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Metrical Weyl almost automorphy and applications

Métrique de Weyl presque automorphe et applications

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ABSTRACT. In this paper, we reconsider and slightly generalize various classes of Weyl almost automorphic functions ([29], [33]). More precisely, we consider here various classes of metrically Weyl almost automorphic functions of the form $F : \mathbb{R}^n \times X \to Y$ and metrically Weyl almost automorphic sequences of the form $F : \mathbb{Z}^n \times X \to Y$, where X and Y are complex Banach spaces. The main structural characterizations for the introduced classes of metrically Weyl almost automorphic functions and sequences are established. In addition to the above, we provide several illustrative examples, useful remarks and applications of the theoretical results.
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1 Introduction and preliminaries

The class of almost automorphic functions, which generalizes the well-known class of almost periodic functions, was introduced by S. Bochner in 1955 ([14]). If $F : \mathbb{R}^n \to X$ is a continuous function, where $(X, \| \cdot \|)$ is a complex Banach space, then $F(\cdot)$ is said to be almost automorphic if and only if for every sequence $(\mathbf{b}_k)$ in $\mathbb{R}^n$ there exist a subsequence $(\mathbf{a}_k)$ of $(\mathbf{b}_k)$ and a mapping $G : \mathbb{R}^n \to X$ such that

$$\lim_{k\to\infty} F\big(\mathbf{t} + \mathbf{a}_k\big) = G(\mathbf{t}) \text{ and } \lim_{k\to\infty} G\big(\mathbf{t} - a_k\big) = F(\mathbf{t}), \tag{1.1}$$

pointwisely for $\mathbf{t} \in \mathbb{R}^n$. In this case, the range of $F(\cdot)$ is relatively compact in X and the limit function $G(\cdot)$ is bounded on $\mathbb{R}^n$ but not necessarily continuous on $\mathbb{R}^n$. Furthermore, if the convergence of limits appearing in (1.1) is uniform on compact subsets of $\mathbb{R}^n$, then $F(\cdot)$ is said to be compactly almost automorphic. For further information about almost periodic functions and almost automorphic functions, the reader may consult the monographs [13, 18, 24, 26, 29, 39, 43].

On the other hand, the theory of (abstract) Volterra difference equations is a rapidly growing field of research; for more details in this direction, the reader may consult the monographs [5, 6, 7, 21]. Various classes of (multi-dimensional) Bohr ρ-almost periodic sequences and their Weyl, Besicovitch and Doss generalizations have recently been considered in [36]; in that paper, we have also provided several applications of generalized ρ-almost periodic sequences to the (abstract) Volterra difference equations. Concerning the applications of one-dimensional almost automorphic type sequences in this field, we can recommend for the reader the following research articles [1, 3, 4, 8, 12, 15, 40]. In [38], we have recently analyzed the multi-dimensional almost automorphic sequences of the form $F : \mathbb{Z}^n \times X \to Y$,

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where $(Y, \| \cdot \|_Y)$ is a complex Banach space, and provided several applications to the abstract Volterra difference equations depending on several variables.

The main aim of this research article is to introduce and analyze several new classes of metrically Weyl almost automorphic functions of the form $F : \mathbb{R}^n \times X \to Y$ and metrically Weyl almost automorphic sequences of the form $F : \mathbb{Z}^n \times X \to Y$, where X and Y are complex Banach spaces. In such a way, we continue our recent investigations of metrically Stepanov almost periodic functions [23], metrically Stepanov almost automorphic functions [16], Weyl almost automorphic type functions [33] (let us recall that the class of Weyl almost automorphic functions was introduced by S. Abbas [2] in 2012; for more details about Weyl almost periodic type functions, we refer the reader to [22]) and Bochner almost automorphic sequences [38]. We provide many illustrative examples, useful remarks and applications to the abstract fractional integro-differential equations and the abstract fractional difference equations.

The organization of this paper, which is written in a semi-heuristical manner, can be briefly described as follows. After explaining the notion and terminology used throughout the paper as well as the most important function spaces needed for our further work, we recall the basic definitions and facts about the recently considered notion of metrical Stepanov almost automorphy (see Subsection 1.1). Various classes of metrically Weyl almost automorphic type functions and metrically Weyl almost automorphic type sequences are introduced and thoroughly analyzed in Section 2. The main purpose of this section is to clarify the metrical generalizations of the structural results presented in our recent research article [33] (Proposition 2.5 and Proposition 2.7 are new; see also Example 2.8 for the slight improvements of the conclusions established in [33] as well as Remark 2.6 and Remark 2.9, where we provide several important observations about the notion under our consideration).

