Advances in Pure and Applied Mathematics
2025, vol. 16, n° 3, 1-19 pages, DOI: 10.21494/ISTE.OP.2025.1314 ISTE OpenScience

$A(n,m)$-iso-contra-expansive operators and Some applications on Bergman spaces

Les opérateurs $A(n,m)$)-iso-contra-expansifs et quelques applications sur les espaces de Bergman

Gherairi Khadija[1], Sedki Amira[1] and Snoun Safa[1,2]

[1]Laboratory of Mathematics and Applications LR17ES11, Faculty of Sciences of Gabès. University of Gabès, Tunisia
[2]University of Jendouba, National Institute of Technologies and Sciences of Kef, 7100, Le kef, Tunisia
khadija.gherairi@gmail.com sedkiamira@fsg.rnu.tn snoun.safa@fsg.rnu.tn

ABSTRACT. In this paper, we study a new class of operators, so called $A(n,m)$-iso-contra-expansive operators. These new families of operators are considered as a generalization that combines the m-expansive operators as well as m-contractive operators and the classes of (A,m)-expansive and (A,m)-contractive operators and we recover the notion of n-quasi-(A,m)-isometric operators. Some spectral properties of these kind of operators are provided, we derive also a conditions to have the single-valued extension property (SVEP) and we finish by an application of Toeplitz operators on Bergman spaces.
2010 Mathematics Subject Classification. 30H20, 47A65,47B35, 47B37.
KEYWORDS. m-expansive, m-contractive, Toeplitz operators, Bergman spaces.

1. Preliminaries and Introduction

During many years, the theory of m-isometric operators on Hilbert space has witnessed a remarkable development. There was a period of strong research on those concepts at the close of the twentieth century, and there are currently many papers on the subject. By gathering the old and the new findings found in the literature related to this topic, we focus in addition to the several scientific papers, on two repositories in the domain: the famous paper of Agler and Stankus in 1995 [1] and the paper of Saddi and Sid Ahmed in 2012 [15]. In spite of the relatively important time lapse between the two publications, the second one, develops over more the findings of the first reference . It tackles the issues of the previous publications on semi-Hilbertian spaces.

In the works of Agler and Stankus [1, 2, 3, 4], authors considered that T is an m-isometry on a Hilbert space if

$$\beta_m(T) := \sum_{k=0}^{m}(-1)^k\binom{m}{k}T^{*k}T^k = 0.$$

In the former setting, the m-isometries operators had been generalized from various points of view. In particular, the class of the m-expansive and m-contractive operators have gained a great deal of attention in the works of G. Exen [7], G. Exen, I.Bong Jung, C. Li [8], G. Exen, I.Bong Jung, S. Soo Park [9], O A M Sid Ahmed [16, 17, 18] and G.Caixing [5]. However, only a few years ago, the class of m-isometries has been generalized to a larger sets the so-called n-quasi-m-isometric operators and for more details on these classes of operators, you consult the following references [14, 18] and the references therein.

The investigation of m-isometry operators on semi-Hilbertian spaces has been initiated by Saddi and Sid Ahmed [15]. In this beautiful piece of work, T is called an (A, m)-isometry if

$$\beta_m(A,T) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} T^{*k} A T^k = 0,$$

where A is a positive operator. Afterwords, such numerous results in the semi-Hilbertian space setting [20, 13, 6] provide insights into the study of some kinds of generalization of the later class of operators such as the (A, m)-expansive and (A, m)-contactive operators and the n-quasi-(A, m)-isometric operators.

Throughout this paper $\mathcal{H}$ stands for a separable complex Hilbert space with the norm $\|.\|$ induced by the inner product $\langle ., . \rangle$ and $\mathcal{L}(\mathcal{H})$ is the algebra of all bounded linear operators on $\mathcal{H}$. For every $T \in \mathcal{L}(\mathcal{H})$, its null space and its range are denoted by $\mathcal{N}(T)$ and $\mathcal{R}(T)$, respectively.

Now, for $T, A \in \mathcal{L}(\mathcal{H})$ such that $\overline{\mathcal{R}(T)}$ is invariant subspace for A, we set

$$\Theta_{m,n}(A,T,x) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \|AT^{n+k}x\|^2, \quad \forall\, x \in \mathcal{H}. \tag{1.1}$$

The outcomes of this work consist of the study of a new class of operators which we called it the $A(n, m)$-iso-contra-expansive operators class. This label covers two new completely different concepts: the n-quasi-(A, m)-contractive operators which means that the operators satisfy $\Theta_{m,n}(A, T, x) \geq 0$ or the n-quasi-(A, m)-expansive operators named the operators verifying $\Theta_{m,n}(A, T, x) \leq 0$. For more clarification, T is called $A(n, m)$-iso-contra-expansive operator if it belongs to only one of the two classes defined above, but if it is in the two classes simultaneously, then it will belongs to the class of n-quasi-(A, m)-isometric operators [13]. We mention here that throughout the paper when we use this label, we mean that all the properties are true for both concepts (and they are already verified for n-quasi-(A, m)-isometries class) and we will separate the two definitions in some properties when we need it and we will use the following trick: For $T, S \in \mathcal{L}(\mathcal{H})$ and for all n, m the statement

" T is $A(n, m)$-iso-contra-expansive $\Leftrightarrow$ S is $A(n, m)$-iso-contra-expansive"

is equivalent to say

" T is n-quasi-(A, m)-expansive (respectively n-quasi-(A, m)-contractive)
$\Leftrightarrow$ S is n-quasi-(A, m)-expansive (respectively n-quasi-(A, m)-contractive)"

It is very important to point out that for a positive operator A and $n = 0$, we recover the definition of (A, m)-expansive ((A, m)-contractive) operators [20].

At this stage, it might be necessary to notify that the $A(0, m)$-iso-contra-expansive operator is an $A(n, m)$-iso-contra-expansive operator. But the converse is not true, as the following example shows:

Example 1.1. *Let* $A = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$ *and* $T = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$
Then, T is quasi-$(A, 2)$-contractive. But, it is not $(A, 2)$-contractive.

It is very interesting to mention here that there is another famous class of operators, so-called m-symmetric operator class, which has seen develpment and interest by many researchers. This class was generalized to the k-quasi-m-symmetric operator by Salah Mechri and Zuo in [21]. The point of mentioning this is whether we can extend our new class to the k-quasi-m-symmetric operator class.

The paper is organized as follows. We introduce the concept of $A(n,m)$-iso-contra-expansive operators in Section 2. For the readers convenience, the elementary and the basic properties of this class are collected in this section. Section 3 is devoted to the concrete description of the spectral properties of $A(n,m)$-iso-contra-expansive operators. Several S.V.E.P properties are given in this section by exploiting the matrix representation. Toeplitz operators prove to be a very interesting class of operators on Bergman space that one can be investigate. So, as a part of our investigation, it might be a good research direction to characterize this operator to be $A(n,m)$-iso-contra-expansive for a special conditions which is the aim of the last section.

2. Basic properties of $A(n,m)$-iso-contra-expansive operators

In this section, we study $A(n,m)$-iso-contra-expansive operators by highlighting their basic properties. We begin by the next proposition which is an immediate consequence for the definitions so we omit the proof.

Proposition 2.1. *Let $T, A \in \mathcal{L}(\mathcal{H})$. Then T is*

(i) n-quasi-(A,m)-expansive if and only if $\Theta_{m,n}(A,T,x) \leq 0$ for all $x \in \mathcal{H}$.

(ii) n-quasi-(A,m)-contractive if and only if $\Theta_{m,n}(A,T,x) \geq 0$ for all $x \in \mathcal{H}$.

(iii) n-quasi-(A,m)-isometry if and only if $\Theta_{m,n}(A,T,x) = 0$ for all $x \in \mathcal{H}$.

