. For arbitrary elements $f, h \in \mathfrak{H}$ the following integral is defined in the Riemann sense

$$(f, g)_\mathfrak{H} = \int_0^\infty (dP_\lambda f, g)_\mathfrak{H}.$$

It is clear that the last relation, in accordance with the properties of the Hilbert space, gives us the unit decomposition formula. It is remarkable that there is another approach generalizing the case corresponding to the self-adjoint operator with discrete spectrum. Using formula (2) we can prove the following relation (see [17])

$$(\zeta I - B)^{-1} e_{q_\xi+i} = \sum_{j=0}^{i} \frac{e_{q_\xi+j}}{(\zeta - \mu_q)^{i-j+1}},\ 0 \le i \le k, \tag{3}$$

where $\zeta$ belongs to the resolvent set of the operator $B$. The Riesz integral operator is defined as follows

$$\mathcal{P}_{\mu_q} f := -\frac{1}{2\pi i} \oint_{\Gamma_{\mu_q}} (B - \lambda I)^{-1} f \, d\lambda,\ f \in \mathfrak{H},\ q \in \mathbb{N},$$

 

where $\Gamma_{\mu_q}$ is a closed contour bounding a domain containing the eigenvalue $\mu_q$ only. The properties of the Riesz integral operator are described in detail in §1.3, Chapter I, [5], below we represent another forms of this operator and prove that it preserves the root vectors. Using (3), we have

$$\mathcal{P}_{\mu_q} e_{q_\xi+i} = \frac{1}{2\pi i} \oint\limits_{\Gamma_{\mu_q}} (\lambda I - B)^{-1} e_{q_\xi+i} d\lambda = \frac{1}{2\pi i} \sum_{j=0}^{i} \oint\limits_{\Gamma_{\mu_q}} \frac{e_{q_\xi+j}}{(\lambda - \mu_q)^{i-j+1}} d\lambda = e_{q_\xi+i},$$

here we applied the Cauchy integral formula to the latter integral. Thus, the Riesz integral operator is the projector onto the subspace $\mathfrak{N}_q$ alone the subspace $\mathfrak{M}_q$. It is clear that

$$-\frac{1}{2\pi i} \oint\limits_{\Gamma_{\lambda_q}} B(I - zB)^{-1} f dz = -\frac{1}{2\pi i} \oint\limits_{\Gamma_{\lambda_q}} \frac{1}{z^2} \left\{ (z^{-1}I - B)^{-1} - zI \right\} f dz$$

$$= \frac{1}{2\pi i} \oint\limits_{\Gamma_{\lambda_q}} (z^{-1}I - B)^{-1} f dz^{-1} = \frac{1}{2\pi i} \oint\limits_{\Gamma_{\mu_q}} (\lambda I - B)^{-1} f d\lambda = \mathcal{P}_{\mu_q} f,$$

where $\Gamma_{\lambda_q}$ is the image of the contour $\Gamma_{\mu_q}$ under the mapping $z = \lambda^{-1}$. Thus, we obtain another form of the Riesz integral operator. Now, consider an operator with discrete spectrum $W$. Observe that the operator $B$ has a commutative property $B(I - \lambda B)^{-1} = (I - \lambda B)^{-1} B$, see Problem 5.4 [7, p.36]. This leads to the equality $(W - \lambda I)^{-1} = B(I - \lambda B)^{-1}$, $\lambda \in \mathrm{P}(W)$. Therefore in accordance with the above, we have

$$\mathcal{P}_{\mu_q} = -\frac{1}{2\pi i} \oint\limits_{\Gamma_{\lambda_q}} (W - zI)^{-1} f dz, \; f \in \mathfrak{H}. \tag{4}$$

Here, since this fact is valuable in the context, we should stress that the root vectors of the operators $B$ and $W$ are coincided. To prove this fact consider relation (2), we have

$$B e_{q_\xi+1} = \mu_q e_{q_\xi+1} + e_{q_\xi}, \; q \in \mathbb{N},$$

applying the operator $W$ to the both sides, we get

$$e_{q_\xi+1} = \mu_q W e_{q_\xi+1} + \lambda_q e_{q_\xi}; \; (W - \lambda_q) e_{q_\xi+1} = -\lambda_q^2 e_{q_\xi},$$

from what follows the fact that a root vector $e_{q_\xi+1}$ is the root vector of the operator $W$ of the height 2. Having established the formulas

$$\lambda_q^2 (W - \lambda_q) e_{q_\xi+1} = -\lambda_q^4 e_{q_\xi}; \; (W - \lambda_q)^n e_{q_\xi+n} = -\lambda_q^2 (W - \lambda_q)^{n-1} e_{q_\xi+n-1}, \; n = 2, 3, \ldots,$$

we get the formula showing that $e_{q_\xi+n}$ is the root vector of the operator $W$ of the height $n+1$, i.e.

$$(W - \lambda_q)^n e_{q_\xi+n} = (-1)^n \lambda_q^{2n} e_{q_\xi}.$$

On the other hand, the inverse statement establishing the fact that the root vectors of the operator $W$ are the root vectors of the operator $B$ with the same height is based upon the absolutely analogous reasonings. Thus, we can consider invariant subspaces $\mathfrak{N}_q$ and $\mathfrak{M}_q$ of the operator $W$, where $\mathcal{P}_{\mu_q}$ is the projector onto $\mathfrak{N}_q$. Apparently, in such terms, we have a description of the operator restriction to the subspaces given by Theorem 6.17 [7, p.178] wherein the most valuable fact is that the restriction of the resolvent to $\mathfrak{M}_q$ is holomorphic in the closed domain enclosed with the contour $\Gamma_{\lambda_q}$.

 

Further, using the made assumption $\Theta(W) \subset \mathfrak{L}_0(\theta)$, we put the following contour into correspondence with the operator

$$\vartheta := \{\lambda : \ |\lambda| = r_0, \ |\mathrm{arg}\lambda| \le \theta_0\} \cup \{\lambda : \ |\lambda| > r_0, \ |\mathrm{arg}\lambda| = \theta_0\}, \ r_0 := |\lambda_1| - \varsigma, \ \theta_0 := \theta + \varsigma, \quad (5)$$

where $\varsigma$ is an arbitrary small positive fixed number. We put the contour $C\vartheta$ into correspondence with the operator $CW$. Denote by $\Omega$ the set of the complex plane enclosed with the contour $\vartheta$ so that the set does not contain the origin. Consider the following improper integral

$$I_\vartheta f = \frac{1}{2\pi i} \int\limits_{\vartheta} B(I - \lambda B)^{-1} f d\lambda, \ f \in \mathfrak{H},$$

where the integration direction is chosen so that the inside of the sector appears at the right-hand side while the point is going along the contour. However, it may happen that the improper integral does not exist but the Cauchy principal value does exist. This idea motivates us to introduce the contour

$$R^+ = \tilde{R} \cup \hat{R},$$

where

$$\tilde{R} := \{\lambda : \ |\lambda| = r_0, \ |\mathrm{arg}\lambda| \le \theta_0\} \cup \{\lambda : \ r_0 < |\lambda| < R, \ |\mathrm{arg}\lambda| = \theta_0\},$$

and

$$\hat{R} := \{\lambda : \ |\lambda| = R, \ |\mathrm{arg}\lambda| \le \theta_0\}.$$

Consider the integral

$$I_{R^+} f = \frac{1}{2\pi i} \int\limits_{\hat{R}} B(I - \lambda B)^{-1} f d\lambda + \frac{1}{2\pi i} \int\limits_{\tilde{R}} B(I - \lambda B)^{-1} f d\lambda = I_{\tilde{R}} + I_{\hat{R}}.$$

It is clear that if we are able to arrange a sequence $\{R_k\}_1^\infty$ so that $I_{\tilde{R}_k} \to 0$ and

$$P_{\hat{R}_k} f \xrightarrow{\mathfrak{H}} h \in \mathfrak{H},$$

then we have the convergence of the contour integral in some sense what we define by the following relation

$$\text{p.v.} \frac{1}{2\pi i} \int\limits_{\vartheta} B(I - \lambda B)^{-1} f d\lambda < \infty.$$

At the same time, we have

$$I_{R_k^+} f = \sum_{\nu=0}^{k} \sum_{q=N_\nu+1}^{N_{\nu+1}} \sum_{\xi=1}^{m(q)} \sum_{i=0}^{k(q_\xi)} e_{q_\xi+i} c_{q_\xi+i},$$

where $N_k$ is a number of the eigenvalues fallen inside the contour $R_k^+$. Apparently, we can write

$$\text{p.v.} \frac{1}{2\pi i} \int\limits_{\vartheta} B(I - \lambda B)^{-1} f d\lambda = \sum_{\nu=0}^{\infty} \sum_{q=N_\nu+1}^{N_{\nu+1}} \sum_{\xi=1}^{m(q)} \sum_{i=0}^{k(q_\xi)} e_{q_\xi+i} c_{q_\xi+i} = If.$$

The latter relation gives us the analog of the unit decomposition corresponding to the sectorial operator with the discrete spectrum.

 

In order to make a comparison analysis, let us apply the above reasonings to the special case fallen in the issue - positive self-adjoint operator with the discreet spectrum. In this case we can claim

$$\int\limits_{0}^{\infty} dP_{\lambda} f = \text{p.v.} \frac{1}{2\pi i} \int\limits_{\vartheta} R_W(\lambda) f d\lambda, \tag{6}$$

where in accordance with the given conditions $W$ is a self-adjoint operator with the discrete spectrum such that $(Wf, f)_{\mathfrak{H}} \geq 0$, the operator $B$ is the inverse operator that is compact due to the definition of the operator with a discrete spectrum. Apparently, the improper integrals represent the unit decomposition formula in the considered case. To prove relation (6), we should show that the considered self-adjoint operator does not have root vectors. Consider the adjoint operator with respect to $\mathcal{P}_{\mu_q},$ using relation (4), we get

$$(\mathcal{P}_{\mu_q} f, g)_{\mathfrak{H}} = \left( f, \frac{1}{2\pi i} \oint\limits_{\Gamma_{\lambda_q}} R_W(\bar{\lambda}) g \, d\bar{\lambda} \right)_{\mathfrak{H}} = - \left( f, \frac{1}{2\pi i} \oint\limits_{\Gamma_{\lambda_q}} R_W(\lambda) g \, d\lambda \right)_{\mathfrak{H}} = (f, \mathcal{P}_{\mu_q} g)_{\mathfrak{H}}, \ f, g \in \mathfrak{H}.$$

Taking into account the fact $\mathcal{P}_{\mu_q} \mathcal{P}_{\mu_p} = 0, \ q \neq p$ (see paragraph 1.3, Chapter I, [5]), we obtain the fact that root vectors corresponding to the different eigenvalues are orthogonal. Thus, we have a sequence of the orthogonal subspaces $\{\mathfrak{N}_q\}_1^{\infty}$. Having noticed the fact that $\mathfrak{N}_q$ are finite-dimensional and chosen in each $\mathfrak{N}_q$ an orthogonal basis due to the Schmidt orthogonalisation procedure, we obtain the orthogonal system in $\mathfrak{H}$. Here, we should remark that if we choose the order in the procedure corresponding to the hight of the root vectors, i.e. the first basis vector is an eigenvector, the second basis vector is obtained as a linear combination of the first one and the root vector of the height equals two etc. then the root vectors preserve their heights after the Schmidt procedure. In this way, we obtain the fact that the inner product of a root vector with an arbitrary eigenvector equals to zero. Taking into account the fact that the eigenvectors of the operator $B$ form a complete orthogonal system in $\mathfrak{H}$, we obtain the desired result, i.e. the self-adjoint operator with the discrete spectrum does not have root vectors. It follows that the given formula of the unit decomposition for the sectorial operator does not contain root vectors. Taking into account the uniqueness property of the decomposition on the complete orthogonal system in the Hilbert space, we obtain the desired result.