Section 3, where we provide many supporting examples, investigates the extensions of metrically Weyl almost automorphic sequences. The main structural results established in this section are Theorem 3.2 and Theorem 3.3. The main purpose of Section 4 is to present some applications of the obtained theoretical results to the abstract fractional integro-differential equations and the abstract fractional difference equations. In this section, we first examine the convolution invariance of joint Weyl almost automorphy and Weyl almost automorphy of type 2; see Proposition 4.1 and Theorem 4.2 for more details. The second part of this section is devoted to the study of applications to the abstract (fractional) difference equations. In the appendix section of paper, we consider vectorial Weyl almost automorphic type functions; the final section of paper is reserved for the final comments and remarks about the introduced notion.

Notation and terminology. We will always assume henceforth that $(X, \| \cdot \|)$ and $(Y, \| \cdot \|_Y)$ are complex Banach spaces, $\mathcal{B}$ is a non-empty collection of non-empty subsets of X and R is a non-empty collection of sequences in $\mathbb{R}^n$ $[\mathbb{Z}^n]$. Furthermore, we will always assume henceforth that for each $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$. By $L(X, Y)$ we denote the Banach space of all bounded linear operators from X into Y; $L(X, X) \equiv L(X)$ and I denotes the identity operator on Y. Define $\mathbb{N}_0 := \{0, 1, ..., m, ...\}$, $\mathbb{N}_m := \{1, ..., m\}$ and $\mathbb{N}_m^0 := \{0, 1, ..., m\}$ $(m \in \mathbb{N})$. If A and B are non-empty sets, then we set $\mathrm{B}^{\mathrm{A}} := \{f | f : \mathrm{A} \to \mathrm{B}\}$; $\chi_A(\cdot)$ $[A^c]$ stands for the characteristic function of the set A [the complement of A]. Define $\lfloor s \rfloor := \sup\{k \in \mathbb{Z} : s \geq k\}$ and $\lceil s \rceil := \inf\{k \in \mathbb{Z} : s \leq k\}$ $(s \in \mathbb{R})$; if $A \subseteq \mathbb{R}^n$, then its convex hull is denoted by $CH(A)$.

Assume that $0 < p < 1$ and Ω is any Lebesgue measurable subset of $\mathbb{R}^n$ with positive Lebesgue measure. Then $L^p(\Omega : X)$ consists of all Lebesgue measurable functions $f : \Omega \to X$ such that

$\int_\Omega \|f(\mathbf{u})\|^p\, d\mathbf{u} < +\infty$; the metric on $L^p(\Omega : X)$ is given by $d(f,g) := \int_\Omega \|f(\mathbf{u}) - g(\mathbf{u})\|^p\, d\mathbf{u}$ for all $f,\ g \in L^p(\Omega : X)$. Let us recall that $(L^p(\Omega : X), d)$ is a complete quasi-normed metric space; the notion and properties of the metric space $(L^p(\Omega : X), d)$ are well-known if $p \geq 1$. Concerning the Lebesgue spaces with variable exponents $L^{p(x)}$, we will use the same notion and notation as in the monograph [29] and the research article [33]. For further information in this direction, we refer the reader to the important research monograph [19] by L. Diening et al.

In this paper, we deal with the following classes of weighted function spaces:

1. Suppose that the set $I \subseteq \mathbb{R}^n$ is Lebesgue measurable and $\nu : I \to (0,\infty)$ is a Lebesgue measurable function. Of concern is the Banach space

$$L_\nu^{p(\mathbf{t})}(I : Y) := \{u : I \to Y\ ;\ u(\cdot) \text{ is measurable and } ||u||_{p(\mathbf{t})} < \infty\},$$

where $p \in \mathcal{P}(I)$, the collection of all measurable functions from I into $[1,+\infty]$, and

$$\|u\|_{p(\mathbf{t})} := \|u(\mathbf{t})\nu(\mathbf{t})\|_{L^{p(\mathbf{t})}(I:Y)}.$$

We similarly define the space $L_\nu^p(I : Y)$ with $p > 0$.

2. If $\nu : I \to (0,\infty)$ is any function such that the function $1/\nu(\cdot)$ is locally bounded, then the vector space $C_{0,\nu}(I : Y)$ $[C_{b,\nu}(I : Y)]$ consists of all continuous functions $u : I \to Y$ satisfying that $\lim_{|\mathbf{t}|\to\infty, \mathbf{t}\in I} \|u(\mathbf{t})\|_Y \nu(\mathbf{t}) = 0$ $[\sup_{\mathbf{t}\in I} \|u(\mathbf{t})\|_Y \nu(\mathbf{t}) < +\infty]$. When equipped with the norm $\|\cdot\| := \sup_{\mathbf{t}\in I} \|\cdot(\mathbf{t})\nu(\mathbf{t})\|_Y$, $C_{0,\nu}(I : Y)$ $[C_{b,\nu}(I : Y)]$ is a Banach space.