It should be noted that

$$\Theta_{m,n}(A,T,x) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \|AT^k(T^n x)\|^2 = 0, \quad \forall\, x \in \mathcal{H}$$

Therefore, T is n-quasi-(A,m)-isometry on $\mathcal{H}$ if and only if T is (A,m)-isometry on $\overline{\mathcal{R}(T^n)}$. Similarly for n-quasi-(A,m)-expansive and n-quasi-(A,m)-contractive operators.

Remark 2.1. *It's clear to see that it is an equivalence between the two concepts in the previous proposition when we suppose that $\mathcal{R}(T^n)$ is dense.*

The following theorem shows that the notion of $A(n,m)$-iso-contra-expansive has an ascending nature with respect to the parameter n. While the next proposition gives a condition for this notion to be descending with respect to n.

Theorem 2.2. *Let T be a $A(n,m)$-iso-contra-expansive operator, then T is $A(p,m)$-iso-contra-expansive operator, for all $p \geq n$.*

Proof. The proof is a direct application of the equivalence in Proposition 2.1 keeping in mind that $\overline{\mathcal{R}(T^p)} \subset \overline{\mathcal{R}(T^n)}$ for every $p \geq n$. $\square$

Proposition 2.3. *Let T be a $A(n,m)$-iso-contra-expansive operator. If there exists $1 \leq p \leq n-1$ such that $\overline{\mathcal{R}(T^p)} = \overline{\mathcal{R}(T^{p+1})}$, then T is $A(p,m)$-iso-contra-expansive operator.*

Proof. Suppose that there exists $p \in \{1, \cdots, n-1\}$ such that $\overline{\mathcal{R}(T^p)} = \overline{\mathcal{R}(T^{p+1})}$. Then, we can check that $\overline{\mathcal{R}(T^p)} = \overline{\mathcal{R}(T^n)}$. On the other hand, since T is $A(n,m)$-iso-contra-expansive on $\mathcal{H}$ and by using Proposition 2.1, T will be $A(0,m)$-iso-contra-expansive operator on $\overline{\mathcal{R}(T^n)} = \overline{\mathcal{R}(T^p)}$.

Hence, the desired result attains by applying again Proposition 2.1. □

In the following, we deal with some elementary properties.

Proposition 2.4. *Let $T, S \in \mathcal{L}(\mathcal{H})$ be commuting operators such that $\mathcal{R}(S) \subset \mathcal{N}(A)$. Then, the following assertions are equivalent:*

1. *T is $A(n,m)$-iso-contra-expansive.*
2. *$T+S$ is $A(n,m)$-iso-contra-expansive.*

Proof. Let $x \in \mathcal{H}$. Since $ST = TS$, we have

$$\begin{aligned}
\Theta_{m,n}(A, T+S, x) &= \sum_{k=0}^{m}(-1)^k \binom{m}{k} \left\|A(T+S)^{k+n}x\right\|^2 \\
&= \|A(T+S)^n x\|^2 + \sum_{k=1}^{m}(-1)^k \binom{m}{k} \left\|A(T+S)^{k+n}x\right\|^2 \\
&= \left\|A\sum_{i=0}^{n}\binom{n}{i} T^i S^{n-i}x\right\|^2 + \sum_{k=1}^{m}(-1)^k \binom{m}{k} \left\|A\sum_{i=0}^{n+k}\binom{n+k}{i} T^i S^{n+k-i}x\right\|^2 \\
&= \left\|\sum_{i=0}^{n}\binom{n}{i} AS^{n-i}T^i x\right\|^2 \\
&\quad + \sum_{k=1}^{m}(-1)^k \binom{m}{k} \left\|AT^{n+k}x + \sum_{i=0}^{n+k-1}\binom{n+k}{i} AS^{n+k-i}T^i x\right\|^2
\end{aligned}$$

By the fact that $\mathcal{R}(S) \subset \mathcal{N}(A)$, we obtain $AS^{n-i}Tx = 0$ and $AS^{n+k-i}T^i x = 0$ for all $i \in \{0, \cdots, n+k-1\}$. Then

$$\begin{aligned}
\Theta_{m,n}(A, T+S, x) &= \sum_{k=0}^{m}(-1)^k \binom{m}{k} \|AT^{n+k}x\|^2 \\
&= \Theta_{m,n}(A, T, x).
\end{aligned}$$

Hence, this completes the proof. □

Proposition 2.5. *Let $T \in \mathcal{L}(\mathcal{H})$ and $\lambda \in \mathbb{C}$, then the following statement hold*

1. *T is $A(n,m)$-iso-contra-expansive if and only if λT is $A(n,m)$-iso-contra-expansive for $|\lambda| = 1$.*
2. *If T is $A(n,2)$-iso-contra-expansive and λT^2 is $A(n,1)$-iso-contra-expansive, then λT is $A(n,2)$-iso-contra-expansive for $|\lambda| < 1$.*

3. *If T is n-quasi-$(A,2)$-expansive (resp n-quasi-$(A,2)$-contractive) and λT^2 is n-quasi-$(A,1)$-contractive (resp n-quasi-$(A,1)$-expansive), then λT is n-quasi-$(A,2)$-expansive (resp n-quasi-$(A,2)$-contractive) for $|\lambda|>1$.*

Proof. The first statement is obvious since $\Theta_{m,n}(A,\lambda T,x)=\Theta_{m,n}(A,T,x)$ for $|\lambda|=1$. For the second and the last one, we suppose that T is n-quasi-$(A,2)$-expansive. Then,

$$\|AT^nx\|^2-2\|AT^{n+1}x\|^2+\|AT^{n+2}x\|^2\leq 0$$

implying

$$-2\|AT^{n+1}x\|^2\leq -\|AT^nx\|^2-\|AT^{n+2}x\|^2.$$

Multiplying the inequality by $|\lambda|^{2(n+1)}$, we obtain

$$-2|\lambda|^{2(n+1)}\|AT^{n+1}x\|^2\leq -|\lambda|^{2(n+1)}\|AT^nx\|^2-|\lambda|^{2(n+1)}\|AT^{n+2}x\|^2.$$

On the other hand, we have

$$\begin{aligned}
\Theta_{2,n}(A,\lambda T,x) &= |\lambda|^{2n}\|AT^nx\|^2-2|\lambda|^{2(n+1)}\|AT^{n+1}x\|^2+|\lambda|^{2(n+2)}\|AT^{n+2}x\|^2\\
&\leq |\lambda|^{2n}\|AT^nx\|^2-|\lambda|^{2(n+1)}\|AT^nx\|^2-|\lambda|^{2(n+1)}\|AT^{n+2}x\|^2\\
&\quad+|\lambda|^{2(n+2)}\|AT^{n+2}x\|^2\\
&\leq |\lambda|^{2n}(|\lambda|^2-1)(|\lambda|^2\|AT^{n+2}x\|^2-\|AT^nx\|^2)\\
&\leq |\lambda|^{2n}(|\lambda|^2-1)(|\lambda|^2\|AT^2T^nx\|^2-\|AT^nx\|^2)\\
&\leq |\lambda|^{2n}(|\lambda|^2-1)(|\lambda|^2\|AT^2y\|^2-\|Ay\|^2), \qquad \forall y\in\overline{\mathcal{R}(T^n)}\supset\overline{\mathcal{R}(T^{2n})}\\
&\leq -|\lambda|^{2n}(|\lambda|^2-1)\Theta_1(A,\lambda T^2,y)
\end{aligned}$$

Using the fact that an $(A,1)$-contractive operator on $\overline{\mathcal{R}(T^n)}$ is n-quasi-$(A,1)$-contractive operator on $\mathcal{H}$ (View Proposition 2.1) and a some discussion on $|\lambda|$ ($|\lambda|<1$ and $|\lambda|>1$), we obtain the desired results.