Having motivated by the previously made generalization, we will consider the unit decomposition for the sectorial operator with a discrete spectrum

$$I = \text{p.v.} \frac{1}{2\pi i} \int\limits_{\vartheta} R_W(\lambda) f d\lambda,$$

where the sequence of contours is chosen in a special way based upon the location of the operator eigenvalues. In accordance with the main principles of the spectral theorem, we can construct a functional calculus having most valuable applications in the differential equation theory. Moreover, we can consider the Riesz integral in the generalized sense, where the generalization is given due to the Abel method of summation represented in detail in [6] and described in the next subparagraph.

**2.** The Abel method of summation is described in the monograph [6], it is represented by the following integral construction regarding the Riesz integral

$$\frac{1}{2\pi i} \int\limits_{\vartheta} e^{-\lambda^{\alpha} t} R_W(\lambda) f d\lambda, \ \alpha, t > 0, \ f \in \mathfrak{H}, \tag{7}$$

 

where the exponential function under the integral guarantees the absolute convergence of the integral in the improper sense. It is remarkable that the problem of the integral absolute convergence is solvable due to the following lemma representing a main technical tool in evaluating the subintegral expression (Lidskii V.B. [27]).

**Lemma 2.** *Assume that $B$ is a compact operator, $\Theta(B) \subset \mathfrak{L}_0(\theta)$, $0 < \theta < \pi$, then on each ray $\zeta$ containing the point zero and not belonging to the sector $\mathfrak{L}_0(\theta)$ as well as the real axis, we have*

$$\|(I - \lambda B)^{-1}\| \leq \frac{1}{\sin\psi}, \, \lambda \in \zeta,$$

*where* $\psi = \min\{|\arg\zeta - \theta|, |\arg\zeta + \theta|\}$.

However, the idea by Lidskii V.B. is to connect the integral construction (7) and the series on the root vectors of the operator. Using notations corresponding to (2), define the operators

$$\mathcal{P}_q(\alpha, t)f = \sum_{\xi=1}^{m(q)} \sum_{i=0}^{k(q_\xi)} e_{q_\xi+i} c^{(\alpha)}_{q_\xi+i}(t), \, q \in \mathbb{N},$$

where

$$c^{(\alpha)}_{q_\xi+i}(t) = e^{-\lambda_q^\alpha t} \sum_{m=0}^{k(q_\xi)-i} H_m(\alpha, \lambda_q, t) c_{q_\xi+i+m}, \, i = 0, 1, 2, ..., k(q_\xi),$$

$$H_m(\alpha, \lambda, t) := \frac{e^{\lambda^\alpha t}}{m!} \cdot \lim_{\zeta \to 1/\lambda} \frac{d^m}{d\zeta^m} \left\{ e^{-\zeta^{-\alpha} t} \right\}, \, m = 0, 1, 2, ..., .$$

The following relation gives us a connection between the improper integral and the series

$$\frac{1}{2\pi i} \int\limits_{R_n^+} e^{-\lambda^\alpha t} R_W(\lambda) f d\lambda = \sum_{q=1}^{n} \mathcal{P}_q(\alpha, t) f, \tag{8}$$

where the sequence $\{R_k\}_1^\infty$ is chosen so that the characteristic numbers only $\lambda_1, \lambda_2, ..., \lambda_n$ lie inside the contour $R_n^+$. The latter idea leads us to the lacunae method elaborated by Lidskii V.B. who introduced the splitting of the sum (8) corresponding to subsets of the characteristic numbers $\lambda_{N_\nu+1}, \lambda_{N_\nu+2}, ..., \lambda_{N_\nu+1}$, where $\{N_\nu\}_0^\infty$ is a subsequence of the natural numbers. The following relation was studied by Lidskii V.B. [27]

$$\frac{1}{2\pi i} \int\limits_{\vartheta} e^{-\lambda^\alpha t} R_W(\lambda) f d\lambda = \sum_{\nu=0}^{\infty} \sum_{q=N_\nu+1}^{N_{\nu+1}} \mathcal{P}_q(\alpha, t) f, \, f \in \mathfrak{H}.$$

We have previously mentioned that the improper integral may be absolutely divergent but the principal value does exist, however it does not happen in the case of the generalized integral construction (7) due the norm estimate given in Lemma 2. Thus, the properties of the resolvent of the considered operator class, the involved exponential function in the integral construction allow to avoid the Cauchy principal value sense.

Apparently, the main concept of the root vectors decomposition in the Abel-Lidskii sense can be resumed as follows.

(S1) *Under the assumptions* $B\in\mathfrak{S}_\sigma,\ 0<\sigma<\infty,\ \Theta(B)\subset\mathfrak{L}_0(\theta),\ \alpha>0$ *a sequence of natural numbers* $\{N_\nu\}_0^\infty$ *can be chosen so that*

$$\frac{1}{2\pi i}\int\limits_{\vartheta}e^{-\lambda^{\alpha}t}B(I-\lambda B)^{-1}fd\lambda=\sum_{\nu=0}^{\infty}\sum_{q=N_\nu+1}^{N_{\nu+1}}\mathcal{P}_q(\alpha,t)f,\ f\in\mathfrak{H},$$

*the latter series is absolutely convergent in the sense of the norm. Moreover*

$$\lim_{t\to 0}\frac{1}{2\pi i}\int\limits_{\vartheta}e^{-\lambda^{\alpha}t}B(I-\lambda B)^{-1}fd\lambda=f\in\mathrm{D}(W).$$

Note that the main difficulty in the proof of the first relation appeals to estimating of the Fredholm determinant introduced by Lidskii V.B. in [27].

It is rather reasonable to consider a modified version of the above integral construction (7) defined as follows

$$\tilde{f}(t)=\frac{1}{2\pi i}\int\limits_{\vartheta}e^{-t\varphi^{\alpha}(\lambda)}R_W(\lambda)fd\lambda,\ f\in\mathfrak{H},\ \alpha,t>0,\tag{9}$$

where $\varphi(\lambda),\ \lambda\in\mathbb{C}$ is a function of the complex variable. Certainly, we can involve notions related to the integral convergence such as Cauchy principal value etc. Here and further, for the simplicity of writing we omit the indexes indicating the function $\varphi$ and the parameter $\alpha$ assuming that their absence will not lead to any misunderstanding. Remark that the parameter $\alpha$ is involved due to the wide spectrum of applications in the differential equation theory given by the Cauchy problem represented here in the formal form

$$\mathfrak{D}_-^{1/\alpha}f(t)=\varphi(W)f(t),\ f(t)\to f\in\mathfrak{H},\ t\to 0,\ \alpha>0,\tag{10}$$

where the function $\varphi$ is supposed to have a decomposition on the series convergent point-wise regarding the argument of the operator $W$, i.e.

$$\varphi(W)f=\sum_{k=-\infty}^{\infty}c_kW^kf,\ c_k\in\mathbb{C},\ f\in\mathrm{D}(W^\infty),\tag{11}$$

the element-function $\tilde{f}(t)$ is a solution of problem (10).

**Schatten-von Neumann classes with limit index**

Consider an operator class

$$\mathfrak{S}^{\star}_{\sigma}(\mathfrak{H}):=\{B\in\mathfrak{S}_{\sigma+\delta},\ B\bar{\in}\,\mathfrak{S}_\sigma,\ \delta>0\},\ 0\le\sigma<\infty.$$

Recall that an arbitrary compact operator $B$ can be represented as a series on the basis vectors due to the so-called polar decomposition $B=U|B|$, where $U$ is a concrete unitary operator, $|B|:=(B^*B)^{1/2}$, i.e. using the system of eigenvectors $\{e_n\}_1^\infty$, we have

$$Bf=\sum_{n=1}^{\infty}s_n(B)(f,e_n)g_n,\tag{12}$$

 

where $e_n, s_n$ are the eigenvectors and eigenvalues of the operator $|B|$ respectively, $g_n = Ue_n$. Consider the following condition

$$(\ln^{1+\kappa} x)'_{s_n^{-1}(B)} = o(n^{-\kappa}),\ \kappa > 0. \tag{13}$$

The given relation can be said as an asymptotics more subtle than one of the power type. It can be verified easily (see [15]) that there exists an artificially constructed operator (12) belonging to the class $\mathfrak{S}^{\star}_{1/\kappa}(\mathfrak{H})$ and having singular numbers satisfying the latter relation. Below, we produce the corresponding set of the singular numbers.

**i**) Consider a sequence $s_n = n^{-\kappa} \ln^{-\kappa}(n+q) \cdot \ln^{-\kappa} \ln(n+q),\ q > e^e - 1,\ n \in \mathbb{N}$. Substituting, we get

$$\frac{\ln^{\kappa} s_n^{-1}}{s_n^{-1}} \leq \frac{C \ln^{\kappa}(n+q)}{n^{\kappa} \ln^{\kappa}(n+q) \cdot \ln^{\kappa} \ln(n+q)} = \frac{C}{n^{\kappa} \cdot \ln^{\kappa} \ln(n+q)},$$

what gives us the fulfilment of condition (13), whereas using the integral test for convergence, we can easily see that

$$\sum_{n=1}^{\infty} s_n^{-1/\kappa} = \infty.$$

Consider condition (13) in detailed, it gives us

$$\frac{n \ln s_n^{-1}(B)}{s_n^{-1/\kappa}(B)} \leq C \cdot \alpha_n,\ \alpha_n \to 0,\ n \to \infty.$$

Taking into account the facts $n(s_j^{-1}, |B|) = j,\ s_j^{-1} < r < s_{j+1}^{-1}$, using the monotonous property of the functions, we get

$$\ln r \frac{n(r, |B|)}{r^{1/\kappa}} < C \cdot \alpha_n,\ s_n^{-1} < r < s_{n+1}^{-1}, \tag{14}$$

i.e. we obtain the following implication

$$(\ln^{1+\kappa} x)'_{s_n^{-1}(B)} = o(n^{-\kappa}), \Longrightarrow \left( \ln r \frac{n(r, |B|)}{r^{1/\kappa}} \to 0 \right).$$

In accordance with Lemma 3 [15], we have

$$\ln r \frac{n(r, |B|)}{r^{\rho}} \to 0, \Longrightarrow \ln r \left( \int_0^r \frac{n(t, |B|)}{t^{p+1}} dt + r \int_r^{\infty} \frac{n(t, |B|)}{t^{p+2}} dt \right) r^{p-\rho} \to 0,\ r \to \infty, \tag{15}$$

where $\rho$ is a convergence exponent (non integer) corresponding to the counting function $n(r, |B|)$, i.e. $\rho := \inf \xi$,

$$\sum_{n=1}^{\infty} s_n^{\xi}(B) < \infty,$$

the number $p+1$ is the smallest natural number for which the latter series is convergent.

Remark that relation (15) is the main tool in the proof of the fulfilment of condition S1 (see estimates for the Fredholm determinant given in Theorems 3,4 [15]). In this case the main advantage in comparison with the Lidskii V.B. results [27] is that we can expand the range assuming $\alpha = \rho$ producing a sequence of contours of the power type instead of undefined one, what follows from the results [27].

Let us make a comparison analysis with the previously used concepts. Recall that the order of the operator with a discrete spectrum $W$ was introduced in [45] as a value $\mu > 0$ for which $s_n(B) \leq Cn^{-\mu}$. Apparently, this definition is rather vague for we are compelled to consider $\inf \mu$ to obtain the correct information on the singular number asymptotics. On the contrary, note that the sequence (i) satisfying condition (13) admits the estimate from below, i.e.

$$\forall \varepsilon > 0, \, \exists N(\varepsilon): \; n^{-\kappa-\varepsilon} < s_n < \frac{C}{n^\kappa \ln^\kappa s_n^{-1}}, \, n > N(\varepsilon). \tag{16}$$

Thus, we can observe that the notion of the order is not applicable to the artificially constructed operator (12) with the singular numbers satisfying condition (16) as well as the asymptotics of the power type is spoiled. At the same time, the asymptototics given by relation (13) allows to deploy fully technicalities in estimating the norm of Fredholm determinant. However, the artificial construction of the operator (12) creates a prerequisite for the following question.