3. Suppose that $\nu : I \to [0,\infty)$ is any non-trivial function. Then we define the vector space $C_{b,\nu}(I : Y)$ as above; equipped with the pseudometric $d(\cdot,\cdot\cdot) := \sup_{\mathbf{t}\in I} \|\nu(\mathbf{t})[\cdot(\mathbf{t}) - \cdot\cdot(\mathbf{t})]\|_Y$, $(C_{b,\nu}(I : Y), d)$ is a pseudometric space.

1.1 Metrical Stepanov almost automorphy

Suppose that $\Omega \subseteq \mathbb{R}^n$ is a fixed compact set with positive Lebesgue measure. Let $Z \subseteq Y^\Omega$, $0 \in Z$ and let (Z, d_Z) be a pseudometric space. Set $\|f\|_Z := d_Z(f, 0)$, $f \in Z$.

In [16, Definition 2.1], we have recently introduced the following notion:

Definition 1.1. Suppose that $F : \mathbb{R}^n \times X \to Y$ is a given function and R is a collection of sequences in $\mathbb{R}^n$. Then we say that the function $F(\cdot;\cdot)$ is Stepanov $(\Omega, \mathrm{R}, \mathcal{B}, Z^{\mathcal{P}})$-multi-almost automorphic if and only if, for every $B \in \mathcal{B}$ and for every sequence $(\mathbf{b}_k = (b_k^1, b_k^2, \cdots, b_k^n)) \in \mathrm{R}$, there exist a subsequence $(\mathbf{b}_{k_m} = (b_{k_m}^1, b_{k_m}^2, \cdots, b_{k_m}^n))$ of $(\mathbf{b}_k)$ and a function $F_B^* : \mathbb{R}^n \times X \to Z$ such that, for every $\mathbf{t} \in \mathbb{R}^n$, $m \in \mathbb{N}$ and $x \in B$, we have $F(\mathbf{t} + \cdot + (b_{k_m}^1, \cdots, b_{k_m}^n); x) - [F_B^*(\mathbf{t}; x)](\cdot) \in Z$, $[F_B^*(\mathbf{t} - (b_{k_m}^1, \cdots, b_{k_m}^n); x)](\cdot) - F(\mathbf{t} + \cdot; x) \in Z$,

$$\lim_{m\to+\infty} \Big\| F\big(\mathbf{t} + \cdot + (b_{k_m}^1, \cdots, b_{k_m}^n); x\big) - [F_B^*(\mathbf{t}; x)](\cdot) \Big\|_Z = 0,$$

and

$$\lim_{m\to+\infty} \Big\| \big[F_B^*\big(\mathbf{t} - (b_{k_m}^1, \cdots, b_{k_m}^n); x\big)\big](\cdot) - F(\mathbf{t} + \cdot; x) \Big\|_Z = 0.$$

We omit the term "$\mathcal{B}$" from the notation for the functions of the form $F : \mathbb{R}^n \to Y$. Let $p \in (0,\infty)$; then we say that the function $F : \mathbb{R}^n \to Y$ is Stepanov-p-almost automorphic if and only if $F(\cdot)$ is

Stepanov $(\Omega, \mathrm{R}, Z^{\mathcal{P}})$-multi-almost automorphic with $\Omega = [0,1]^n$, $P = L^p([0,1]^n : Y)$ and R being the collection of all sequences in $\mathbb{R}^n$. The notion of Stepanov-p-almost automorphy with a general exponent $p > 0$ has recently been introduced by M. Kostić and W.-S. Du in [37], while the notions of Stepanov-p-almost periodicity and equi-Weyl-p-almost periodicity with a general exponent $p > 0$ were introduced by H. D. Ursell [42] in 1931. Although not directly connected with these notions, we would like to mention here the research article [25] by C. G. Gal, S. G. Gal and G. M. N'Guérékata, where the authors have considered a class of almost automorphic functions with values in p-Fréchet spaces, where $0 < p < 1$.

2 Metrically Weyl almost automorphic functions and metrically Weyl almost automorphic sequences

The main aim of this section is to introduce and analyze various classes of metrically Weyl almost automorphic functions and metrically Weyl almost automorphic sequences as well as to slightly generalize the notion introduced recently in [29, Section 8.3] (we will continue citing [29] since the paper [33] is still not published in the final form). Unless stated otherwise, we will always assume henceforth that $\Omega := [-1,1]^n \subseteq \mathbb{R}^n$ [$\Omega := [-1,1]^n \cap \mathbb{Z}^n \subseteq \mathbb{Z}^n$], $\mathbb{F} : (0,\infty) \times \mathbb{R}^n \to (0,\infty)$ [$\mathbb{F} : (0,\infty) \times \mathbb{Z}^n \to (0,\infty)$] as well as that, for every $l > 0$, (P_l, d_l) is a pseudometric space, where $P_l \subseteq Y^{l\Omega}$ [$P_l \subseteq Y^{l\Omega \cap \mathbb{Z}^n}$] is closed under the addition and subtraction of functions, containing the zero-function: $0 \in P_l$. Set $\|f\|_l := d_l(f, 0)$ for all $f \in P_l$ $(l > 0)$. We will always assume henceforth that R is a collection of sequences in $\mathbb{R}^n$ [$\mathbb{Z}^n$]; for simplicity and better understanding, we will not consider here the corresponding classes of functions with the collections R_X of sequences in $\mathbb{R}^n \times X$ [$\mathbb{Z}^n \times X$].