□

Lemma 2.6. *Let $T\in\mathcal{L}(\mathcal{H})$ and $n,m\in\mathbb{N}$. Then, for all $x\in\mathcal{H}$, we have*

1. $\Theta_{m,n}(A,T,x)=\Theta_{m-1,n}(A,T,x)-\Theta_{m-1,n+1}(A,T,x)$.
2. $(-1)^m\Theta_{m,n}(A,T,x)=\|AT^{m+n}x\|^2-\displaystyle\sum_{k=0}^{m-1}\binom{m}{k}(-1)^k\Theta_{k,n}(A,T,x)$
3. $\displaystyle\sum_{k=0}^{m-1}(-1)^k\binom{p}{k}\Theta_{k,n+1}(A,T,x)=\sum_{k=0}^{m-1}(-1)^k\binom{p+1}{k}\Theta_{k,n}(A,T,x)+(-1)^m\binom{p}{m-1}\Theta_{m,n}(A,T,x)$, *for all $p\geq m-1$*

Proof.

1. Since $\binom{m}{k}=\binom{m-1}{k}+\binom{m-1}{k-1}$, we have

$$\Theta_{m,n}(A,T,x) = \sum_{k=0}^{m}(-1)^k\binom{m}{k}\|AT^{n+k}x\|^2$$

$$\begin{aligned}
&= \|AT^nx\|^2 + \sum_{k=1}^{m-1}(-1)^k\binom{m}{k}\|AT^{n+k}x\|^2 + (-1)^m\|AT^{n+m}x\|^2. \\
&= \|AT^nx\|^2 + \sum_{k=1}^{m-1}(-1)^k\binom{m-1}{k}\|AT^{n+k}x\|^2 \\
&\quad + \sum_{k=1}^{m-1}(-1)^k\binom{m-1}{k-1}\|AT^{n+k}x\|^2 + (-1)^m\|AT^{n+m}x\|^2 \\
&= \sum_{k=0}^{m-1}(-1)^k\binom{m-1}{k}\|AT^{n+k}x\|^2 + \sum_{k=1}^{m}(-1)^k\binom{m-1}{k-1}\|AT^{n+k}x\|^2 \\
&= \sum_{k=0}^{m-1}(-1)^k\binom{m-1}{k}\|AT^{n+k}x\|^2 - \sum_{k=0}^{m-1}(-1)^k\binom{m-1}{k}\|AT^{n+1+k}x\|^2 \\
&= \Theta_{m-1,n}(A,T,x) - \Theta_{m-1,n+1}(A,T,x).
\end{aligned}$$

2. To prove this assertion, we use inductive reasoning on m. So, for $m = 1$ it is obvious. Now, we suppose that

$$(-1)^p\Theta_{p,n}(A,T,x) = \|AT^{p+n}x\|^2 - \sum_{k=0}^{p-1}(-1)^k\binom{p}{k}\Theta_{k,n}(A,T,x), \text{ for all } 1 \le p \le m.$$

On the other hand, we have

$$\begin{aligned}
&\|AT^{n+m+1}x\|^2 \\
&= (-1)^{m+1}\Theta_{m+1,n}(A,T,x) - \sum_{k=0}^{m}(-1)^{m+1-k}\binom{m+1}{k}\|AT^{k+n}x\| \\
&= (-1)^{m+1}\Theta_{m+1,n}(A,T,x) - \sum_{k=0}^{m}(-1)^{m+1-k}\binom{m+1}{k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\Theta_{j,n}(A,T,x) \\
&= (-1)^{m+1}\Theta_{m+1,n}(A,T,x) - \sum_{j=0}^{m}(-1)^j\Theta_{j,n}(A,T,x)\sum_{k=j}^{m}(-1)^{m+1-k}\binom{m+1}{k}\binom{k}{j} \\
&= (-1)^{m+1}\Theta_{m+1,n}(A,T,x) \\
&\quad - \sum_{j=0}^{m}(-1)^j\binom{m+1}{j}\Theta_{j,n}(A,T,x)\underbrace{\sum_{k=j}^{m}(-1)^{m+1-k}\binom{m+1-j}{k-j}}_{=-1} \\
&= (-1)^{m+1}\Theta_{m+1,n}(A,T,x) + \sum_{j=0}^{m}(-1)^j\binom{m+1}{j}\Theta_{j,n}(A,T,x).
\end{aligned}$$

Hence, the desired result attains.

3. We compute

$$\begin{aligned}
&\sum_{k=0}^{m-1}(-1)^k\binom{p}{k}\Theta_{k,n+1}(A,T,x) \\
&= \sum_{k=0}^{m-1}(-1)^k\binom{p}{k}\left[\Theta_{k,n}(A,T,x) - \Theta_{k+1,n}(A,T,x)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{m-1}(-1)^k\binom{p}{k}\Theta_{k,n}(A,T,x) - \sum_{k=0}^{m-1}(-1)^k\binom{p}{k}\Theta_{k+1,n}(A,T,x)\\
&= \sum_{k=0}^{m-1}(-1)^k\binom{p}{k}\Theta_{k,n}(A,T,x) - \sum_{k=1}^{m}(-1)^{k+1}\binom{p}{k-1}\Theta_{k,n}(A,T,x)\\
&= \sum_{k=0}^{m-1}(-1)^k\binom{p}{k}\Theta_{k,n}(A,T,x) + \sum_{k=1}^{m}(-1)^{k}\binom{p}{k-1}\Theta_{k,n}(A,T,x)\\
&= \sum_{k=0}^{m-1}(-1)^k\Big[\binom{p}{k}+\binom{p}{k-1}\Big]\Theta_{k,n}(A,T,x) + (-1)^m\binom{p}{m-1}\Theta_{m,n}(A,T,x)\\
&= \sum_{k=0}^{m-1}(-1)^k\binom{p+1}{k}\Theta_{k,n}(A,T,x) + (-1)^m\binom{p}{m-1}\Theta_{m,n}(A,T,x).
\end{aligned}
$$

□

As an immediate consequence of the first assertion of Lemma 2.6, we have the following proposition

Proposition 2.7. *Let $T \in \mathcal{L}(\mathcal{H})$. If T is $A(n,m)$-iso-contra-expansive operator and $A(n,m-1)$-iso-contra-expansive on $\overline{\mathcal{R}(T)}$, then T is $A(n,m-1)$-iso-contra-expansive on $\mathcal{H}$.*

We claim here that the hypothesis $A(n,m-1)$-iso-contra-expansive on $\overline{\mathcal{R}(T)}$ in the previous proposition is necessary condition as shown in the example below:

Example 2.1. *Let $A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ and $T = \begin{pmatrix} 4 & -1 \\ 3 & 2 \end{pmatrix}$*
Then, T is quasi-$(A,2)$-contractive. But, it is not quasi-$(A,1)$-contractive.

An operator T is called $A(n,m)$-hyper-iso-contra-expansive if it is $A(n,k)$-iso-contra-expansive for all $k = 1,\cdots,m$.

Proposition 2.8. *Let $T \in \mathcal{L}(\mathcal{H})$, we have*

1. *If T is $A(n,1)$, $A(n,2)$ and $A(n,m)$-iso-contra-expansive, then T is $A(n,m)$-hyper-iso-contra-expansive.*
2. *If T is n-quasi-(A,m)-expansive (resp n-quasi-(A,m)-contractive) and n-quasi-$(A,1)$-contractive (resp n-quasi-$(A,1)$-expansive), then T is n-quasi-(A,k)-expansive (resp n-quasi-(A,k)-contractive) for all $k = 2,\cdots,m$.*

Proof.