- Is there exists an operator represented in the analytical form whose singular numbers satisfy more subtle asymptotics than one of the power type, i.e. an operator belonging to the class $\mathfrak{S}_\rho$ satisfying condition (13) where $\kappa = 1/\rho$?

This question is a central point of the further narrative wherein we consider a functional calculus for sectorial operators with discrete spectrum and on its base create an operator with the asymptotics more subtle than one of the power type.

## 3. Main results

### 3.1. *Operator function*

Assume that generally $\varphi$ is a function of the complex variable and define an operator function in the sense of the functional calculus based upon the spectral theorem as follows

$$\varphi_t(W)f := \frac{1}{2\pi i} \int\limits_{\vartheta} \varphi(\lambda) e^{-\varphi^\alpha(\lambda)t} B(I - \lambda B)^{-1} f d\lambda, \, \alpha, t > 0,$$

where the element $f$ is such that the integral exists in the Riemann sense. Using this notation we assume that the parameter $\alpha$ is fixed. The operator $W$ is called by the operator argument. Assume that the function $\varphi$ is represented by the Laurent series in the neighborhood of the infinitely distant point

$$\varphi(z) = \sum_{k=-\infty}^{\infty} c_k z^k, \, z \in \mathbb{C}, \; c_k := \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{\varphi(\xi)}{\xi^{k+1}} d\xi, \tag{17}$$

where $\Gamma$ is a closed contour within the region of analyticity, thus we can define the operator function in the sense (11). Below, we will show the equivalence between the given definitions under natural assumptions regarding the operator argument, we produce sufficient conditions under which being imposed the operator function exists in both senses, moreover they are equal on the subset of the Hilbert space.

If we assume generally that $\varphi$ is an analytic function then, as it can be clearly seen from the structure of the proofs [27], it is technically insufficient for establishing properties of the operator function leading to the basic statements of the functional calculus. Thus, we need impose some specific conditions. The detailed analysis of the proof of [27], [17] and [18] allows to extract minimal conditions that can be formulated as follows

(P1) Assume that $W$ is an operator with discrete spectrum, $\Theta(W)\subset\mathfrak{L}_0(\theta)$, the function $\varphi$ analytic on the set $\Omega$ maps it into the sector $\mathfrak{L}_0(\varpi),\ \varpi<\pi/2\alpha,$ its zeros do not belong to $\Omega$.

(P2) There exists a sequence $\{R_n\}_1^\infty,\ R_n\uparrow\infty$ such that

$$\forall\varepsilon>0,\ \exists N(\varepsilon):\ m_\varphi(R_n)\geq[M_\varphi(R_n)]^{\cos\pi p-\varepsilon},\ n>N(\varepsilon),\ 0<p<1.$$

(P3) For sufficiently large absolute values of the argument, the following relation holds

$$|\lambda|^{1/\alpha}C_1<|\varphi(\lambda)|<e^{C_2|\lambda|^q},\ \lambda\in\vartheta,\ 0<q<1.$$

Note that the conditions formulated above are not unnatural but represent a genuine nature of the well-known mathematical objects. For instance the operator satisfying conditions of the abstract uniform elipticity H1,H2 satisfies condition P1, see [14]. It is not hard to prove that the polynomial function with the conditions imposed upon the coefficients satisfies condition P1, see [17]. The class of the entire functions of the order less than a one satisfies condition P2 due to the Wieman theorem, see Theorem 30, §18, Chapter I [23]. The entire functions of the order less than a half whose zeros form $R$ - set of the proximate order satisfy condition P3 by virtue of Lemma 1. In its own turn, condition P3 is the so-called condition of the growth regularity, more detailed see §1, Chapter II, [23].

Eventually, we are ready to formulate a number of propositions forming the main technical tool in solving Cauchy problem (10) and constructing a functional calculus for the sectorial operators. The scheme of the following lemma proof is represented in Lemma 3 [18], however we produce the main technical points of the proof along the statement.

**Lemma 3.** *Assume that conditions P1,P2 hold, then*

$$\tilde{f}(t)\xrightarrow{\mathfrak{H}}f,\ t\to0,\ f\in\mathrm{D}(W).$$

*Proof.* The idea of the proof is based upon the following decomposition

$$\frac{1}{2\pi i}\int\limits_{\vartheta}e^{-\varphi^\alpha(\lambda)t}B(I-\lambda B)^{-1}fd\lambda=\frac{1}{2\pi i}\int\limits_{\vartheta}e^{-\varphi^\alpha(\lambda)t}\lambda^{-2}\left(I-\lambda B\right)^{-1}Wfd\lambda-$$
$$-\frac{1}{2\pi i}\int\limits_{\vartheta}e^{-\varphi^\alpha(\lambda)t}\lambda^{-2}(I+\lambda B)Wfd\lambda=I_1(t)+I_2(t),$$

Firstly, we should show that

$$\lim_{t\to0}I_1(t)=\frac{1}{2\pi i}\int\limits_{\vartheta}\lambda^{-2}\left(I-\lambda B\right)^{-1}Wfd\lambda,$$

secondly

$$\frac{1}{2\pi i}\int\limits_{\vartheta}\lambda^{-2}\left(I-\lambda B\right)^{-1}Wfd\lambda=f,$$

and thirdly $I_2(t)=0$. The first relation follows from the condition in accordance with which the function $\varphi$ maps the sector $\mathfrak{L}_0(\theta)$ into the sector $\mathfrak{L}_0(\varpi),\ \varpi<\pi/2\alpha$ for sufficiently large absolute values of the argument. It gives us the relation $|e^{-\varphi^{\alpha}(\lambda)t}| \leq C,\ \lambda \in \vartheta$ what leads us to the uniform convergence of the improper integral $I_1(t)$ with respect to the parameter $t$ and as a consequence provides the substantiation of passing to the limit under the integral. The second relation is based on the fact that the eigenvalues of the operator $W$ belongs to the set $\Omega$ it gives us an opportunity to consider a domain bounded by the contour $R^- = \tilde{R} \cup \check{R}$, where $\tilde{R}$ is defined above, $\check{R} := \{\lambda : |\lambda| = R,\ \theta_0 \leq \arg\lambda \leq 2\pi - \theta_0\}$. Using the analytic property of the function under the integral, calculating the residue, we get

$$\frac{1}{2\pi i}\int\limits_{R^-} \lambda^{-2}\left(I - \lambda B\right)^{-1} W f d\lambda = \operatorname*{Res}_{z=0}\left\{\lambda^{-2}\left(I - \lambda B\right)^{-1} W f\right\} = \lim_{\lambda\to 0}\frac{d(I-\lambda B)^{-1}}{d\lambda} W f = f.$$

Taking into account the following facts, the first one follows from boundedness of the resolvent on the set $\check{R}$

$$\int\limits_{\check{R}} \lambda^{-2}\left(I - \lambda B\right)^{-1} W f d\lambda \to 0,\ \int\limits_{\tilde{R}} \lambda^{-2}\left(I - \lambda B\right)^{-1} W f d\lambda \to \int\limits_{\vartheta} \lambda^{-2}\left(I - \lambda B\right)^{-1} W f d\lambda,\ R\to\infty,$$

we obtain the second relation. The third relation follows from condition P2. Having noticed the fact

$$\frac{1}{2\pi i}\oint\limits_{R^+} \lambda^{-2} e^{-\varphi^{\alpha}(\lambda)t}\left(I + \lambda B\right) W f d\lambda = 0,$$

since the function under the integral is analytic inside the closed contour $R^+$, and considering decomposition

$$\frac{1}{2\pi i}\oint\limits_{R^+} \lambda^{-2} e^{-\varphi^{\alpha}(\lambda)t}\left(I + \lambda B\right) W f d\lambda = \frac{1}{2\pi i}\int\limits_{\tilde{R}} \lambda^{-2} e^{-\varphi^{\alpha}(\lambda)t}\left(I + \lambda B\right) W f d\lambda +$$

$$+\frac{1}{2\pi i}\int\limits_{\hat{R}} \lambda^{-2} e^{-\varphi^{\alpha}(\lambda)t}\left(I + \lambda B\right) W f d\lambda = I_{\tilde{R}} + I_{\hat{R}},$$

we arrive at the following implication. If $I_{\hat{R}_k} \to 0,\ R_k \to \infty$, then

$$\frac{1}{2\pi i}\oint\limits_{R_k^+} \lambda^{-2} e^{-\varphi^{\alpha}(\lambda)t}\left(I + \lambda B\right) W f d\lambda \to \frac{1}{2\pi i}\int\limits_{\vartheta} \lambda^{-2} e^{-\varphi^{\alpha}(\lambda)t}\left(I + \lambda B\right) W f d\lambda.$$

The fact $I_{\hat{R}_k} \to 0$ is based upon the mapping property of the function $\varphi$, in accordance with which for a sufficiently large values of the argument the following relation holds

$$\operatorname{Re}\varphi^{\alpha}(\lambda) \geq |\varphi(\lambda)|^{\alpha}\cos\alpha\varpi,\ \lambda\in\Omega,$$

applying condition P2, we obtain

$$\forall\varepsilon > 0,\ \exists N(\varepsilon) :\ e^{-\operatorname{Re}\varphi^{\alpha}(\lambda)t} \leq e^{-Ct[M_{\varphi}(\lambda)]^{\alpha\cos\pi p - \varepsilon}},\ \lambda\in R_k,\ k > N(\varepsilon),\ 0 < p < 1,$$

where the choice of the sequence $\{R_k\}_1^{\infty}$ is dictated by condition P2. Using these estimates, we can easily obtain the desired result. Here, the detailed reasonings related to estimation of the integral constructions are omitted due to simplicity, however the reader can find them in Lemma 3 [18]. □

**Lemma 4.** *Assume that conditions P1,P3 are satisfied, then the operator function is defined, i.e.*

$$\varphi_t(W) f = \frac{1}{2\pi i}\int\limits_{\vartheta} \varphi(\lambda) e^{-\varphi^{\alpha}(\lambda)t} B(I - \lambda B)^{-1} f d\lambda \in \mathfrak{H},\ f\in \mathrm{D}(W).$$

*Proof.* Using the mapping properties of the function $\varphi$ formulated in condition P1, applying Lemma 1 to estimate the operator, and condition P3 to estimate the function under the integral, we have

$$\left\| \int_{\vartheta} \varphi(\lambda) e^{-\varphi^{\alpha}(\lambda)t} B(I-\lambda B)^{-1} f d\lambda \right\|_{\mathfrak{H}} \leq \int_{\vartheta} e^{-\mathrm{Re}\varphi^{\alpha}(\lambda)t} |\varphi(\lambda)| \cdot \left\| B(I-\lambda B)^{-1} f \right\|_{\mathfrak{H}} |d\lambda| \leq$$

$$\leq C \|f\|_{\mathfrak{H}} \int_{\vartheta} e^{-|\varphi^{\alpha}(\lambda)t| \cos \alpha\varpi} |\varphi(\lambda)| \, |d\lambda| \leq C \|f\|_{\mathfrak{H}} \int_{\vartheta} e^{C_2 |\lambda|^q - C_1^{\alpha} |\lambda| t \cos \alpha\varpi} |d\lambda| < \infty.$$

The latter relation proves the desired result. □

The following lemma represents conditions on the function of the complex variable $\varphi$ under which being imposed the operator function $\varphi(W)$ has the extension on the set $\mathrm{D}(W)$.