We will first introduce the following notion, which generalizes the corresponding notion from [29, Definition 8.3.17, Definition 8.3.18, Definition 8.3.28]:

Definition 2.1. Suppose that $F : \mathbb{R}^n \times X \to Y$ [$F : \mathbb{Z}^n \times X \to Y$] satisfies that for each $x \in X$, $l > 0$ and $\mathbf{t} \in \mathbb{R}^n$ [$\mathbf{t} \in \mathbb{Z}^n$] we have $F(\mathbf{t} + \cdot; x) \in P_l$. Let for every $l > 0$, $B \in \mathcal{B}$ and $(\mathbf{b}_k = (b_k^1, b_k^2, \cdot\cdot\cdot, b_k^n)) \in \mathrm{R}$ there exist a subsequence $(\mathbf{b}_{k_m} = (b_{k_m}^1, b_{k_m}^2, \cdot\cdot\cdot, b_{k_m}^n))$ of $(\mathbf{b}_k)$ and a function $F^* : \mathbb{R}^n \times X \to P_l$ [$F^* : \mathbb{Z}^n \times X \to P_l$] such that for each $x \in B$, $l > 0$ and $\mathbf{t} \in \mathbb{R}^n$ [$\mathbf{t} \in \mathbb{Z}^n$] we have:

(i)

$$\lim_{m\to+\infty} \lim_{l\to+\infty} \mathbb{F}(l,\mathbf{t}) \Big\| F(\mathbf{t} + \cdot + (b_{k_m}^1, \cdot\cdot\cdot, b_{k_m}^n); x) - \big[F^*(\mathbf{t}; x)\big](\cdot) \Big\|_l = 0 \tag{2.1}$$

and

$$\lim_{m\to+\infty} \lim_{l\to+\infty} \mathbb{F}(l,\mathbf{t}) \Big\| \big[F^*(\mathbf{t} - (b_{k_m}^1, \cdot\cdot\cdot, b_{k_m}^n); x)\big](\cdot) - F(\mathbf{t} + \cdot; x) \Big\|_l = 0, \tag{2.2}$$

pointwise for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$ [$\mathbf{t} \in \mathbb{Z}^n$], or

(ii)

$$\lim_{l\to+\infty} \lim_{m\to+\infty} \mathbb{F}(l,\mathbf{t}) \Big\| F(\mathbf{t} + \cdot + (b_{k_m}^1, \cdot\cdot\cdot, b_{k_m}^n); x) - \big[F^*(\mathbf{t}; x)\big](\cdot) \Big\|_l = 0 \tag{2.3}$$

and

$$\lim_{l\to+\infty} \lim_{m\to+\infty} \mathbb{F}(l,\mathbf{t}) \Big\| \big[F^*(\mathbf{t} - (b_{k_m}^1, \cdot\cdot\cdot, b_{k_m}^n); x)\big](\cdot) - F(\mathbf{t} + \cdot; x) \Big\|_l = 0, \tag{2.4}$$

pointwise for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$ [$\mathbf{t} \in \mathbb{Z}^n$], or

(iii)

$$\lim_{(l,m)\to+\infty}\mathbb{F}(l,\mathbf{t})\Bigl\|F(\mathbf{t}+\cdot+(b_{k_m}^1,\cdot\cdot\cdot,b_{k_m}^n);x)-\bigl[F^*(\mathbf{t};x)\bigr](\cdot)\Bigr\|_l=0$$

and

$$\lim_{(l,m)\to+\infty}\mathbb{F}(l,\mathbf{t})\Bigl\|\bigl[F^*(\mathbf{t}-(b_{k_m}^1,\cdot\cdot\cdot,b_{k_m}^n);x)\bigr](\cdot)-F(\mathbf{t}+\cdot;x)\Bigr\|_l=0,$$

pointwise for all $x\in B$ and $\mathbf{t}\in\mathbb{R}^n$ [$\mathbf{t}\in\mathbb{Z}^n$].