1. Without loss of generality, we assume that T is n-quasi-(A,m)-expansive and n-quasi-$(A,2)$-hyper-expansive, then
$$\|AT^nx\|^2 - \|AT^{n+1}x\|^2 \le 0 \text{ and } \|AT^nx\|^2 - 2\|AT^{n+1}x\|^2 + \|AT^{n+2}x\|^2 \le 0.$$
Let $(U_p)_{p\ge 0}$ be the sequence defined by $U_p = \|AT^{n+p+1}x\|^2 - \|AT^{n+p}x\|^2 \ge 0$. Moreover,
$$U_{p+1} - U_p = \|AT^{n+p+2}x\|^2 - 2\|AT^{n+p+1}x\|^2 + \|AT^{n+p}x\|^2.$$

Since T is a n-quasi-$(A,2)$-expansive, then

$$\|AT^nx\|^2 - 2\|AT^{n+1}x\|^2 + \|AT^{n+2}x\|^2 \leq 0.$$

This implies that

$$\|AT^{n+p}x\|^2 - 2\|AT^{n+1+p}x\|^2 + \|AT^{n+2+p}x\|^2 \leq 0.$$

Then, $(U_p)_{p\geq 0}$ is a decreasing sequence and bounded by 0. Therefore, there exist a positive constant C such that $\lim\limits_{p\to+\infty} U_p = C$. Now, according to Lemma 2.6 we have

$$\Theta_{m,n}(A,T,x) = \Theta_{m-1,n}(A,T,x) - \Theta_{m-1,n+1}(A,T,x).$$

Since $\Theta_{m,n}(A,T,x) \leq 0$ with $m \geq 3$ and by iteration on n, we get

$$\Theta_{m-1,n}(A,T,x) \leq \Theta_{m-1,n+1}(A,T,x) \leq \cdots \leq \Theta_{m-1,n+p}(A,T,x)$$

implying

$$\Theta_{m-1,n}(A,T,x) \leq \Theta_{m-1,n+p}(A,T,x), \ \ p \geq 1.$$

Thanks to Lemma 2.6, we have

$$\begin{aligned}
\Theta_{m-1,n+p}(A,T,x) &= \Theta_{m-2,n+p}(A,T,x) - \Theta_{m-2,n+p+1}(A,T,x) \\
&= \sum_{k=0}^{m-2}(-1)^k\binom{m-2}{k}\|AT^{n+p+k}x\|^2 \\
&\quad - \sum_{k=0}^{m-2}(-1)^k\binom{m-2}{k}\|AT^{n+p+1+k}x\|^2 \\
&= \sum_{k=0}^{m-2}(-1)^k\binom{m-2}{k}(\|AT^{n+p+k}x\|^2 - \|AT^{n+p+1+k}x\|^2) \\
&\underset{p\to+\infty}{\longrightarrow} C\sum_{k=0}^{m-2}(-1)^k\binom{m-2}{k} = 0.
\end{aligned}$$

Therefore, $\Theta_{m-1,n}(A,T,x) \leq 0$. Hence T is n-quasi-(A,m)-hyper-expansive.

For the case where T is n-quasi-(A,m)-contractive, we keep the same approach just we choose the sequence $(-U_p)_{p\geq 0}$.

2. For both cases, if T is a n-quasi-(A,m)-contractive or n-quasi-(A,m)-expansive, we define the sequence $(U_p)_{p\geq 0}$ by $U_p = \|AT^{n+p}x\|^2$. Since T is n-quasi-$(A,1)$-contractive, it is so $n+p$-quasi-$(A,1)$-contractive for each positive integer p, and hence we get that the sequence U_p is positive and decreasing and so it converges to a positive constant C. By the same idea of (1), we obtain

$$\begin{aligned}
\Theta_{m-1,n}(A,T,x) &\leq \Theta_{m-1,n+p}(A,T,x) \ \forall p \geq 0, \ m \geq 3 \\
&= \Theta_{m-2,n+p}(A,T,x) - \Theta_{m-2,n+p+1}(A,T,x) \\
&= \sum_{k=0}^{m-2}(-1)^k\binom{m-2}{k}(\|AT^{n+p}x\|^2 - \|AT^{n+p+1}x\|^2) \\
&\underset{p\to+\infty}{\longrightarrow} \sum_{k=0}^{m-2}(-1)^k\binom{m-2}{k}(U_p - U_{p+1}) = 0.
\end{aligned}$$

Hence, $\Theta_{m-1,n}(A,T,x) \leq 0$. By repeating the same steps $m-2$ times, we complete the proof.

$\square$

Lemma 2.9. *Let $T \in \mathcal{L}(\mathcal{H})$ and $p \geq m \geq 1$. Then, the following assertions hold:*

1. $(-1)^m\Theta_{m,n}(A,T,x) \leq 0$ *if and only if* $\|AT^{n+p}\|^2 \leq \sum_{k=0}^{m-1}(-1)^k\binom{p}{k}\Theta_{k,n}(A,T,x)$.
2. $(-1)^m\Theta_{m,n}(A,T,x) \geq 0$ *if and only if* $\|AT^{n+p}\|^2 \geq \sum_{k=0}^{m-1}(-1)^k\binom{p}{k}\Theta_{k,n}(A,T,x)$.

Proof. Let $p \geq m$. We will prove the assertion by induction on p. For $p=m$, by applying (2) of Lemma 2.6 and the fact that $(-1)^m\Theta_{m,n}(A,T,x) \leq 0$ we have

$$\|AT^{m+n}x\|^2 - \sum_{k=0}^{m-1}(-1)^k\binom{m}{k}\Theta_{k,n}(A,T,x) \leq 0.$$

Now, we suppose that $\|AT^{n+p}\|^2 \leq \sum_{k=0}^{m-1}(-1)^k\binom{p}{k}\Theta_{k,n}(A,T,x)$ and we will prove it for $p+1$. So, by applying (3) of Lemma 2.6, we get

$$\begin{aligned}
\|AT^{n+p+1}\|^2 &= \|AT^{(n+1)+p}\|^2 \\
&\leq \sum_{k=0}^{m-1}(-1)^k\binom{p}{k}\Theta_{k,n+1}(A,T,x) \\
&\leq \sum_{k=0}^{m-1}(-1)^k\binom{p+1}{k}\Theta_{k,n}(A,T,x) + \binom{p}{m-1}(-1)^m\Theta_{m,n}(A,T,x) \\
&\leq \sum_{k=0}^{m-1}(-1)^k\binom{p+1}{k}\Theta_{k,n}(A,T,x).
\end{aligned}$$

Hence, we obtain the desired result. Similarly we prove the second assertion. $\square$

Lemma 2.10. *Let $T \in \mathcal{L}(\mathcal{H})$. Then, the following assertion holds*

$$(-1)^m\Theta_{m,n}(A,T,x) \leq 0 \Longrightarrow (-1)^{m-1}\Theta_{m-1,n}(A,T,x) \geq 0.$$

Proof. Let $p \geq m$ and suppose that $(-1)^m\Theta_{m,n}(A,T,x) \leq 0$. Then, according to Lemma 2.9, we have

$$\|AT^{n+p}\|^2 \leq \sum_{k=0}^{m-1}(-1)^k\binom{p}{k}\Theta_{k,n}(A,T,x),$$

implying

$$\|AT^{n+p}\|^2(-1)^{m-1} \leq \binom{p}{m-1}\Theta_{m-1,n}(A,T,x) + \sum_{k=0}^{m-2}(-1)^k\binom{p}{k}\Theta_{k,n}(A,T,x).$$

Thus

$$(-1)^{m-1}\binom{p}{m-1}\Theta_{m-1,n}(A,T,x) \geq \|AT^{n+p}\|^2 - \sum_{k=0}^{m-2}(-1)^k\binom{p}{k}\Theta_{k,n}(A,T,x).$$

Multiplying the last inequality by $\frac{1}{p^{m-1}} \geq 0$, we obtain

$$\frac{(-1)^{m-1}}{p^{m-1}}\binom{p}{m-1}\Theta_{m-1,n}(A,T,x) \geq \frac{1}{p^{m-1}}\|AT^{n+p}\|^2 - \sum_{k=0}^{m-2}(-1)^k\frac{1}{p^{m-1}}\binom{p}{k}\Theta_{k,n}(A,T,x)$$

and since $\lim\limits_{p\to+\infty}\frac{1}{p^{m-1}}\binom{p}{k} = 0$ for all $0 \leq k \leq m-2$, we infer that

$$(-1)^{m-1}\Theta_{m-1,n}(A,T,x) \geq \lim_{p\to+\infty}\frac{\|AT^{n+p}\|^2}{\binom{p}{m-1}} \geq 0.$$

□

As a consequence of Lemma 2.10, we derive the following result.