**Lemma 5.** *Assume that an entire function satisfies conditions P1-P3, then the following relation holds*

$$\varphi_t(W) f = \varphi(W) \tilde{f}(t), \; t > 0, \; f \in \mathrm{D}(W^{\infty}).$$

*Proof.* Firstly, we establish the formula

$$\int_{\vartheta} \varphi(\lambda) e^{-\varphi^{\alpha}(\lambda)t} B(I-\lambda B)^{-1} f d\lambda = \sum_{n=0}^{\infty} c_n \int_{\vartheta} e^{-\varphi^{\alpha}(\lambda)t} \lambda^n B(I-\lambda B)^{-1} f d\lambda.$$

Secondly, we obtain the decomposition formula

$$\int_{\vartheta} e^{-\varphi^{\alpha}(\lambda)t} \lambda^n B(I-\lambda B)^{-1} f d\lambda$$

$$= W^n \int_{\vartheta} e^{-\varphi^{\alpha}(\lambda)t} B(I-\lambda B)^{-1} f d\lambda - \int_{\vartheta} e^{-\varphi^{\alpha}(\lambda)t} \sum_{k=0}^{n-1} \lambda^k B^{k+1} W^n f d\lambda = I_1(t) + I_2(t).$$

Thirdly, we will show that $I_2(t) = 0$. Taking into account the fact that the series representing an analytic function is absolutely convergent, we have

$$\int_{\tilde{R}_k} \varphi(\lambda) e^{-\varphi^{\alpha}(\lambda)t} B(I-\lambda B)^{-1} f d\lambda = \sum_{n=0}^{\infty} c_n \int_{\tilde{R}_k} e^{-\varphi^{\alpha}(\lambda)t} \lambda^n B(I-\lambda B)^{-1} f d\lambda.$$

Using Lemma 2, the mapping properties of the function $\varphi$, we get

$$\left\| \int_{\tilde{R}_k} e^{-\varphi^{\alpha}(\lambda)t} \lambda^n B(I-\lambda B)^{-1} f d\lambda \right\|_{\mathfrak{H}} \leq C \|f\|_{\mathfrak{H}} \int_{\tilde{R}_k} e^{-\mathrm{Re}\varphi^{\alpha}(\lambda)t} |\lambda|^n |d\lambda| \leq$$

$$\leq C \|f\|_{\mathfrak{H}} \int_{\tilde{R}_k} e^{-|\varphi(\lambda)|^{\alpha} t \cos \varpi\alpha} |\lambda|^n |d\lambda|.$$

The latter integral is estimated due to condition P3, we have

$$\int_{\tilde{R}_k} e^{-|\varphi(\lambda)|^{\alpha} t \cos \varpi\alpha} |\lambda|^n |d\lambda| \leq C \int_{\tilde{R}_k} e^{-C_1^{\alpha} |\lambda| t \cos \varpi\alpha} |\lambda|^n |d\lambda| \leq C t^{-n} \Gamma(n+1). \tag{18}$$

Using the standard formula establishing the estimate for the Taylor coefficients of the entire function, then applying the Stirling formula, we get

$$|c_n| < (e\sigma_{\varphi}\varrho)^{n/\varrho}\, n^{-n/\varrho} < (2\pi)^{1/2\varrho}(\sigma_{\varphi}\varrho)^{n/\varrho}\left(\frac{\sqrt{n}}{n!}\right)^{1/\varrho},$$

where $0 < \sigma_{\varphi} < \infty$ is the type of the function $\varphi$. Therefore

$$\sum_{n=1}^{\infty}|c_n|\left\|\int\limits_{\tilde{R}_k} e^{-\varphi^{\alpha}(\lambda)t}\lambda^n B(I-\lambda B)^{-1} f d\lambda\right\| \leq C\sum_{n=1}^{\infty}(\sigma_{\varphi}\varrho)^{n/\varrho}t^{-n}(n!)^{1-1/\varrho}n^{1/2\varrho}.$$

The latter series is convergent for an arbitrary fixed $t > 0$, what proves the uniform convergence of the series with respect to $k$. Therefore, reformulating the well-known theorem of calculus applicably to the norm of the Hilbert space, we obtain the desired result. The decomposition formula represented in the second step is obtained by virtue of the relation

$$\lambda^k B^k(I-\lambda B)^{-1} = (I-\lambda B)^{-1} - (I+\lambda B+...+\lambda^{k-1}B^{k-1}),\ k \in \mathbb{N}, \tag{19}$$

and the fact that the operators $W^k$ and $B(I-\lambda B)^{-1}$ commute. The third relation follows from the implication

$$\int\limits_{\hat{R}_k} e^{-\varphi^{\alpha}(\lambda)t}\lambda^k d\lambda \to 0, \Rightarrow \oint\limits_{R_k^+} e^{-\varphi^{\alpha}(\lambda)t}\lambda^k d\lambda \to \int\limits_{\vartheta} e^{-\varphi^{\alpha}(\lambda)t}\lambda^k d\lambda,$$

where $\{R_k\}_1^{\infty}$, $R_k \uparrow \infty$, the premise can be proved by virtue of condition P2, see reasonings of Lemma 3. The proof is complete. □

In accordance with Lemmas 4,5, we can consider an extension of the operator function $\varphi(W)$ (11) to the set $\{\tilde{f}(t) :\ f \in \mathrm{D}(W),\, t > 0\}$. Having noticed this fact, we will use the unified notation $\varphi(W)$ for the extended operator function at the same time taking into account the fact that the decomposition on the series holds for the elements $\{\tilde{f}(t) :\ f \in \mathrm{D}(W^{\infty}),\, t > 0\}$. Observe that under the Lemma 3 conditions it seems rather natural to extend the operator function $\varphi(W)$ to the closure of the set $\{\tilde{f}(t) :\ f \in \mathrm{D}(W),\, t > 0\}$ that is dense in $\mathrm{D}(W)$. The following lemma establishes the fact that the operator function restriction admits the closure.

**Lemma 6.** *Assume that conditions P1,P3 hold, then the operator function $\varphi(W)$ is closable on the image of the closed ball in the Hilbert space*

$$\left\{\tilde{f}(t) :\ f \in \mathrm{D}(W),\, \|f\|_{\mathfrak{H}} \leq C,\, t \geq 0\right\}.$$

*Proof.* Consider sequences

$$\tilde{f}_k(t) \to g(t),\ \tilde{g}_k(t) \to g(t),$$

where $g(t)$ is an element-function of the Hilbert space with unknown dependence on the parameter $t$, see paragraph 1.2. Observe that for an arbitrary element $h \in \mathfrak{H}$, we have

$$(\tilde{f}_k(t), h)_{\mathfrak{H}} = -\left(f_k, \frac{1}{2\pi i}\int\limits_{\vartheta}\overline{e^{-\varphi^{\alpha}(\lambda)t}}R_{W^*}(\bar{\lambda})h\, d\bar{\lambda}\right)_{\mathfrak{H}} = \left(f_k, \frac{1}{2\pi i}\int\limits_{\vartheta}\overline{e^{-\varphi^{\alpha}(\bar{\lambda})t}}R_{W^*}(\lambda)h\, d\lambda\right)_{\mathfrak{H}}.$$

Note that the function $\overline{\varphi(\bar{\lambda})}$ is analytic in $\Omega$. It follows from the fact that the function $\varphi(\bar{\lambda})$ is antianalytic in the domain symmetric with respect to the real axis. In accordance with Lemma 2, we have

$$\lim_{t\to 0}\frac{1}{2\pi i}\int\limits_{\vartheta}\overline{e^{-\varphi^{\alpha}(\bar{\lambda})t}}R_{W^*}(\lambda)h\,d\lambda = h,\ h\in \mathrm{D}(W^*).$$

Since $\mathrm{D}(W^*)$ is dense in $\mathfrak{H}$, then the set

$$\left\{\frac{1}{2\pi i}\int\limits_{\vartheta}\overline{e^{-\varphi^{\alpha}(\bar{\lambda})t}}R_{W^*}(\lambda)h:\ h\in \mathrm{D}(W^*)\right\}$$

is dense in $\mathfrak{H}$. It is not hard to prove that if

$$(f_k,\psi)_{\mathfrak{H}}\to C_{\psi},\ \psi\in\mathfrak{M},\ \|f_k\|_{\mathfrak{H}}\leq C,$$

where $C_{\psi}=\mathrm{const}$, $\mathfrak{M}$ is a dense set in $\mathfrak{H}$, then there exists an element $f_0\in\mathfrak{H}$ such that

$$(f_k,\psi)_{\mathfrak{H}}\to (f_0,\psi)_{\mathfrak{H}},\ k\to\infty,\ \psi\in\mathfrak{H},$$

i.e. the sequence $\{f_k\}_1^{\infty}$ is weakly convergent. To prove this fact, we should consider a sequence of the functionals $\{f_k\}_1^{\infty}$ defined on the dense subset of the Hilbert space, we can prove that the functionals are linear and bounded. Extending them on the whole Hilbert space due to the continuity property, applying the Riesz representation theorem, we obtain the desired result. Therefore

$$(\tilde{f}_k(t),h)_{\mathfrak{H}}\to(\tilde{f}_0(t),h)_{\mathfrak{H}},\ k\to\infty,\ h\in\mathrm{D}(W^*),$$

her the symbol $\tilde{f}_0(t)$ should be understood in the sense of distributions. At the same time in accordance with the made assumptions

$$(\tilde{f}_k(t),h)_{\mathfrak{H}}\to(g(t),h)_{\mathfrak{H}},\ k\to\infty.$$

Using the uniqueness of the weak limit, we obtain the fact $g(t)=\tilde{f}_0(t)$. Therefore the element $\tilde{f}_0(t)$ exists in the strong sense (9). Now, if we assume that

$$\varphi(W)\tilde{f}_k(t)\to h_1(t),\ \varphi(W)\tilde{g}_k(t)\to h_2(t),\ h_1(t)\neq h_2(t),$$

then we will arrive at the contradiction since

$$\big(\varphi(W)\tilde{f}_k(t),h\big)_{\mathfrak{H}}=\big(f_k,\varphi^*(W)h(t)\big)_{\mathfrak{H}},\ \big(\varphi(W)g_k(t),h\big)_{\mathfrak{H}}=\big(g_k,\varphi^*(W)h(t)\big)_{\mathfrak{H}},\ k\in\mathbb{N},\ h\in\mathrm{D}(W^*),$$

where in accordance with Lemma 4

$$\varphi^*(W)h(t):=\frac{1}{2\pi i}\int\limits_{\vartheta}\overline{\varphi(\bar{\lambda})e^{-\varphi^{\alpha}(\bar{\lambda})t}}R_{W^*}(\lambda)h\,d\lambda,$$

the latter equality is understood in the strong sense, from what follows

$$\big(h_1(t),h\big)_{\mathfrak{H}}=\big(h_2(t),h\big)_{\mathfrak{H}},\ h\in\mathrm{D}(W^*),\ \Rightarrow h_1(t)=h_2(t).$$

Moreover, we obtain the fact

$$\lim_{k\to\infty}\varphi(W)g_k(t)=\lim_{k\to\infty}\varphi(W)f_k(t)=\varphi(W)f_0(t),$$

 

since the strong and weak limits are coincided. Thus, we have proved the syllogism

$$\tilde{f}_k(t) \to g(t), \Rightarrow \exists f_0 \in \mathfrak{H} : g(t) = \tilde{f}_0(t), \Rightarrow \varphi(W) f_k(t) \to \varphi(W) f_0(t).$$

Now consider a case when $f_{jk} \in \mathrm{D}(W),\ j = 1, 2,\ k = 1, 2, ...$ , there exist simultaneous limits $\tilde{f}_{jk}(t) \to f_0,\ \varphi(W)\tilde{f}_{jk}(t) \to f_j,\ k \to \infty,\ t \to 0$. Let us prove that $f_1 = f_2$. Note that in accordance with Lemma 3, for each $k \in \mathbb{N}$, we get $\tilde{f}_{jk}(t) \to f_{jk},\ t \to 0$. Applying the theorem which gives the connection between simultaneous limits and repeated limits, we get $f_{jk} \to f_0,\ k \to \infty$. Using condition P1, applying Lemma 1 to estimate the operator and condition P3 to estimate the function under the integral, we obtain

$$\left\| \int\limits_{\vartheta} \varphi(\lambda) e^{-\varphi^{\alpha}(\lambda)t} R_W(\lambda) \left\{ f_{jk} - f_0 \right\} d\lambda \right\|_{\mathfrak{H}} \leq C \left\| f_{jk} - f_0 \right\|_{\mathfrak{H}} \int\limits_{\vartheta} |\varphi(\lambda) e^{-\varphi^{\alpha}(\lambda)t}| \cdot |d\lambda| \leq$$

$$\leq C \left\| f_{jk} - f_0 \right\|_{\mathfrak{H}}.$$

Therefore, there exist the coincident limits

$$\varphi(W)\tilde{f}_{jk}(t) \xrightarrow{\mathfrak{H}} \frac{1}{2\pi i} \int\limits_{\vartheta} \varphi(\lambda) e^{-\varphi^{\alpha}(\lambda)t} B(I - \lambda B)^{-1} f_0 d\lambda,\ k \to \infty,\ j = 1, 2.$$

Applying the theorem on the connection between simultaneous limits and repeated limits, we obtain the desired result. □

Apparently, we have established the fact that the operator $\varphi(W)$ restriction is closeable and therefore naturally arrive at its extension, in particular if we have a limit

$$\lim_{t \to 0} \varphi(W)\tilde{f}(t) = h \in \mathfrak{H},\ f \in \mathrm{D}(W),$$

then in accordance with Lemma 6, we can put $\varphi(W) f = h$.