In the case that (i), resp. [(ii); (iii)], holds, then we say that $F(\cdot;\cdot)$ is Weyl-$(\mathbb{F},\mathcal{P},\mathrm{R},\mathcal{B})$-multi-almost automorphic, resp. [Weyl-$(\mathbb{F},\mathcal{P},\mathrm{R},\mathcal{B})$-multi-almost automorphic of type 1; jointly Weyl-$(\mathbb{F},\mathcal{P},\mathrm{R},\mathcal{B})$-multi-almost automorphic].

Definition 2.2. Suppose that $\emptyset\neq W\subseteq\mathbb{R}^n$ [$\emptyset\neq W\subseteq\mathbb{Z}^n$], $\mathbb{F}:(0,\infty)\times\mathbb{R}^n\to(0,\infty)$ [$\mathbb{F}:(0,\infty)\times\mathbb{Z}^n\to(0,\infty)$] and $F:\mathbb{R}^n\times X\to Y$ [$F:\mathbb{Z}^n\times X\to Y$] satisfies that for each $x\in X$, $l>0$ and $\mathbf{t}\in\mathbb{R}^n$ [$\mathbf{t}\in\mathbb{Z}^n$] we have $F(\mathbf{t}+\cdot;x)\in P_l$. If for every $B\in\mathcal{B}$ and $(\mathbf{b}_k=(b_k^1,b_k^2,\cdot\cdot\cdot,b_k^n))\in\mathrm{R}$ there exists a subsequence $(\mathbf{b}_{k_m}=(b_{k_m}^1,b_{k_m}^2,\cdot\cdot\cdot,b_{k_m}^n))$ of $(\mathbf{b}_k)$ such that for each $\epsilon>0$, $x\in B$ and $\mathbf{t}\in\mathbb{R}^n$ [$\mathbf{t}\in\mathbb{Z}^n$] there exists $m_0\in\mathbb{N}$ such that, for every $m,\ m'\in\mathbb{N}$ with $m\geq m_0$ and $m'\geq m_0$, there exists $l_0>0$ such that, for every $l\geq l_0$ and $w\in lW$, we have

$$\Bigl\|F\bigl(\mathbf{t}+\cdot+(b_{k_m}^1,\cdot\cdot\cdot,b_{k_m}^n)-w;x\bigr)-F\bigl(\mathbf{t}+\cdot+(b_{k_{m'}}^1,\cdot\cdot\cdot,b_{k_{m'}}^n)-w;x\bigr)\Bigr\|_l<\epsilon/\mathbb{F}(l,\mathbf{t}-w),$$

then we say that $F(\cdot;\cdot)$ is Weyl-$(\mathbb{F},\mathcal{P},\mathrm{R},\mathcal{B},W)$-multi-almost automorphic of type 2.

Definition 2.3. Suppose that $\emptyset\neq W\subseteq\mathbb{R}^n$ [$\emptyset\neq W\subseteq\mathbb{Z}^n$], $\mathbb{F}:(0,\infty)\times\mathbb{R}^n\to(0,\infty)$ [$\mathbb{F}:(0,\infty)\times\mathbb{Z}^n\to(0,\infty)$] and $F:\mathbb{R}^n\times X\to Y$ [$F:\mathbb{Z}^n\times X\to Y$] satisfies that for each $x\in X$, $l>0$ and $\mathbf{t}\in\mathbb{R}^n$ [$\mathbf{t}\in\mathbb{Z}^n$] we have $F(\mathbf{t}+\cdot;x)\in P_l$. Let for every $l>0$, $B\in\mathcal{B}$ and $(\mathbf{b}_k=(b_k^1,b_k^2,\cdot\cdot\cdot,b_k^n))\in\mathrm{R}$ there exist a subsequence $(\mathbf{b}_{k_m}=(b_{k_m}^1,b_{k_m}^2,\cdot\cdot\cdot,b_{k_m}^n))$ of $(\mathbf{b}_k)$ and a function $F^*:\mathbb{R}^n\times X\to P_l$ [$F^*:\mathbb{Z}^n\times X\to P_l$] such that for each $\epsilon>0$, $x\in B$ and $\mathbf{t}\in\mathbb{R}^n$ [$\mathbf{t}\in\mathbb{Z}^n$], there exists $p>0$ such that, for every $l\in[p,+\infty)$, $m\in\mathbb{N}$ with $m\geq p$ and $w\in lW$, we have

$$\mathbb{F}(l,\mathbf{t}-w)\Bigl\|F(\mathbf{t}+\cdot+(b_{k_m}^1,\cdot\cdot\cdot,b_{k_m}^n)-w;x)-\bigl[F^*(\mathbf{t}-w;x)\bigr](\cdot)\Bigr\|_l<\epsilon$$

and

$$\mathbb{F}(l,\mathbf{t}-w)\Bigl\|\bigl[F^*(\mathbf{t}-(b_{k_m}^1,\cdot\cdot\cdot,b_{k_m}^n)-w;x)\bigr](\cdot)-F(\mathbf{t}+\cdot-w;x)\Bigr\|_l<\epsilon,$$

then we say that $F(\cdot;\cdot)$ is jointly Weyl-$(\mathbb{F},\mathcal{P},\mathrm{R},\mathcal{B},W)$-multi-almost automorphic.