Theorem 2.11. *Let $T \in \mathcal{L}(\mathcal{H})$.*

1. *If m is even and T is n-quasi-(A,m)-expansive, then T is n-quasi-$(A,m-1)$-expansive.*
2. *If m is odd and T is n-quasi-(A,m)-contractive, then T is n-quasi-$(A,m-1)$-contractive.*

Remark 2.2. *The conditions on m that be even and odd in the above Theorem are necessary, the results are not true in the general cases. Indeed,*

- *For the first assertion, let $m=3$, $A = \begin{pmatrix}1 & -1\\ 1 & 2\end{pmatrix}$ and $T = \begin{pmatrix}-4 & -1\\ 3 & -2\end{pmatrix}$.*

Then, T is quasi-$(A,3)$-expansive. But, it is not quasi-$(A,2)$-expansive

- *For the second, let $m=2$, $A = \begin{pmatrix}1 & 1\\ -1 & 2\end{pmatrix}$ and $T = \begin{pmatrix}-4 & -1\\ 3 & -2\end{pmatrix}$.*

By simple calculation T is quasi-$(A,2)$-contractive. But, it is not quasi-$(A,1)$-contractive.

3. S.V.E.P and Spectral properties of $A(n,m)$-iso-contra-expansive operators

We start this section by providing the most important spectral properties of $A(n,m)$-iso-contra-expansive operators, but we present first the basic notations concerning the concept. We denote by $\sigma(T)$, $\sigma_p(T)$ and $\sigma_{ap}(T)$ the spectrum, the point spectrum and the approximate point spectrum of an operator $T \in \mathcal{L}(\mathcal{H})$, respectively. The results we described below are similar to the work [20].

Proposition 3.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be a n-quasi-(A,m)-expansive such that $0 \notin \sigma_{ap}(A)$. The following properties hold.*

1. *If m is odd, then $\sigma_{ap}(T) \subset (\mathbb{C}\setminus\mathbb{D}(0,1)) \cup \{0\}$*

2. *If m is even, then we have*

   (i) $\sigma_{ap}(T) \subset \mathcal{C}(0,1) \cup \{0\}$

   (ii) $\sigma(T) \subset \mathcal{C}(0,1)$ *or* $\sigma(T) = \overline{\mathbb{D}(0,1)}$

Proof. Let $\lambda \in \sigma_{ap}(T)$, then there exists a sequence $(x_p)_{p\geq 0}$ such that $\|x_p\| = 1$ and $(T-\lambda I)x_p \underset{p\to+\infty}{\longrightarrow} 0$, which implies that $(T^{n+k} - \lambda^{n+k} I)x_p \underset{p\to+\infty}{\longrightarrow} 0$ for all $n, k \in \mathbb{N}$. Since T is a n-quasi-(A,m)-expansive, we get

$$\begin{aligned}
0 &\geq \Theta_{m,n}(A,T,x_p)\\
&= \sum_{k=0}^{m}(-1)^k\binom{m}{k}\|AT^{n+k}x_p\|^2\\
&= \sum_{k=0}^{m}(-1)^k\binom{m}{k}\left(\|AT^{n+k}x_p\|^2 - \|A\lambda^{n+k}x_p\|^2 + \|A\lambda^{n+k}x_p\|^2\right)\\
&= \sum_{k=0}^{m}(-1)^k\binom{m}{k}\|A\lambda^{n+k}x_p\|^2 + \sum_{k=0}^{m}(-1)^k\binom{m}{k}\left(\|AT^{n+k}x_p\|^2 - \|A\lambda^{n+k}x_p\|^2\right)\\
&= |\lambda|^{2n}\left(1-|\lambda|^2\right)^m\|Ax_p\|^2 + \sum_{k=0}^{m}(-1)^k\binom{m}{k}\left(\|AT^{n+k}x_p\|^2 - \|A\lambda^{n+k}x_p\|^2\right).
\end{aligned}$$

Then the sum of the right-hand of the last inequality tends to 0. So, under the condition that $0 \notin \sigma_{ap}(A)$, we derive that $|\lambda|^{2n}\left(1-|\lambda|^2\right)^m \leq 0$. Notice that 0 is a solution of the above inequality, thus $0 \in \sigma_{ap}(T)$. Assume now that $|\lambda| > 0$ means $\left(1-|\lambda|^2\right)^m \leq 0$, two cases are discussed:

1. If m is odd, then $|\lambda| \geq 1$.

2. If m is even, we get

   (i) $\left(1-|\lambda|^2\right)^m = 0$ namely $|\lambda| = 1$.

(ii) Moreover, by recalling that

$$\partial\sigma(T) \subset \sigma_{ap}(T) \subset \left(\mathcal{C}(0,1) \cup \{0\}\right).$$

We deal with two cases: $\sigma(T) \subset \mathcal{C}(0,1)$ or $\sigma(T) = \overline{\mathbb{D}(0,1)}$.

□

By repeating the same approach of the above proof we cite the following (we omit so the proof).

Proposition 3.2. *Let $T \in \mathcal{L}(\mathcal{H})$ be a n-quasi-(A,m)-contractive such that $0 \notin \sigma_{ap}(A)$ and m is odd. Then we have*

(i) $\sigma_{ap}(T) \subset \overline{\mathbb{D}(0,1)}$

(iii) If $0 \in \sigma(T)$ *then* $\sigma(T) = \overline{\mathbb{D}(0,1)}$

We drive your attention that for a positive operator A the definition of $A(n,m)$-iso-contra-expansive is equivalent to say that the function

$$\Theta_{m,n}(T,A) := \sum_{k=0}^{m}(-1)^k\binom{m}{k}T^{*k+n}AT^{k+n}$$

is positive or negative or equal to 0. In the rest of this section, we use this definition to characterize the $A(n,m)$-iso-contra-expansive operators with A is a positive operator.

Now, we say that an operator $T \in \mathcal{L}(\mathcal{H})$ have the single-valued extension property at a point $\xi_0 \in \mathbb{C}$ or we use simply the abbreviation S.V.E.P at ξ_0 if for every neighborhood V of ξ_0 and for every analytic function $f : V \to \mathcal{H}$, we have the following statement

$$(T-\xi)f(\xi) \equiv 0 \Longrightarrow f(\xi) \equiv 0.$$

The operator T is said then that to have the S.V.E.P if T has the S.V.E.P at every point $\xi \in \mathbb{C}$. In the next theorem, we give several conditions for $A(n,m)$-iso-contra-expansive to have the single-valued extension property but we need first for the proof the following matrix representation of this class of operators.

Proposition 3.3. *Let $T \in \mathcal{L}(\mathcal{H})$ such that $\overline{\mathcal{R}(T^n)}$ is a reducing subspace for A. Then, T is $A(n,m)$-iso-contra-expansive operator if and only if*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \qquad on \quad \mathcal{H} = \overline{\mathcal{R}(T^n)} \oplus \mathcal{N}(T^{*n})$$

where T_1 is $A_1(0,m)$-iso-contra-expansive such that $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ and $T_3^n = 0$.

Proof. " $\Rightarrow$ " For the proof we just adopt that T is n-quasi-(A,m)-expansive operator, the other case when T is n-quasi-(A,m)-contractive can be handled easily by using the same steps. Based on the inspirations cited on [13], we can decompose the operators T and A as

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad and \quad A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

Now, we denote by P the projection onto $\overline{\mathcal{R}(T^n)}$, namely $PT = T_1$ and $PA = A_1$. Since T is n-quasi-(A,m)-expansive then

$$P\left(\sum_{k=0}^{m}(-1)^k\binom{m}{k}T^{*k}AT^k\right)P \leq 0$$

which implies that

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}T_1^{*k}A_1T_1^k \leq 0.$$

Thus T_1 is $A_1(0,m)$-expansive operator.