**Theorem 1.** *Assume that the function of the complex variable has a decomposition in the Laurent series in the neighborhood of the infinitely distant point*

$$\varphi(z) = \sum_{k=-\infty}^{\infty} c_k z^k,\ z \in \mathbb{C},$$

*where $c_k$ are the Laurent coefficients, the regular part is an entire function satisfying conditions P1-P3. Then for a sufficiently large constant $a > 0$, the following relation holds*

$$\varphi_t(aW) f = \varphi(aW)\tilde{f}(t),\ t > 0,\ f \in \mathrm{D}(W^{\infty}).$$

*Proof.* Let us prove that the function $\varphi$ and the operator $aW$ satisfy condition P1. Apparently, in accordance with the made assumptions, the following relation holds for sufficiently large absolute values of the argument

$$\left| \sum_{k=0}^{\infty} c_k z^k \right| > C_1 r^{1/\alpha},\ z \in \vartheta,\ r = |z|.$$

Estimating the principal part, we get

$$\left| \sum_{k=-\infty}^{-1} c_k z^k \right| \leq \sum_{k=-\infty}^{-1} |c_k| r^k \leq C_2 \cdot r^{-1}.$$

 

In accordance with condition P1, we have the fact that the regular part maps the set $\Omega$ into the sector $\mathfrak{L}_0(\varpi),\ \varpi < \pi/2\alpha$ for sufficiently large values of the argument. Applying the obtained estimates, we conclude that there exists a number $N > 0$ so that the function $\varphi(z),\ z \in \Omega,\ |z| > N$ does not have zeros and its values belong to the sector $\mathfrak{L}_0(\varpi)$. Suppose $a = N/r_0$, then it is clear that condition P1 holds with respect to the operator $aW$, domain $a\Omega := \left\{z \in \mathbb{C} :\ z \cdot a^{-1} \in \Omega\right\}$, and the function $\varphi$. Here, we should note that the domain $a\Omega$ corresponds to the contour $a\vartheta$. Note that the function $\varphi$ satisfies condition P3 due to the estimate

$$|\varphi(z)| \geq \left| \left|\sum_{k=0}^{\infty} c_k z^k\right| - \left|\sum_{k=-\infty}^{-1} c_k z^k\right| \right| > C_1 r^{1/\alpha} - C_2 \cdot r^{-1} \geq C r^{1/\alpha},\ r > K,$$

where $K > 0$ is a sufficiently large number. Without loss of generality, we assume that $N \geq K$, for in the contrary case we can suppose $a = K/r_0$, preserving the scheme of the reasonings. Therefore, in accordance with Lemma 4 the operator function $\varphi_t(aW)$ is defined. Having repeated the reasonings of Lemma 5, we get

$$\frac{1}{2\pi i}\int\limits_{a\vartheta} \varphi_1(\lambda) e^{-\varphi^{\alpha}(\lambda)t} R_{aW}(\lambda) f d\lambda = \sum_{n=0}^{\infty} c_n a^n W^n \tilde{f}(t),\ f \in \mathrm{D}(W^{\infty}). \tag{20}$$

where $\varphi_1$ denotes the regular part of the Laurent series. Thus, we have in the reminder the proof of the analogous relation for the principal part. Let us rewrite relation (19) in the following form

$$\lambda^{-k} B(I - \lambda B)^{-1} = B^{k+1}(I - \lambda B)^{-1} + \lambda^{-k} B(I + \lambda B + ... + \lambda^{k-1} B^{k-1}),\ k \in \mathbb{N}.$$

Substituting, we get

$$\frac{1}{2\pi i}\int\limits_{a\vartheta} \lambda^{-k} e^{-\varphi^{\alpha}(\lambda)t} R_{aW}(\lambda) f d\lambda = a^{-k} B^k \tilde{f}(t) + \frac{1}{2\pi i}\sum_{j=1}^{k} B^j f \int\limits_{a\vartheta} \lambda^{j-k-1} e^{-\varphi^{\alpha}(\lambda)t} d\lambda = a^{-k} B^k \tilde{f}(t), \tag{21}$$

since

$$\int\limits_{a\vartheta} \lambda^{j-k-1} e^{-\varphi^{\alpha}(\lambda)t} d\lambda = 0,\ j = 1, 2, ... k,$$

we should recall that the proof the latter equality is represented in the third step of the Lemma 4 proof. In accordance with the absolute convergence of the Laurent series, we have

$$\int\limits_{a\tilde{R}} \varphi_2(\lambda) e^{-\varphi^{\alpha}(\lambda)t} R_{aW}(\lambda) f d\lambda = \sum_{k=1}^{\infty} c_{-k} \int\limits_{a\tilde{R}} \lambda^{-k} e^{-\varphi^{\alpha}(\lambda)t} R_{aW}(\lambda) f d\lambda. \tag{22}$$

Applying Lemma 2, we get

$$\left\| \int\limits_{a\tilde{R}} \lambda^{-k} e^{-\varphi^{\alpha}(\lambda)t} R_{aW}(\lambda) f d\lambda \right\| \leq \int\limits_{a\tilde{R}} |\lambda|^{-k} e^{-\mathrm{Re}\varphi^{\alpha}(\lambda)t} \|R_{aW}(\lambda) f\| \cdot |d\lambda| \leq$$

$$\leq C \int\limits_{a\tilde{R}} |\lambda|^{-k} e^{-\mathrm{Re}\varphi^{\alpha}(\lambda)t} |d\lambda|.$$

Using the mapping property of the function $\varphi$ based upon the condition P1, applying condition P3, we have

$$\mathrm{Re}\varphi^{\alpha}(\lambda) \geq |\varphi(\lambda)|^{\alpha} \cos \alpha\varpi > |\lambda| C_1^{\alpha} \cos \alpha\varpi.$$

Therefore

$$\int\limits_{a\tilde{R}} |\lambda|^{-k} e^{-\mathrm{Re}\varphi^{\alpha}(\lambda)t} |d\lambda| \leq \int\limits_{a\tilde{R}} |\lambda|^{-k} e^{-|\lambda| t C_1^{\alpha} \cos \alpha\varpi} |d\lambda| \leq \int\limits_{a\vartheta} |\lambda|^{-k} e^{-|\lambda| t C_1^{\alpha} \cos \alpha\varpi} |d\lambda|$$

$$= 2 \int\limits_{N}^{\infty} x^{-k} e^{-xtC_1^{\alpha} \cos \alpha\varpi} dx \leq 2N^{-k} \int\limits_{N}^{\infty} e^{-xtC_1^{\alpha} \cos \alpha\varpi} dx = \frac{2N^{-k} e^{-NtC_1^{\alpha} \cos \alpha\varpi}}{tC_1^{\alpha} \cos \alpha\varpi}.$$

Hence, the series in the right-side of (22) is uniformly convergent with respect to $R$ and in accordance with the well-know theorem of the calculus, we can pass to the limit in both sides of (22) while $R \to \infty$. Taking into account this fact, we get

$$\int\limits_{a\vartheta} \varphi_2(\lambda) e^{-\varphi^{\alpha}(\lambda)t} R_{aW}(\lambda) f d\lambda = \sum_{k=1}^{\infty} c_{-k} \int\limits_{a\vartheta} \lambda^{-k} e^{-\varphi^{\alpha}(\lambda)t} R_{aW}(\lambda) f d\lambda,$$

where $\varphi_2$ denotes the principal part of the Laurent series. Applying formula (21), we get

$$\frac{1}{2\pi i} \int\limits_{a\vartheta} \varphi_2(\lambda) e^{-\varphi^{\alpha}(\lambda)t} R_{aW}(\lambda) f d\lambda = \sum_{k=1}^{\infty} a^{-k} c_{-k} B^k \tilde{f}(t),\ f \in \mathfrak{H}.$$

Combining this formula with the previously obtained analogous one for the regular part (20), we achieve the desired result. □

### 3.2. Operator function with more subtle asymptotics

In order to create an efficient tool for study we produce the following preliminary reasonings, which, however may be of interest themselves. Consider condition (13), applying Lemma 1 [19] to the compact sectorial operator $\Theta(B) \subset \mathfrak{L}_0(\theta)$, we get

$$s_{2n-1}(B) \leq \sqrt{2} \sec\theta \cdot \lambda_n(\mathfrak{Re} B), \quad s_{2n}(B) \leq \sqrt{2} \sec\theta \cdot \lambda_n(\mathfrak{Re} B), \quad n = 1, 2, \dots.$$

Note, that for monotonically increasing sequences $\{a_n\}_1^{\infty}, \{b_n\}_1^{\infty},\ a_n \leq C \cdot b_n$, we have

$$\frac{\ln a_n}{a_n^q} \leq C \cdot \frac{\ln b_n}{b_n^q},\ q > 0. \tag{23}$$

Therefore

$$\frac{\ln s_{2n}^{-1}(B)}{s_{2n}^{-1/\kappa}(B)} \leq C \cdot \frac{\ln \lambda_n^{-1}(\mathfrak{Re} B)}{\lambda_n^{-1/\kappa}(\mathfrak{Re} B)},\ \kappa > 0.$$

In accordance with the results [12], conditions H1, H2 guarantee

$$\lambda_n^{-1}(\mathfrak{Re} W) \asymp \lambda_n(\mathfrak{Re} B).$$

Apparently, we have arrived at the implication

$$\left\{ (\ln^{1+\kappa} x)'_{\lambda_n(\mathfrak{Re} W)} = o(n^{-\kappa}) \right\} \Rightarrow \left\{ (\ln^{1+\kappa} x)'_{s_n^{-1}(B)} = o(n^{-\kappa}) \right\}. \tag{24}$$

If additionally, we assume that

$$\sum_{n=1}^{\infty}\lambda_n^{-2}(\mathfrak{Re}W)\|\mathfrak{Im}We_n\|^2_{\mathfrak{H}}<1, \tag{25}$$

where $\{e_n\}_1^\infty$ is the orthonormal sequence of the eigenvectors of the operator $\mathfrak{Re}W$, then using Lemma 2 [19], we get

$$\lambda_{2n}^{-1}(\mathfrak{Re}W)\le Cs_n(B),\, n\in\mathbb{N}. \tag{26}$$

Thus, applying the reasonings given above, we obtain

$$\left\{(\ln^{1+\kappa}x)'_{\lambda_n(\mathfrak{Re}W)}=o(n^{-\kappa})\right\}\Leftrightarrow\left\{(\ln^{1+\kappa}x)'_{s_n^{-1}(B)}=o(n^{-\kappa})\right\}.$$

Moreover, the following equivalence follows easily from the relations between singular numbers and eigenvalues of the real component, we have

$$\mathfrak{Re}B\in\mathfrak{S}^\star_\rho\Leftrightarrow B\in\mathfrak{S}^\star_\rho.$$

In this paragraph, we aim to construct an operator represented in the analytic form whose singular numbers has more subtle asymptotics than one of the power type.