In the recent research paper [32], we have considered the Besicovitch multi-dimensional almost automorphic type functions and their applications. This is the first paper in the existing literature dealing with the notion of generalized almost automorphy which additionally involves the growth order of limit function $F^*(\cdot;\cdot)$; cf. [32, Definition 3.1]. We will consider henceforth the notion in which the limit function $F^*(\cdot;\cdot)$ from Definition 2.1, resp. Definition 2.3, is bounded by the function $\omega:\mathbb{R}^n\to(0,\infty)$ [$\omega:\mathbb{Z}^n\to(0,\infty)$] in the sense that there exists $M>0$ such that, for every $x\in B$, $l>0$ and $\mathbf{u}\in l\Omega$, we have $\|[F^*(\mathbf{t};x)](\mathbf{u})\|_Y\leq M\omega(|\mathbf{t}|)$, $\mathbf{t}\in\mathbb{R}^n$ [$\|[F^*(\mathbf{t};x)](\mathbf{u})\|_Y\leq M\omega(|\mathbf{t}|)$, $\mathbf{t}\in\mathbb{Z}^n$]. If this is the case, then we say that the function $F(\cdot;\cdot)$ is Weyl-$(\mathbb{F},\mathcal{P},\mathrm{R},\mathcal{B},\omega)$-multi-almost automorphic

[Weyl-$(\mathbb{F}, \mathcal{P}, \mathrm{R}, \mathcal{B}, \omega)$-multi-almost automorphic of type 1; jointly Weyl-$(\mathbb{F}, \mathcal{P}, \mathrm{R}, \mathcal{B}, \omega)$-multi-almost automorphic], resp., jointly Weyl $(\mathbb{F}, \mathcal{P}, \mathrm{R}, \mathcal{B}, W, \omega)$-multi-almost automorphic; furthermore, if $\omega(\cdot) \equiv 1$, then we write "b" in place of "ω".

If $1 \leq p < \infty$, then the notion of a (jointly) Weyl-p-almost automorphic function (of type 1) [(jointly) Weyl-(p, b)-multi-almost automorphic (of type 1)] $F : \mathbb{R}^n \to Y$ is obtained with $\Omega = [-1, 1]^n$, R being the collection of all sequnces in $\mathbb{R}^n$, $\mathbb{F}(l, \mathbf{t}) \equiv l^{-n/p}$ and $P_l = L^p([-l, l]^n : Y)$; the notion of a (jointly) Weyl-p-almost automorphic sequence (of type 1) [(jointly) Weyl-(p, b)-multi-almost automorphic (of type 1)] $F : \mathbb{Z}^n \to Y$ is new and can be obtained with $\Omega = [-1, 1]^n$, R being the collection of all sequences in $\mathbb{Z}^n$, $\mathbb{F}(l, \mathbf{t}) \equiv l^{-n/p}$ and $P_l = L^p([-l, l]^n \cap \mathbb{Z}^n : Y)$ [cf. the first term in the equation (2.6) below with $p_l \equiv p$ and $\nu_l \equiv 1$]. We similarly define the notion of (joint) Weyl-(p, R)-almost automorphy (of type 1) [(joint) Weyl-(p, R, b)-almost automorphy (of type 1)], where R is a general collection of sequences obeying our requirements.

We continue by introducing the corresponding notion in which $0 < p < 1$ (cf. also [30, Subsection 4.3.1] for the notion of metrical Weyl distance; if $1 \leq p < \infty$, then we have $\mathbb{F}(l, \mathbf{t}) \equiv l^{-n/p}$):

Definition 2.4. Suppose that $0 < p < 1$ and $F : \mathbb{R}^n \to Y$ [$F : \mathbb{Z}^n \to Y$]. Then we say that $F(\cdot)$ is (jointly) Weyl-p-almost automorphic function (of type 1) [(jointly) Weyl-(p, b)-almost automorphic sequence (of type 1)] if and only if $F(\cdot)$ is (jointly) Weyl-$(\mathbb{F}, \mathcal{P}, \mathrm{R})$-multi-almost automorphic (of type 1) [(jointly) Weyl-$(\mathbb{F}, \mathcal{P}, \mathrm{R}, b)$-multi-almost automorphic (of type 1)], where $\Omega = [-1, 1]^n$, R is the collection of all sequences in $\mathbb{R}^n$ [$\mathbb{Z}^n$], $\mathbb{F}(l, \mathbf{t}) \equiv l^{-n}$ and $P_l = L^p([-l, l]^n : Y)$ [$P_l = l^p([-l, l]^n \cap \mathbb{Z}^n : Y)$; cf. the second term in the equation (2.6) below with $\nu_l \equiv 1$]. If R is a general collection of sequences obeying our requirements, then we also say that $F(\cdot)$ is (jointly) Weyl-(p, R)-almost automorphic (of type 1) [(jointly) Weyl-(p, R, b)-almost automorphic (of type 1)].