The statement $T_3^n = 0$ readily follows making a direct computation keeping in mind that every $x \in \mathcal{H}$ can be written as $x = x_1 + x_2$ with $x_1 \in \overline{\mathcal{R}(T^n)}$ and $x_2 \in \mathcal{N}(T^{*n})$. So

$$\begin{aligned} \langle T_3^n x_2, x_2\rangle &= \langle T^n(I-P)x, (I-P)x\rangle \\ &= \langle (I-P)x, T^{*n}(I-P)x\rangle \\ &= 0. \end{aligned}$$

" $\Leftarrow$ " Assume that $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\overline{\mathcal{R}(T^n)} \oplus \mathcal{N}(T^{*n})$, such that T_1 is a $A_1(0,m)$-expansive operator and $T_3^n = 0$. We know that for all $r \in \mathbb{N}$, we have

$$T^r = \begin{pmatrix} T_1^r & \sum_{i=0}^{r-1} T_1^i T_2 T_3^{r-1-i} \\ 0 & T_3^r \end{pmatrix}.$$

Therefore

$$T^nT^{*n} = \begin{pmatrix} T_1^nT_1^{*n} & 0 \\ 0 & 0 \end{pmatrix},$$

since $T_3^n = 0$. On the other hand, by a simple calculation we get

$$\Theta_{m,0}(T,A) = \begin{pmatrix} \Theta_{m,0}(T_1,A_1) & C \\ B & D \end{pmatrix}$$

where C, B and $D \in \mathcal{L}(\mathcal{H})$. Multiplying $\Theta_{m,0}(T,A)$ on the left and right by T^nT^{*n}, we get

$$T^nT^{*n}\Theta_{m,0}(T,A)T^nT^{*n} = \begin{pmatrix} T_1^nT_1^{*n}\Theta_{m,0}(T,A_1)T_1^nT_1^{*n} & 0 \\ 0 & 0 \end{pmatrix}.$$

Then for all $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{H}$, we have

$$\langle T^nT^{*n}\Theta_{m,0}(T,A)T^nT^{*n}x, x\rangle = \langle \Theta_{m,n}(T,A)\left(T^{*n}x\right), \left(T^{*n}x\right)\rangle \tag{3.2}$$

On the other hand

$$\begin{aligned}\langle T^nT^{*n}\Theta_{m,0}(T,A)T^nT^{*n}x, x\rangle &= \left\langle \begin{pmatrix} T_1^nT_1^{*n}\Theta_{m,0}(T,A_1)T_1^nT_1^{*n} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right\rangle \\ &= \langle T_1^nT_1^{*n}\Theta_{m,0}(T_1,A_1)T_1^nT_1^{*n}x_1, x_1\rangle \\ &= \langle T_1^{*n}\Theta_{m,0}(T_1,A_1)T_1^n\left(T_1^{*n}x_1\right), \left(T_1^{*n}x_1\right)\rangle\end{aligned}$$

Since T_1 is a $A_1(0,m)$-expansive operator, then $\Theta_{m,0}(T_1,A_1) \leq 0$ and by the Proposition 2.1 we get that $T_1^{*n}\Theta_{m,0}(T_1,A_1)T_1^n \leq 0$. This implies, by the equation (4.4), that $\Theta_{m,n}(T,A) \leq 0$. $\square$

Theorem 3.4. *Let T and S be respectively a n-quasi-(A,m)-expansive and n-quasi-(A,m)-contractive operators such that A is left invertible and $\overline{\mathcal{R}(T^n)}, \overline{\mathcal{R}(S^n)}$ are reducing subspace for A. Then the following statements hold:*

1. *If m is even, then T has the S.V.E.P at each $\xi \in \mathbb{C}$.*
2. *If m is odd, then T has the S.V.E.P at $\xi \in \mathbb{C}$ such that $|\xi| \leq 1$ or $|\xi| > \|T\|$.*
3. *For every m, S has the S.V.E.P at $\xi \in \mathbb{C}$ such that $|\xi| \geq \min(1, \|T\|)$.*

Proof. The result can be proved using the matrix representation of T and S presented in Proposition 3.3. Indeed, we start by assuming that T is a n-quasi-(A,m)-expansive then

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

where T_1 is $A_1(0,m)$-expansive and $T_3^n = 0$ keeping in mind the matrix representation of A. We take now an analytic function f on $\mathcal{H} = \overline{\mathcal{R}(T^n)} \oplus \mathcal{N}(T^{*n})$ where $f(\xi) = f_1(\xi) \oplus f_2(\xi)$ that satisfies

$$(T-\xi)f(\xi) \equiv 0.$$

This is equivalent to write

$$T = \begin{pmatrix} T_1-\xi & T_2 \\ 0 & T_3-\xi \end{pmatrix} \begin{pmatrix} f_1(\xi) \\ f_2(\xi) \end{pmatrix} = \begin{pmatrix} (T_1-\xi)f_1(\xi) + T_2 f_2(\xi) \\ (T_3-\xi)f_2(\xi) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We deduce, from the fact that T_3 is nilpotent, that $f_2 \equiv 0$ and it follows that $(T_1-\xi)f_1(\xi) \equiv 0$. Under the condition that T_1 is $A_1(0,m)$-expansive, the results come out successively based on [16, Theorem 3.1] and this completes the proof of (1) and (2).

Now, [20, Theorem 3.1] and the left invertibility of A which imply in particular that $0 \notin \sigma_p(A)$ give the proof of (3). $\square$

4. Application of Toeplitz operators on Bergman space

In the current section, we are concerned about a characterization of Toeplitz operators on the β-modified Bergman space $\mathcal{A}^2_{\alpha,\beta} = \mathcal{A}^2_{\alpha,\beta}(\mathbb{D}^*)$ which consists of all holomorphic functions f on the unit disk $\mathbb{D}^*$ such that

$$\|f\|_{\alpha,\beta} = \left(\int_{\mathbb{D}} |f(z)|^2 \, d\mu_{\alpha,\beta}(z) \right)^{1/2} < +\infty,$$

for $-1 < \alpha < +\infty$ and $-1 < \beta = \beta_0 + p < +\infty$ with $-1 < \beta_0 \le 0$ and $p \in \mathbb{N}$ where

$$d\mu_{\alpha,\beta}(z) = \frac{1}{\mathscr{B}(\alpha+1, \beta+1)} |z|^{2\beta} (1-|z|^2)^{\alpha} dA(z), \tag{4.3}$$

$\mathscr{B}$ is the beta function and dA is the normalized area measure on $\mathbb{D}$. The space $\mathcal{A}^2_{\alpha,\beta}$ is a closed subspace of the Hilbert space $\mathbf{L}^2(\mathbb{D}, d\mu_{\alpha,\beta})$ with inner product given by

$$\langle f, g \rangle_{\alpha,\beta} = \int_{\mathbb{D}} f(z) \, \overline{g(z)} \, d\mu_{\alpha,\beta}(z), \tag{4.4}$$

for $f, g \in \mathbf{L}^2(\mathbb{D}, d\mu_{\alpha,\beta})$. For more details of this space, we can refer to [10, 11, 12].