The approach implemented below is based upon the ordinary properties of operators acting in the Hilbert space. We denote by $\lambda_n, e_n,\ n\in\mathbb{N}$ the eigenvalues and eigenvectors of the sectorial operator $W$ respectively, where the numbering is chosen so that each eigenvalue is counted as many times as its geometric multiplicity. Consider the invariant space $\mathfrak{N}$ generated by eigenvectors of the operator, we mentioned above that it is the infinite dimensional space endowed with the stricture of the initial Hilbert space, hence we can consider a restriction of the operator $R_W(\lambda)$ on the space $\mathfrak{N}$, where $\lambda$ does not take values of eigenvalues. Assume that conditions of Lemma 4 hold with respect to a function of the complex variable $\varphi$. Using simple reasonings involving properties of the resolvent, Cauchy integral formula, clock-wise direction, etc., we get

$$\varphi(W)e_n=\lim_{t\to 0}\frac{1}{2\pi i}\int\limits_{\vartheta}e^{-\varphi^\alpha(\lambda)t}\varphi(\lambda)R_W(\lambda)e_n d\lambda=e_n\lim_{t\to 0}\frac{1}{2\pi i}\int\limits_{\vartheta}e^{-\varphi^\alpha(\lambda)t}\frac{\varphi(\lambda)}{\lambda_n-\lambda}d\lambda$$
$$=e_n\lim_{t\to 0}e^{-\varphi^\alpha(\lambda_n)t}\varphi(\lambda_n)=e_n\varphi(\lambda_n).$$

This property can be taken as a concept since by virtue of such an approach and uniqueness of the decomposition on basis vectors in the Hilbert space, we can represent the operator function defined on elements of $\mathfrak{N}$ in the form of the series on the eigenvectors

$$\varphi(W)f=\sum_{n=1}^{\infty}e_n\varphi(\lambda_n)f_n,\, f\in\mathrm{D}_1(\varphi), \tag{27}$$

where

$$\mathrm{D}_1(\varphi):=\left\{f\in\mathfrak{N}:\ \sum_{n=1}^{\infty}|\varphi(\lambda_n)f_n|^2<\infty\right\}.$$

To prove this fact, we should note that the resolvent $R_W(\lambda)$ is defined on $\mathfrak{N}$ and admits the following decomposition

$$R_W(\lambda)f=\sum_{n=1}^{\infty}\frac{f_n}{\lambda_n-\lambda}e_n,\, f\in\mathfrak{N},$$

having substituted the latter relation to the formula of the operator function, we obtain (27), i.e.

$$\varphi(W)f=\lim_{t\to 0}\frac{1}{2\pi i}\int\limits_{\vartheta}e^{-\varphi^{\alpha}(\lambda)t}\varphi(\lambda)\sum_{n=1}^{\infty}e_n\frac{f_n}{\lambda_n-\lambda}d\lambda$$

$$=\lim_{t\to 0}\frac{1}{2\pi i}\sum_{n=1}^{\infty}e_nf_n\int\limits_{\vartheta}\frac{e^{-\varphi^{\alpha}(\lambda)t}\varphi(\lambda)}{\lambda_n-\lambda}d\lambda=\lim_{t\to 0}\sum_{n=1}^{\infty}e_nf_n\operatorname*{Res}_{z=\lambda_n}\{e^{-\varphi^{\alpha}(z)t}\varphi(z)(z-\lambda_n)^{-1}\}$$

$$=\lim_{t\to 0}\sum_{n=1}^{\infty}e_nf_ne^{-\varphi^{\alpha}(\lambda_n)t}\varphi(\lambda_n)=\sum_{n=1}^{\infty}e_nf_n\varphi(\lambda_n),\ f\in \mathrm{D}_1(\varphi).$$

Here, we ought to explain that we have managed to pass to the limit dealing with the contour integrals by virtue of the growth regularity of the function, a complete scheme of reasonings is represented in Lemma 4.

Suppose $\mathfrak{H}:=\mathfrak{N}$ and let us construct a space $\mathfrak{H}_+$ satisfying the condition of compact embedding $\mathfrak{H}\subset\subset\mathfrak{H}_+$ and suitable for spreading the condition H2 upon the operator $\varphi(W)$. For this purpose, define

$$\mathfrak{H}_+:=\left\{f\in\mathfrak{N}:\ \|f\|^2_{\mathfrak{H}_+}=\sum_{n=1}^{\infty}|\varphi(\lambda_n)||f_n|^2<\infty\right\},$$

and let us prove the fact $\mathfrak{H}\subset\subset\mathfrak{H}_+$. The idea of the proof is based on the application of the criterion of compactness in Banach spaces, let us involve an operator $Q:\mathfrak{H}\to\mathfrak{H}$ defined as follows

$$Qf=\sum_{n=1}^{\infty}|\varphi(\lambda_n)|^{-1/2}f_ne_n,$$

Observe that, having imposed a condition of the growth regularity upon the operator function, applying Lemma 1, we have $|\varphi(\lambda_n)|\uparrow\infty$. We have made the latter assumption without loss of generality since we can rearrange the sequence in the order of its absolute values increasing. Observe that if $\|f\|<C_1$, then

$$\|R_kQf\|=\sum_{n=k}^{\infty}|\varphi(\lambda_n)|^{-1}|f_n|^2\le\frac{\|f\|^2}{|\varphi(\lambda_k)|}<\frac{C_1^2}{|\varphi(\lambda_k)|},\ k\in\mathbb{N}.$$

Therefore, in accordance with the compactness criterion in Banach spaces the operator $Q$ is compact. Now, consider a set bounded in the sense of the norm $\mathfrak{H}_+$, we will denote its elements by $f$, thus in accordance with the above, we have

$$\sum_{n=1}^{\infty}|\varphi(\lambda_n)||f_n|^2<C.$$

It is clear that the element $g:=\{|\varphi(\lambda_n)|^{1/2}f_n\}_1^{\infty}$ belongs to $\mathfrak{H}$ and the set of elements from $\mathfrak{H}$ corresponding to the bounded set of elements from $\mathfrak{H}_+$ is bounded also. This is why the operator $Q$ image of the set of elements $g$ is compact, but we have $Qg=f$. The latter relation proves the fact that the set of elements $f$ bounded in the sense of the norm $\mathfrak{H}_+$ is a compact set in the sense of the norm $\mathfrak{H}$. Thus, we have established the fulfilment of condition H1, it is obvious that we can choose a span of $\{e_n\}_1^{\infty}$ as the mentioned linear manifold $\mathfrak{M}$.

The verification of the first relation of H2 is implemented due to the direct application of the Cauchy-Schwarz inequality, we have

$$|\,(\varphi(W)f,h)_{\mathfrak{H}}\,|=\left|\sum_{n=1}^{\infty}\varphi(\lambda_n)f_n\overline{h}_n\right|\le\left(\sum_{n=1}^{\infty}|\varphi(\lambda_n)||f_n|^2\right)^{1/2}\left(\sum_{n=1}^{\infty}|\varphi(\lambda_n)||h_n|^2\right)^{1/2}.$$

 

Let us verify the fulfilment of the second condition, here we need impose a sectorial condition upon the analytic function $\varphi$, i.e. it should preserve in the open right-half plain the closed sector belonging to the latter, then we have

$$\sum_{n=1}^{\infty}|\varphi(\lambda_n)||f_n|^2 \leq \sec\beta \cdot \mathrm{Re}\sum_{n=1}^{\infty}\varphi(\lambda_n)|f_n|^2,$$

where $\beta$ is the semi-angle of the sector contenting the image of the analytic function $\varphi$. Thus, the condition H2 is fulfilled. Observe that

$$\varphi^*(W)f = \sum_{n=1}^{\infty}\overline{\varphi(\lambda_n)}f_n e_n,\ f \in \mathrm{D}_1(\varphi),$$

it follows easily from the representation of the inner product in terms of the Fourier coefficients. Therefore, we get

$$\mathfrak{Re}\varphi(W)f = \sum_{n=1}^{\infty}\mathrm{Re}\varphi(\lambda_n)f_n e_n,\ f \in \mathrm{D}_1(\varphi).$$

It is clear that

$$\mathfrak{Re}\varphi(W)e_n = \mathrm{Re}\varphi(\lambda_n)e_n,\ n \in \mathbb{N}.$$

The fact that there does not exist additional eigenvalues of the operator $\mathfrak{Re}\varphi(W)$ follows from the fact that $\{e_n\}_1^{\infty}$ forms a basis in $\mathfrak{H}$ and the fact that eigenvectors corresponding to different eigenvalues of a self-adjoint operator are orthogonal. Thus, we obtain

$$\lambda_n\{\mathfrak{Re}\varphi(W)\} = \mathrm{Re}\varphi(\lambda_n),\ \lambda_n\{\mathfrak{Im}\varphi(W)\} = \mathrm{Im}\varphi(\lambda_n),\ n \in \mathbb{N}. \tag{28}$$

Recall that condition (13) plays the distinguished role in the refinement of the Lidskii results [15] since it guaranties the equality of the convergence exponent $\rho$ and the order $\alpha$ of summation in the Abel-Lidskii sense. At the same time we can choose a sequence of contours of the power type in the integral construction. It is rather reasonable to expect that we are highly motivated to produce a concrete example of the operator satisfying condition (13) since it would stress the significance and novelty of the papers [11, 12, 13, 14, 15, 16, 17, 18].

Having been inspired by the idea, by virtue of the comparison test for convergence, we can use the function considered in example (i) as an indicator to find the desired operator function. Consider the following condition: for sufficiently large numbers $n \in \mathbb{N}$, we have

$$C_1 < \frac{(n\ln n \cdot \ln\ln n)^{\kappa}}{\mathrm{Re}\varphi(\lambda_n)} < C_2,\ \kappa > 0. \tag{29}$$

Applying formulas (28), relation (23), using the comparison test for convergence, we conclude

$$(\ln^{1+\kappa}x)'_{\lambda_n(\mathfrak{Re}\varphi)} = o(n^{-\kappa}),\ \mathfrak{Re}\varphi(W)^{-1} \in \mathfrak{S}^{\star}_{1/\kappa},$$

therefore due to implication (24), we obtain

$$\left\{(\ln^{1+\kappa}x)'_{s_n^{-1}(\varphi^{-1})} = o(n^{-\kappa})\right\}. \tag{30}$$

Using formulas (28), we can rewrite condition (25) in the following form

$$\sum_{n=1}^{\infty}\left\{\frac{\mathrm{Im}\varphi(\lambda_n)}{\mathrm{Re}\varphi(\lambda_n)}\right\}^2<1, \tag{31}$$

from what follows $\varphi(W)^{-1}\in\mathfrak{S}^{\star}_{1/\kappa}$ due to relation (26) and conditions H1,H2.