We can further generalize the notion introduced in the above three definitions following our approach from [29, Definition 8.1.2]; cf. also [29, Remark 8.3.19(i)] and [32, Example 2.7]. The notion in which $P_l = L^p([-l, l]^n : Y)$ [$P_l = l^p([-l, l]^n \cap \mathbb{Z}^n : Y)$] and $\mathbb{F}(l, \mathbf{t}) \equiv l^{-n/p}$, if $1 \leq p < +\infty$, resp., $\mathbb{F}(l, \mathbf{t}) \equiv l^{-n}$, if $0 < p < 1$, is extremely important; if this is the case, then we have the following result which can be simply formulated for the general function spaces introduced in Definition 2.1-Definition 2.3, as well:

Proposition 2.5. *Let P_l and $\mathbb{F}(l, \cdot)$ be as above ($l > 0$). Then the following holds:*

- *(i) Suppose that $0 < p \leq q < +\infty$. Then any (jointly) Weyl-q-almost automorphic function $F : \mathbb{R}^n \to Y$ [$F : \mathbb{Z}^n \to Y$] (of type 1) is (jointly) Weyl-p-almost automorphic (of type 1); furthermore, the same holds for the corresponding classes of (jointly) Weyl-(q, b)-almost automorphic functions (of type 1) and (jointly) Weyl-(p, b)-almost automorphic functions (of type 1).*
- *(ii) Suppose that $0 < p \leq q < +\infty$. Then $F : \mathbb{R}^n \to Y$ [$F : \mathbb{Z}^n \to Y$] is essentially bounded, (jointly) Weyl-(q, b)-almost automorphic (of type 1) if and only if $F(\cdot)$ is essentially bounded, (jointly) Weyl-(p, b)-almost automorphic (of type 1).*

Proof. The proof of (i) is very simple and follows from an elementary application of the Hölder inequality. Keeping this in mind, the statement (ii) follows immediately if we prove that any essentially bounded, (jointly) Weyl-(p, b)-almost automorphic function $F : \mathbb{R}^n \to Y$ [$F : \mathbb{Z}^n \to Y$] (of type

1) is (jointly) Weyl-(q,b)-almost automorphic (of type 1). For the sake of brevity, we will consider here the essentially bounded, jointly Weyl-(p,b)-almost automorphic functions $F:\mathbb{R}^n\rightarrow Y$, only. Let (b_k) be a given sequence. Then, for every $l>0$, there exist a subsequence (b_{k_m}) of (b_k), a function $F^*:\mathbb{R}^n\rightarrow L^p([-l,l]^n:Y)$ and a finite real number $M>0$ such that $\|[F^*(\mathbf{t})](x)\|_Y\leq M$ for all $\mathbf{t}\in\mathbb{R}^n$ and $x\in[-l,l]^n$ as well as that, for every $\mathbf{t}\in\mathbb{R}^n$ and $\epsilon>0$, there exists $s>0$ such that, for every $l\geq s$ and for every $m\in\mathbb{N}$ with $m\geq s$, we have

$$l^{-n}\int_{[-l,l]^n}\Bigl\|F\bigl(\mathbf{t}+x+b_{b_m}\bigr)-\bigl[F^*(\mathbf{t})\bigr](x)\Bigr\|_Y^p\,dx<\epsilon$$

and

$$l^{-n}\int_{[-l,l]^n}\Bigl\|\bigl[F^*(\mathbf{t}-b_{b_m})\bigr](x)-F\bigl(\mathbf{t}+x\bigr)\Bigr\|_Y^p\,dx<\epsilon. \tag{2.5}$$

Then the final conclusion simply follows from the estimate

$$l^{-n}\int_{[-l,l]^n}\Biggl(\Bigl\|F\bigl(\mathbf{t}+x+b_{b_m}\bigr)-\bigl[F^*(\mathbf{t})\bigr](x)\Bigr\|_Y/\bigl(\|F\|_\infty+M\bigr)\Biggr)^q dx$$
$$\leq l^{-n}\int_{[-l,l]^n}\Biggl(\Bigl\|F\bigl(\mathbf{t}+x+b_{b_m}\bigr)-\bigl[F^*(\mathbf{t})\bigr](x)\Bigr\|_Y/\bigl(\|F\|_\infty+M\bigr)\Biggr)^p dx$$

and the corresponding estimate for the term appearing in (2.5). □

We continue by providing some useful observations:

Remark 2.6. (i) We omit the term "$\mathcal{B}$" from the notation for the functions [sequences] of the form $F:\mathbb{R}^n\rightarrow Y$ [$F:\mathbb{Z}^n\rightarrow Y$]; furthermore, we omit the term "R" if R denotes the collection of all sequences in $\mathbb{R}^n$ [$\mathbb{Z}^n$], and we omit the term "-multi" if $n=1$.