For $\varphi \in L^\infty(\mathbb{D})$, the Toeplitz operator on $\mathcal{A}^2_{\alpha,\beta}$ with symbol φ is defined by

$$T_\varphi(f) = \mathbb{P}_{\alpha,\beta}(\varphi f)$$

where $\mathbb{P}_{\alpha,\beta}$ be the orthogonal projection from $\mathbf{L}^2(\mathbb{D}, d\mu_{\alpha,\beta})$ onto $\mathcal{A}^2_{\alpha,\beta}$ which is a bounded operator given by the integral

$$\mathbb{P}_{\alpha,\beta} f(z) = \int_{\mathbb{D}} f(w) \mathbb{K}_{\alpha,\beta}(z,w) d\mu_{\alpha,\beta}(w)$$

with $\mathbb{K}_{\alpha,\beta}(w,z) := \mathcal{K}_{\alpha,\beta}(w\overline{z})$ is the reproducing (Bergman) kernel given by

$$\mathcal{K}_{\alpha,\beta}(\xi) = \frac{\mathscr{B}(\alpha+1, \beta+1)}{\mathscr{B}(\alpha+1, \beta_0+1)} \xi^{-p} \, {}_2F_1 \left(\begin{array}{c} 1, \alpha+\beta_0+2 \\ \beta_0+1 \end{array} \middle| \xi \right).$$

Here ${}_2F_1$ is the hypergeometric function. The third auther proved in [19] that the orthogonal projection $\mathbb{P}_{\alpha,\beta}$ satisfies the following calculus:

Lemma 4.1. *Let $-1 < \alpha, \beta < +\infty$ with $\beta = \beta_0 + p$ and $p \in \mathbb{N}$. Then, for s, t be nonnegative integers, we have*

$$\textit{(i)}\ \mathbb{P}_{\alpha,\beta}(\overline{z}^s z^t) = \begin{cases} \dfrac{(\alpha+\beta+t+2)_{-s}}{(\beta+t+1)_{-s}} z^{t-s} & if \quad t+p \geq s \\ 0 & if \quad t+p < s. \end{cases}$$

$$\textit{(ii)}\ \left\| \mathbb{P}_{\alpha,\beta}(\overline{z}^s \sum_{j=-p}^{+\infty} a_j z^j) \right\|^2_{\alpha,\beta} = \sum_{j=-p}^{+\infty} \frac{(\beta+1)_{j-s}}{(\alpha+\beta+2)_{j-s}} \frac{(\alpha+\beta+j+2)^2_{-s}}{(\beta+j+1)^2_{-s}} |a_j|^2,$$

where $(a)_n = a(a+1)\dots(a+n-1) = \dfrac{\Gamma(a+n)}{\Gamma(a)}$ is the Pochhammer symbol

Here, we give an overview of the proof of this lemma.

Proof. Recall that

$${}_2F_1\left(\begin{array}{c} a, b \\ c \end{array} \middle| \xi \right) = \sum_{n=0}^{+\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{\xi^n}{n!}.$$

By the definition of $\mathbb{P}_{\alpha,\beta}$ given above, we have

$$\mathbb{P}_{\alpha,\beta}(\overline{z}^s z^t) = \frac{z^{-p}}{\mathscr{B}(\alpha+1, \beta_0+1)} \sum_{n=0}^{+\infty} \frac{(\alpha+\beta_0+2)_n}{(\beta_0+1)_n} z^n \int_{\mathbb{D}} \overline{w}^{s-p+n} w^t |w|^{2\beta} (1-|w|^2)^\alpha dA(w).$$

Using now the polar coordinates, we get

$$\begin{aligned} \int_{\mathbb{D}} \overline{w}^{s-p+n} w^t |w|^{2\beta} (1-|w|^2)^\alpha dA(w) &= \frac{1}{\pi} \int_0^1 r^{s+t-p+2\beta+n+1} (1-r^2)^\alpha dr \int_0^{2\pi} e^{i(t+p-s-n)\theta} d\theta \\ &= \begin{cases} \mathscr{B}(\alpha+1, t+\beta+1) & if \quad n = t+p-s \\ 0 & if \quad n \neq t+p-s \end{cases} \end{aligned}$$

and this finishes the proof of the first statement. The second one is a simple consequence of the first statement by using a classical computation and the polar coordinates. □

Now, as an application for what we have studied in this paper we focus on the n-quasi-(A,m)-expansive and n-quasi-(A,m)-contractive Toeplitz operators with analytic and coanalytic symbol and with a special values of n and m and the operator A. Let us point out that the results we described below can be extended for the general cases but in our opinion this requires further investigations which will be our next project. In this context, we take $f = \sum_{j=-p}^{+\infty} a_j z^j \in \mathcal{A}^2_{\alpha,\beta}$ with $\beta = \beta_0 + p$ and $A = \mathbb{P}_{\alpha,\beta}$. So, we cite the following three examples:

Example 1: Suppose that φ is analytic function, then we have

$$\begin{aligned}
\left\langle \sum_{k=0}^{m}(-1)^k\binom{m}{k}T_{\varphi}^{*k+n}\mathbb{P}_{\alpha,\beta}^{*}\mathbb{P}_{\alpha,\beta}T_{\varphi}^{k+n}f,f\right\rangle_{\alpha,\beta} &= \left\langle \sum_{k=0}^{m}(-1)^k\binom{m}{k}T_{\varphi}^{*k+n}T_{\varphi}^{k+n}f,f\right\rangle_{\alpha,\beta} \\
&= \left\langle \sum_{k=0}^{m}(-1)^k\binom{m}{k}T_{|\varphi|^{2(k+n)}}f,f\right\rangle_{\alpha,\beta} \\
&= \left\langle T_{\sum_{k=0}^{m}(-1)^k\binom{m}{k}|\varphi|^{2(k+n)}}f,f\right\rangle_{\alpha,\beta} \\
&= \left\langle \sum_{k=0}^{m}(-1)^k\binom{m}{k}|\varphi|^{2(k+n)}f,f\right\rangle_{\alpha,\beta} \\
&= \left\langle |\varphi|^{2n}(1-|\varphi|^2)^m f,f\right\rangle_{\alpha,\beta}
\end{aligned}$$

This leads to conclude that:

- The Toeplitz operator T_φ is n-quasi-(A,m)-expansive for odd m and $|\varphi|\geq 1$.
- The Toeplitz operator T_φ is n-quasi-(A,m)-contractive for all m and $|\varphi|\leq 1$ or for even m and $|\varphi|\geq 1$.

Example 2: Suppose that $\varphi = az^N$ so $\varphi f\in\mathcal{A}^2_{\alpha,\beta}$. Using the properties of Toeplitz operators cited above and the polar coordinates we obtain

$$\begin{aligned}
&\sum_{k=0}^{m}(-1)^k\binom{m}{k}\left\|\varphi^{n+m}f\right\|^2_{\alpha,\beta} \\
&\qquad = \sum_{k=0}^{m}(-1)^k\binom{m}{k}|a|^{2(n+k)}\sum_{j=-p}^{+\infty}\frac{\Gamma(N(n+k)+j+\beta+1)\Gamma(\alpha+\beta+2)}{\Gamma(N(n+k)+\alpha+\beta+j+2)\Gamma(\beta+1)}|a_j|^2.
\end{aligned}$$

By a straightforward computations, we can conclude

- The Toeplitz operator T_φ is 2-quasi-$(A,1)$-expansive if and only if
$$|a|^2\geq\sup_{j\geq 0}\frac{(\alpha+\beta_0+j+2N+2)_N}{(\beta_0+j+2N+1)_N}=\frac{(\alpha+\beta_0+2N+2)_N}{(\beta_0+2N+1)_N}.$$
- The Toeplitz operator T_φ is 2-quasi-$(A,1)$-contractive if and only if
$$|a|^2\leq\inf_{j\geq 0}\frac{(\alpha+\beta_0+j+2N+2)_N}{(\beta_0+j+2N+1)_N}=1.$$

The values of the above supremum and infimum due to the fact that the function $j\mapsto\frac{(\alpha+\beta_0+j+2N+2)_N}{(\beta_0+j+2N+1)_N}$ is a positive and decreasing function on $[0,+\infty[$.