Consider a function of the complex variable

$$\varphi(z)=(z^{\xi}\ln z\cdot\ln\ln z)^{\kappa},\ 0<\xi\leq 1,\ \kappa>0 \tag{32}$$

and define $\psi(z):=z^{\xi}\ln z\cdot\ln\ln z,\ 0<\xi\leq 1$ in the sector $|\arg z|\leq\theta$, where for the simplicity of reasonings the branch of the power function has been chosen so that it acts onto the sector and we have chosen the branch of the logarithmic function corresponding to the value $\phi:=\arg z$. Let us find the real and imaginary parts of the function $\psi(z)$, we have

$$\psi(z)=z^{\xi}\ln z\cdot\ln\ln z=|z|^{\xi}e^{i\xi\phi}\left(\ln|z|+i\phi\right)\left(a+i\arctan\frac{\phi}{\ln|z|}\right),$$

where, we denote $a:=\ln|\ln|z|+i\phi|$. Separating the real and imaginary parts of the function $\psi(z)$, we have

$$\psi(z)=|z|^{\xi}e^{i\xi\phi}\left\{a\ln|z|-\phi\arctan\frac{\phi}{\ln|z|}+i\left(a\phi+\ln|z|\arctan\frac{\phi}{\ln|z|}\right)\right\}$$

$$=|z|^{\xi}\cos\xi\phi\left(a\ln|z|-\phi\arctan\frac{\phi}{\ln|z|}\right)-|z|^{\xi}\sin\xi\phi\left(a\phi+\ln|z|\arctan\frac{\phi}{\ln|z|}\right)+$$

$$+i\left\{|z|^{\xi}\sin\xi\phi\left(a\ln|z|-\phi\arctan\frac{\phi}{\ln|z|}\right)+|z|^{\xi}\cos\xi\phi\left(a\phi+\ln|z|\arctan\frac{\phi}{\ln|z|}\right)\right\}.$$

It gives us

$$\frac{\mathrm{Im}\,\psi(z)}{\mathrm{Re}\,\psi(z)}=\frac{\tan\xi\phi\left(a\ln|z|-\phi\arctan\ln^{-1}|z|^{1/\phi}\right)+\left(a\phi+\ln|z|\arctan\ln^{-1}|z|^{1/\phi}\right)}{\left(a\ln|z|-\phi\arctan\ln^{-1}|z|^{1/\phi}\right)-\tan\xi\phi\left(a\phi+\ln|z|\arctan\ln^{-1}|z|^{1/\phi}\right)}.$$

Taking into account the fact

$$\frac{a\phi+\ln|z|\arctan\ln^{-1}|z|^{1/\phi}}{a\ln|z|-\phi\arctan\ln^{-1}|z|^{1/\phi}}\to 0,\ |z|\to\infty,$$

we get

$$\frac{\mathrm{Im}\,\psi(z)}{\mathrm{Re}\,\psi(z)}\to\tan\xi\arg z,\ |z|\to\infty,$$

from what follows $\arg\psi(z)\to\xi\arg z,\ |z|\to\infty$. Moreover, we have an opportunity to claim that for an arbitrary $\varepsilon>0$, there exists $R(\varepsilon)$ such that the following estimate holds

$$\frac{\mathrm{Im}\,\psi(z)}{\mathrm{Re}\,\psi(z)}<(1+\varepsilon)\tan\xi\theta,\ |z|>R(\varepsilon).$$

Apparently, we can claim that the function $\psi(z)$ nearly preserves the sector $|\arg z|\leq\theta$, what is completely sufficient for our reasonings since we are dealing with the neighborhood of the infinitely distant point. Let us calculate the absolute value, we have

$$|\psi(z)|^2=|z|^{2\xi}\left|(\ln|z|+i\phi)\right|^2\cdot\left|\left(a+i\arctan\frac{\phi}{\ln|z|}\right)\right|^2=|z|^{2\xi}(\ln^2|z|+\phi^2)(a^2+\arctan^2\ln^{-1}|z|^{1/\phi}),$$

 

the latter relation establishes the growth regularity of the function $\psi(z)$ and therefore this concerns the function $\varphi(z)$, i.e. condition P3 is satisfied if $\xi\kappa > 1/\alpha$. Consider the following formulas

$$\mathrm{Re}\,\varphi(z) = |\varphi(z)|\cos(\kappa \arg\psi),\;\; \mathrm{Im}\,\varphi(z) = |\varphi(z)|\sin(\kappa\arg\psi).$$

Having noticed that

$$\psi(|z|) = |z|^{\xi}\ln|z|\cdot\ln\ln|z|,$$

taking into account the proved above fact $\arg\psi(z)\to\xi\arg z,\; |z|\to\infty$, we get

$$\mathrm{Re}\,\varphi(z)\sim\varphi(|z|)\cos(\xi\kappa\arg z),\;\; \mathrm{Im}\,\varphi(z)\sim\varphi(|z|)\sin(\xi\kappa\arg z),\; |z|\to\infty. \tag{33}$$

Therefore, the values of the function $\varphi(z)$ belong to the sector $\mathfrak{L}_0(\kappa\,\xi\theta+\varepsilon)$ for sufficiently large absolute values of the argument, where $\varepsilon$ is an arbitrary small positive value. It is clear that to satisfy the mapping properties of the function formulated in condition P1, we must assume that $\kappa\,\xi\theta < \pi/2\alpha$. Consider an operator argument $W$ such that

$$\Theta(W)\in\mathfrak{L}_0(\theta),\;\; \theta/\alpha<\xi\kappa\theta<\pi/2\alpha,\; |\lambda_n(W)|\asymp n^{1/\xi}. \tag{34}$$

Eventually, the reasonings given above lead us to the conclusion that the conditions of Lemma 4 holds. The first formula (33) guarantees that relation (29) holds and we obtain the operator with more subtle asymptotics of the real component than the asymptotics of the power type generated by the function of the complex variable (32).

The implication (24) leads to the asymptotical formula (30). Let us prove the fact $\varphi(W)^{-1}\in\mathfrak{S}^{\star}_{1/\kappa}$. Since the fulfilment of conditions H1,H2 has been previously established we should notice that (31) can be obtained by virtue of relation (33) and due to the location of the eigenvalues of the operator $W$. For instance it may be a domain of the parabolic type that characterizes the spectrum of uniformly elliptic differential operators with a self-adjoint senior term (see [1], [34]). In order to show this, assume that

$$|\mathrm{Im}(Wf,f)_{\mathfrak{H}}|\le b\cdot\mathrm{Re}^{\gamma}(Wf,f)_{\mathfrak{H}},\; b>0,\; 0<\gamma<1.$$

Then, we get

$$\sum_{n=1}^{\infty}\left\{\frac{\mathrm{Im}\varphi(\lambda_n)}{\mathrm{Re}\varphi(\lambda_n)}\right\}^2\le b^2\sum_{n=1}^{\infty}\{\mathrm{Re}\varphi(\lambda_n)\}^{2(\gamma-1)}.$$

It is clear that the latter series is convergent if $\kappa(1-\gamma)>1/2$. The rest part of the relation (31) proof becomes clear since we can choose an arbitrary small value of the parameter $b$.

Here, we should make a short digression from the narrative and remind that we pursue a rather particular aim to produce an example of an operator so that we are free in some sense to choose an operator as an object for our theoretical needs. At the same time the given above reasonings origin from the fundamental scheme and as a result allow to construct a fundamental theory.

### 3.3. *Operator argument*

In this paragraph, we preserve notations of the previous paragraph expressing the asymptotics of the eigenvalues in terms

$$|\lambda_n(W)|\asymp n^{1/\xi},$$

where $W$ is an operator argument of the operator function defined by the function of the complex variable

$$\varphi(z) = (z^{\xi} \ln z \cdot \ln\ln z)^{\kappa},\ 0 < \xi \leq 1,\ \kappa > 0.$$

**1.** Consider a remarkably showing case $\xi = 1$ corresponding to the so-called quasi-trace operator class $\mathfrak{S}_1^{\star}$. Consider an operator

$$W := -a_2\Delta + a_0,$$

with a constant complex coefficients acting in $L_2(I)$, here $I \subset \mathbb{E}^2$ is a bounded domain with a sufficiently smooth boundary. The operator is understood as a closure of the given differential construction defined on the set $C_0^{\infty}(I)$. We assume that the coefficients are chosen so that the condition (34) holds, i.e. $\Theta(W) \subset \mathfrak{L}_0(\theta),\ \theta/\alpha < \kappa\,\theta < \pi/2\alpha$. It is clear that the operator is normal. Let us remind well-known properties of a normal operator: a singular number of the normal operator coincides with its eigenvalue absolute value, $We = \lambda e,\ We = \overline{\lambda}\, e$ where $e, \lambda$ is an arbitrary eigenvector and the corresponding eigenvalue of the operator, the system of the eigenvectors is complete in $\overline{\mathrm{R}(W)}$. Note that by virtue of the well-known statement of the operator theory, we have an orthogonal decomposition of the Hilbert space

$$L_2(I) = \mathrm{N}(W^*) \oplus \overline{\mathrm{R}(W)}.$$

It is not hard to prove that $\mathrm{N}(L^*) = 0$, since we have

$$(W^* f, f)_{L_2(I)} = a_2 \|f\|^2_{H_0^1(I)} + a_0 \|f\|^2_{L_2(I)},\ f \in \mathrm{D}(W^*).$$

Therefore $\overline{\mathrm{R}(W)} = L_2(I)$. Using this fact, we can claim that $\mathfrak{Re}W$ and $W$ have the same eigenvectors. Indeed, it is clear due to the normal property of the operator the system of eigenvectors of $W$ is a subsystem of eigenvectors of $\mathfrak{Re}W$. The coincidence can be established easily if we recall that the eigenvectors corresponding to different eigenvalues of the self-adjoint operator are orthogonal. Thus, if we assume that there exists one more eigenvector of the operator $\mathfrak{Re}W$ then we obtain the fact that it is a zero element since it is orthogonal to the complete system but it is impossible. It is well-known fact (see [39]) that under the conditions imposed upon $I$, we have an asymptotics

$$\lambda_n(\mathfrak{Re}W) \asymp n^{1/\xi},\ \xi = n/m,$$

where $m$ is the highest derivative and $n$ is a dimension of the Euclidian space. Thus, we consider the case $n = m$. The sectorial property allows us to claim that

$$\lambda_n(\mathfrak{Re}W) \asymp n^{1/\xi},\ \Rightarrow |\lambda_n(W)| \asymp n^{1/\xi}.$$

Hence we can implement the above scheme of reasonings and consider the operator function $\varphi(W)$ where $\varphi(z) = \{z \ln z \cdot \ln\ln z\}^{\kappa}$ for which condition (30) holds and at the same time

$$\varphi(W)^{-1} \in \mathfrak{S}^{\star}_{1/\kappa}.$$

It is remarkable that despite of the fact that the hypotheses H1, H2 hold for the operator $W$ in the natural way, the verification is left to the reader, we can use the benefits of the scheme introduced above to construct the required pair of Hilbert spaces.

In addition, we want to represent a concrete domain $I := \{x_j \in [0, \pi],\ j = 1, 2\}$ that gives us an opportunity to construct a concrete complete system of the eigenvectors of the operator $W$. Consider the following functions

$$e_{\bar{l}} = \sin l_1 x_1 \cdot \sin l_2 x_2,\ \bar{l} := \{l_1, l_2\},\ l_1, l_2 \in \mathbb{N}.$$

It is clear that

$$We_{\bar{l}} = \lambda_{\bar{l}}\, e_{\bar{l}},\ \lambda_{\bar{l}} = a_1(l_1^2 + l_2^2) + a_0.$$

Let us show that the system $\{e_{\bar{l}}\}$ is complete in the Hilbert space $L_2(I)$, we will show it if we prove that the element that is orthogonal to every element of the system is a zero. Assume that

$$\int\limits_0^\pi \sin l_1 x_1 dx_1 \int\limits_0^\pi \sin l_2 x_2 f(x_1, x_2) dx_2 = (e_{\bar{l}}, f)_{L_2(I)} = 0.$$

In accordance with the fact that the system $\{\sin lx\}_1^\infty$ is a complete system in $L_2(0, \pi)$, we have

$$\int\limits_0^\pi \sin l_2 x_2 f(x_1, x_2) dx_2 = 0.$$

Having implemented the same reasonings, we obtain the desired result (well-known fact). Let us show that the coefficients of the operator can be chosen so that relation (31) holds. Assume that $\mathrm{Im} a_1 = 0,\ \mathrm{Re} a_0 = 0$, then

$$\frac{\mathrm{Im}\lambda_{\bar{l}}}{\mathrm{Re}\lambda_{\bar{l}}} = \frac{a_0}{a_1(l_1^2 + l_2^2)};\ \ \arg\lambda_{\bar{l}} = \arctan\left\{\frac{a_0}{a_1(l_1^2 + l_2^2)}\right\}.$$

Applying relation (33), we get

$$\frac{\mathrm{Im}\varphi(\lambda_{\bar{l}})}{\mathrm{Re}\varphi(\lambda_{\bar{l}})} \sim \tan\kappa \arctan\left\{\frac{a_0}{a_1(l_1^2 + l_2^2)}\right\} \sim \frac{\kappa\, a_0}{a_1(l_1^2 + l_2^2)}.$$

However, the following series is convergent and it is clear that the coefficients can be chosen so that for an arbitrary $\varepsilon > 0$, we have

$$\left(\frac{\kappa\, a_0}{a_1}\right)^2 \sum_{l_1, l_2 = 1}^{\infty} \frac{1}{(l_1^2 + l_2^2)^2} < \varepsilon.$$

The latter relation gives us the desired result.

**2.** The next case, within the scale of most important and at the same time simple ones, appeals to the so-called quasi Hilbert-Schmidt class $\mathfrak{S}_2^\star$. In this regard, consider the Sturm-Liouville operator

$$Wu := -a_1 u'' + a_0,\ u(0) = u(\pi) = 0,\ a_1, a_0 \in \mathbb{C},$$

acting in $L_2(I)$, where the corresponding Euclidian space is one-dimensional $I := (0, \pi)$. In accordance, with the initial conditions, we obtain the eigenvalues and the eigenvectors respectively $\lambda_n(W) = a_1 n^2 + a_0,\ e_n(x) = \sin nx,\ n \in \mathbb{N}$. Note that the operator is normal and the closure of the linear span of the functions $\sin nx,\ n \in \mathbb{N}$ gives us the Hilbert space $L_2(I)$. In this case, we have $\xi = 1/2$ since

$$|\lambda_n(W)| \asymp n^2.$$

Therefore, we can implement the scheme of reasonings given in the previous paragraph having considered the operator function $\varphi(W)$, where $\varphi(z) = z^{\kappa/2}\{\ln z \cdot \ln\ln z\}^\kappa$. Conditions (31), (34) can be satisfied due to the regulation of the coefficients $a_1, a_0$, absolutely analogously to the previously considered two dimensional case.

Below, we represent a most general operator class satisfying condition P1 generated by strictly continuous semigroups of contractions and therefore considered as a class of operator arguments.

 

### 3.4. *Operators generated by semigroups*

In this paragraph we consider well-known operators generated by the strictly continuous semigroups of contractions from the point of view of the constructed theory. In particular, we can refer to the propositions established in [14] in accordance with which the operators satisfy conditions H1,H2 and therefore can be considered as the operator arguments. Consider the infinitesimal generator $A$ of a strictly continuous semigroup of contractions, we can form an infinitesimal generator transform

$$Z^{\alpha}_{G,F}(A) := A^{*}GA + FA^{\alpha},\ \alpha \in [0,1),$$

where the symbols $G, F$ denote operators acting in $\mathfrak{H}$. Applicably to the infinitesimal generator, taking into account Corollary 3.6 [37, p.11], we can reformulate Theorem 5 [14] as follows

**Theorem 2.** *Assume that $A^{-1}$ is compact, $F \in \mathcal{B}(\mathfrak{H})$, $G$ is bounded, strictly accretive, with a lower bound $\gamma_G > C_\alpha \|A^{-1}\| \cdot \|F\|$, where $C_\alpha = 2\alpha^{-1}\|A^{-1}\| + (1-\alpha)^{-1},\ \mathrm{D}(G) \supset \mathrm{R}(A)$. Then $Z^{\alpha}_{G,F}(A)$ satisfies conditions H1 - H2.*

It is remarkable that Theorem 5 [14] gives us a tool to describe spectral properties of the operator $Z^{\alpha}_{G,F}(A)$. In particular, we can establish the order of the operator and the fact of its belonging to the Schatten-von Neumann class of the convergence exponent by virtue of the Theorem 3 [14]. Having known the index of the Schatten-von Neumann class of the convergence exponent, we can apply results [18], [16], [15] in order to verify fulfilment of condition S1. Now, consider concrete operators generated by infinitesimal generators of strictly continuous semigroups of contractions.

**The linear combination of the differential operator and the Riesz potential**

Consider a space $L_2(I),\ I := (-\infty, \infty)$ and the Riesz potential

$$I^{\beta} f(x) = B_\beta \int\limits_{-\infty}^{\infty} f(s)|s-x|^{\beta-1} ds,\ B_\beta = \frac{1}{2\Gamma(\beta)\cos(\beta\pi/2)},\ \beta \in (0,1),$$

where $f$ is in $L_p(\Omega),\ 1 \leq p < 1/\beta$. It is obvious that $I^{\beta} f = B_\beta \Gamma(\beta)(I^{\beta}_{+} f + I^{\beta}_{-} f)$, where

$$I^{\beta}_{\pm} f(x) = \frac{1}{\Gamma(\beta)} \int\limits_{0}^{\infty} f(s \mp x) s^{\beta-1} ds,$$

these operators are known as fractional integrals on the whole real axis (see [42, p.94]). Assume that the following condition holds $\sigma/2 + 3/4 < \beta < 1$, where $\sigma$ is a non-negative constant. Following the idea of the monograph [42, p.176], consider a sum of a differential operator and a composition of fractional integro-differential operators

$$W := D^2 a D^2 + I^{\sigma}_{+}\, \xi\, I^{2(1-\beta)} D^2 + \delta I,\ \mathrm{D}(W) = C^{\infty}_{0}(I),$$

where $\xi(x) \in L_\infty(I),\ a(x) \in L_\infty(I) \cap C^2(I),\ \mathrm{Re}\, a(x) > \gamma_a (1+|x|)^5$. It is proved in [14] that the product of the Riesz potential and the differential operator of the second order $I^{2(1-\beta)} D^2$ is a fractional power of the infinitesimal generator $-D^2/2$ of the strictly continuous semigroup of contractions

$$T_t f(x) = (2\pi t)^{-1/2} \int\limits_{-\infty}^{\infty} e^{-(x-\tau)^2/2t} f(\tau) d\tau,\ t > 0,\ T_t f(x) = f(x),\ t = 0,\ f \in L_2(\Omega).$$

Note that under the Theorem 2 assumptions related to the coefficients the operator $W$ satisfies conditions H1,H2 and therefore it is a sectorial operator with discrete spectrum, thus it belongs to the class of operator arguments.

**The perturbation of the difference operator**

Consider a space $L_2(I),\ I:=(-\infty,\infty)$, define a family of operators

$$T_t f(x):=e^{-ct}\sum_{k=0}^{\infty}\frac{(c\,t)^k}{k!}f(x-d\mu),\ f\in L_2(I),\ c,d>0,\ t\geq 0,$$

where convergence is understood in the sense of $L_2(I)$ norm. In accordance with Lemma 6 [14], we know that $T_t$ is a strictly continuous semigroup of contractions, the corresponding infinitesimal generator and its adjoint operator are defined by the following expressions

$$Af(x)=c[f(x)-f(x-d)],\ A^*f(x)=c[f(x)-f(x+d)],\ f\in L_2(I).$$

Let us find a representation for fractional powers of the operator $A$. Using formula (45) [14], we get

$$A^{\beta}f=\sum_{k=0}^{\infty}M_k f(x-kd),\ f\in L_2(I),\ M_k=-\frac{\beta\Gamma(k-\beta)}{k!\Gamma(1-\beta)}c^{\beta},\ \beta\in(0,1).$$

Consider the operator

$$W:=A^*aA+bA^{\beta}+Q^*NQ,$$

where $a,b\in L_\infty(I),\ Q$ is a closed operator acting in $L_2(I),\ Q^{-1}\in\mathfrak{S}_\infty(L_2)$, the operator $N$ is strictly accretive, bounded, $\mathrm{R}(Q)\subset\mathrm{D}(N)$. Note that in accordance with Theorem 14 [14], we conclude that the operator $W$ satisfies conditions H1,H2, where we put $\mathfrak{M}:=\mathrm{D}_0(Q)$, if

$$\gamma_N>\left\{4\lambda\|a\|_{L_\infty}+\|b\|_{L_\infty}\frac{\beta\lambda^{\beta}}{\Gamma(1-\beta)}\sum_{k=0}^{\infty}\frac{\Gamma(k-\beta)}{k!}\right\}\|Q^{-1}\|^2.$$

Apparently, we have conditions under which being imposed the operator $W$ belongs to the class of operator arguments.

**General approach to integro-differential equations of the fractional order**

Eventually, we can point out a method in accordance with which the theory of integro-differential equations of the fractional order admits a harmonious theoretical refinment and completion. Consider a case when the operator argument is represented by a fractional power of the infinitesimal generator of a semigroup of contractions, i.e.

$$W=A^{\beta},\ \beta>0.$$

It is clear that in order to integrate harmoniously this paragraph to the previously developed theory, we ought to reformulate condition P1 in terms of the operator $A$. The main obstacle that may appear in this way is preservation of the sectorial property under the exponentiation operation. It is remarkable that a suitable location of the numerical range of values can be obtained directly for many well-known fractional powers of the infinitesimal generators such as the Kipriyanov operator and the Riesz potential [11].

 

Consider a prototypical case when the Laurent series has a finite quantity of numbers, substituting the operator argument, we get

$$\varphi(W)=\sum_{k=-n}^{n}c_kA^{\beta k}. \tag{35}$$

Thus, we can formally consider the Cauchy problem (10) applicably to the operator function and the operator argument rewritten in the form

$$\mathfrak{D}_{-}^{1/\alpha}f(t)=\sum_{k=-n}^{n}c_kA^{\beta k}f(t),\; f(t)\to f\in\mathfrak{H},\; t\to 0.$$

The latter equation is an abstract form of the integro-differential equation of the fractional order. In this regard, the obtained results can be treated as a harmonious completion of the well-known theory at the same time admitting application in the reduced form corresponding to the operator function (35). In order to highlight convexly the relevance of the abstract approach, we should point out that most of the well-known differential operators of the fractional order admit a representation through the infinitesimal generator of a semigroup of contraction [14]. The corresponding examples are given in the previous paragraph. In addition, we should note that we can consider more complicated integro-differential constructions generated by operator function (35) and operator arguments considered in the previous paragraph.

## 4. Conclusions

The paper represents an overview of the abstract theoretical results devoted to the evolution equations of the fractional order. The developed Lidskii V.B. approach allows to solve a wide class of Cauchy problems by virtue of the operator decomposition on the root vectors series. The involved operator function allows to extend techniques and methods to more complicated operators that may be treated as generalizations of the ones representing integro-differential equations of the fractional order. The corresponding scheme providing the application is plain and in its simplest form appeals to linear combinations of the well-known operators. The semigroup approach allows to construct a harmonious abstract theory within the functional analysis applicable to differential equations of the fractional order.

In conclusion, we should add that we can consider arbitrary non-self-adjoint fractional differential and pseudo-differential operators assuming that the functional space is defined on the bounded domain of the Euclidian space with a sufficiently smooth boundary (regular operators). In most of such cases the minimax principle can be applied and we can obtain the asyptotics of the eigenvalues of the real component, here we can refer a detailed description represented in the monograph by Rozenblyum G.V. [39]. The case corresponding to an unbounded domain (irregular operator) is also possible for study, in this regard the Fefferman concept covers such problems [39, p.47]. The given above theoretical results can be applied to the operator class and we can construct in each case a corresponding operator function representing to the reader an operator with more subtle asymptotics than one of the power type.

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