Example 3: Suppose that $\varphi = a\overline{z}^s$, then we have

$$\sum_{k=0}^{1}(-1)^k\binom{1}{k}\left\|AT_\varphi^{2+k}f\right\|_{\alpha,\beta}^2 = \left\|T_\varphi^2 f\right\|_{\alpha,\beta}^2 - \left\|T_\varphi^3 f\right\|_{\alpha,\beta}^2$$

$$= \left\|\mathbb{P}_{\alpha,\beta}\left(\varphi\mathbb{P}_{\alpha,\beta}\left(\varphi f\right)\right)\right\|_{\alpha,\beta}^2 - \left\|\mathbb{P}_{\alpha,\beta}\left(\varphi\mathbb{P}_{\alpha,\beta}\left(\varphi\mathbb{P}_{\alpha,\beta}\left(\varphi f\right)\right)\right)\right\|_{\alpha,\beta}^2$$

$$= \left\|\mathbb{P}_{\alpha,\beta}\left(a\overline{z}^s\mathbb{P}_{\alpha,\beta}\left(a\overline{z}^s\sum_{j=-p}^{+\infty}a_jz^j\right)\right)\right\|_{\alpha,\beta}^2 - \left\|\mathbb{P}_{\alpha,\beta}\left(a\overline{z}^s\mathbb{P}_{\alpha,\beta}\left(a\overline{z}^s\mathbb{P}_{\alpha,\beta}\left(a\overline{z}^s\sum_{j=-p}^{+\infty}a_jz^j\right)\right)\right)\right\|_{\alpha,\beta}^2.$$

By using Lemma 4.1 and by a tedious computation, for $j+p \geq 2s$, we perform

$$\left\|\mathbb{P}_{\alpha,\beta}\left(a\overline{z}^s\mathbb{P}_{\alpha,\beta}\left(a\overline{z}^s\sum_{j=-p}^{+\infty}a_jz^j\right)\right)\right\|_{\alpha,\beta}^2$$

$$= |a|^2\sum_{j=-p}^{+\infty}\frac{(\alpha+\beta+j+2)_{-s}^2}{(\beta+j+1)_{-s}^2}\frac{(\alpha+\beta+j-s+2)_{-s}^2}{(\beta+j-s+1)_{-s}^2}\frac{(\beta+1)_{j-2s}}{(\alpha+\beta+2)_{j-2s}}|a_j|^2.$$

And

$$\left\|\mathbb{P}_{\alpha,\beta}\left(a\overline{z}^s\mathbb{P}_{\alpha,\beta}\left(a\overline{z}^s\mathbb{P}_{\alpha,\beta}\left(a\overline{z}^s\sum_{j=-p}^{+\infty}a_jz^j\right)\right)\right)\right\|_{\alpha,\beta}^2$$

$$= |a|^6\sum_{j=-p}^{+\infty}\frac{(\alpha+\beta+j+2)_{-s}^2}{(\beta+j+1)_{-s}^2}\frac{(\alpha+\beta+j-s+2)_{-s}^2}{(\beta+j-s+1)_{-s}^2}\frac{(\alpha+\beta+j-2s+2)_{-s}^2}{(\beta+j-2s+1)_{-s}^2}$$

$$\times\frac{(\beta+1)_{j-3s}}{(\alpha+\beta+2)_{j-3s}}|a_j|^2.$$

This leads to conclude that

- The Toeplitz operator T_φ is 2-quasi-$(A,1)$-expansive if and only if

  $$\left\|T_\varphi^2 f\right\|_{\alpha,\beta}^2 \leq \left\|T_\varphi^3 f\right\|_{\alpha,\beta}^2$$

  which equivalent to

  $$|a|^4 \geq \sup_{j\geq 0}\frac{(j-2s+\beta_0+1)_{-s}}{(j-2s+\alpha+\beta_0+2)_{-s}} = \frac{(\beta_0-2s+1)_{-s}}{(\alpha+\beta_0-2s+2)_{-s}}.$$

- The Toeplitz operator T_φ is 2-quasi-$(A,1)$-contractive if and only if

  $$\left\|T_\varphi^2 f\right\|_{\alpha,\beta}^2 \geq \left\|T_\varphi^3 f\right\|_{\alpha,\beta}^2$$

  which equivalent to

  $$|a|^4 \leq \inf_{j\geq 0}\frac{(j-2s+\beta_0+1)_{-s}}{(j-2s+\alpha+\beta_0+2)_{-s}} = 1.$$

As in the previous example, the computation of the infimum and the supremum of $\frac{(j-2s+\beta_0+1)_{-s}}{(j-2s+\alpha+\beta_0+2)_{-s}}$ follows from the fact that the function is positive and decreasing on $[0,+\infty[$.

The results of the above examples due by simplification of the coefficients using the well-known formula of Pochhammer symbol $(\lambda)_{m+n} = (\lambda)_m(\lambda+m)_n$.

It should be mentioned here that in order to achieve a deep study, we extend the symbol φ of the form $\varphi(z) = \sum_{k=0}^{+\infty} a_k z^k$ in the second example and $\varphi(z) = \sum_{k=0}^{+\infty} a_k \overline{z}^k$ in the third one.

References

[1] J. Agler, A disconjugacy theorem for Toeplitz operators. Am J. Math. 112(1), 1-14, (1990)

[2] J. Agler, M. Stankus, m-Isometric transformations of Hilbert space. I. Integral Equ. Oper. Theory 21(4), 383-429, (1995)

[3] J. Agler, M. Stankus, m-Isometric transformations of Hilbert space. II. Integral Equ.Oper. Theory 23(1), 1-48, (1995)

[4] J. Agler, M. Stankus, m-Isometric transformations of Hilbert space. III. Integral Equ. Oper. Theory 24(4), 379-421, (1996)

[5] G. Caixing, On (m,p)-expansive and (m,p)-contractive operators on Hilbert and Banach spaces. J. Math. Anal. Appl. 426 (2015), no. 2, 893-916.

[6] B. P. Duggal, Tensor product of n-isometries III. Funct. Anal. Approx. Comput. 4 (2012), no. 2, 61-67.

[7] G. Exner, On n-contractive and n-hypercontractive operators. Integral Equations Operator Theory 56 (2006), no. 4, 451-468.

[8] G. Exner, I. Bong Jung, C. Li, On k-hyperexpansive operators. J. Math. Anal. Appl. 323 (2006), no. 1, 569-582.

[9] G. Exner, I. Bong Jung, S. Soo Park, On n-contractive and n-hypercontractive operators II. Integral Equations Operator Theory, 60 (2008), pp. 451-467

[10] A. Ghanmi and S. Snoun, Integral representations of Bargmann type for β-modified Bergman space on punctured unit disc, Bull. Malays. Math. Sci. Soc. (2022) 45, 1367-1381.

[11] N. Ghiloufi and S. Snoun, Zeros of news Bergman kernels, J. Korean Math. Soc. 59 (2022), 449-468.

[12] N. Ghiloufi and M. Zaway, Meromorphic Bergman spaces, Ukrainian Mathematical Journal, 74 (2023), 1209-1224.

[13] M. Guesba, E. M Ould Beiba, O. A. M Sid Ahmed, n-quasi-(A,m)-isometric operators on a Hilbert space. Bull. Math. Anal. Appl. 12 (2020), no. 5, 8-26.

[14] S. Mecheri, T. Prasad, On n-quasi-m-isometric operators. Asian-European Journal of Mathematics Vol. 9, No. 3, 1650073, (2016)

[15] A. Saddi, O. A. M Sid Ahmed, A-m-isometric operators in semi-Hilbertian spaces. Linear Algebra Appl. 436 (2012), no. 10, 3930-3942.

[16] O. A. M. Sid ahmed, On A(m,p)-expansive and A(m, p)-hyperexpansive operators on Banach spaces-I. Aljouf Science and Engineering Journal. 2014; 1(1), 23-43.

[17] O. A. M. Sid ahmed, On A(m,p)-expansive and A(m, p)-hyperexpansive operators on Banach spaces-II. J. Math. Comput. Sci. 5(2015), No. 2, 123-148.

[18] O. A. M. Sid Ahmed, A. Saddi, K. Gherairi, Some results on Higher orders quasi-isometries, Hacet. J. Math. Stat. (2019)

[19] S. Snoun, Toeplitz operators via Carleson measures On β-modified Bergman Spaces. arXiv preprint arXiv:2310.03519,2023.

[20] J. Sungeun, K. Yoenha, K. Eungil, L. Ji Eun, On (A,m)-expansive operators. Studia Math. 213 (2012), no. 1, 3-23.

[21] Zuo. F , Mecheri. S, A class of operators related to m-symmetric operators. Turkish J. Math. 45 , no. 3, 1300-1309.(2021)