(ii) If we consider the continuous notion from the above three definitions, then the very natural extension of the notion introduced in [29, Definition 8.3.17, Definition 8.3.18, Definition 8.3.28] can be obtained by setting $\|f\|_l\equiv\|f\|_{L^{p_l(\mathbf{t})}_{\nu_l}(l\Omega:Y)}$, where $p_l\in\mathcal{P}(l\Omega)$ [$p_l\equiv p\in(0,1)$ for all $l>0$] and $\nu_l:l\Omega\rightarrow(0,\infty)$ is a Lebesgue measurable function, or $\|f\|_l\equiv\|f\|_{C_{b,\nu_l}(l\Omega:Y)}$, where $\nu_l:l\Omega\rightarrow[0,\infty)$ is any non-zero function ($l>0$). Concerning the discrete notion from the above-mentioned definitions, the natural choices for $\|f\|_l$ can be obtained by setting

$$\|f\|_l\equiv\Biggl[\sum_{j\in l\Omega\cap\mathbb{Z}^n}\bigl\|\nu_l(j)f(j)\bigr\|^{p_l}\Biggr]^{1/p_l}\ \Biggl[\|f\|_l\equiv\sum_{j\in l\Omega\cap\mathbb{Z}^n}\bigl\|\nu_l(j)f(j)\bigr\|^{p}\Biggr], \tag{2.6}$$
$$\text{or } \|f\|_l\equiv\sup_{j\in l\Omega\cap\mathbb{Z}}\bigl\|\nu_l(j)f(j)\bigr\|,$$

where $1\leq p_l<+\infty$ [$p_l\equiv p\in(0,1)$ for all $l>0$] and $\nu_l:l\Omega\cap\mathbb{Z}^n\rightarrow[0,\infty)$ is any non-zero function ($l>0$).

(iii) The notions introduced in the above definitions provide a very general approach to Weyl-p-almost automorphy. For example, if we assume that for each $l>0$ we have $P_l=L^\infty(l\Omega:Y)$ or

$P_l = L^p_\nu(l\Omega : Y)$ with $\nu \in L^p(\mathbb{R}^n : (0,\infty))$ and $p > 0$ as well as that for each $x \in X$ the function $F(\cdot; x)$ is bounded and for each $t \in \mathbb{R}^n$ we have $\lim_{l\to+\infty} \mathbb{F}(l, \mathbf{t}) = 0$, then the function $F(\cdot;\cdot)$ is (jointly) Weyl-$(\mathbb{F}, \mathcal{P}, \mathrm{R}, \mathcal{B})$-multi-almost automorphic [Weyl-$(\mathbb{F}, \mathcal{P}, \mathrm{R}, \mathcal{B})$-multi-almost automorphic of type 1, provided that the limits $\lim_{m\to+\infty} \cdot$ in the equations (2.3)-(2.4) exist], which can be shown by plugging $F^* \equiv F$.

In [33], we have observed that any Stepanov-p-almost automorphic function $f : \mathbb{R} \to Y$ is Weyl-p-almost automorphic of type 1. This result can be simply extended as follows. Suppose that the requirements clarified directly before Definition 1.1 hold with $\Omega = [-1,1]^n$ and $G : (0,\infty) \to (0,\infty)$. Let for each $l > 0$ the non-empty set $Z_l \subseteq Y^{l\Omega}$ be defined as the set of all functions $f : [-l,l]^n \to Y$ such that, for every cube of the form $k + [-1,1]^n$ which belongs to $[-l,l]^n$, where $k \in (2\mathbb{Z}+1)^n$, we have that the restriction of function $f(\cdot - k)$ to the set $[-1,1]^n$ belongs to P; suppose, further, that the set Z_l is equipped with any pseudometric $d_l(\cdot;\cdot)$ such that

$$d_l(f,g) \le G(l) \sum_k d\Big(f(\cdot - k)_{|[-1,1]^n}, g(\cdot - k)_{|[-1,1]^n}\Big), \quad f,\ g \in Z_l,\ l > 0,$$

where the summation is taken over all points $k \in (2\mathbb{Z}+1)^n$ such that $k + [-1,1]^n \subseteq [-l,l]^n$. If for each $l > 0$ there exists a finite $c_l > 0$ such that $\mathbb{F}(l,\mathbf{t})G(l) \le \mathrm{c}_l$ for all $\mathbf{t} \in \mathbb{R}^n$, then the second limits in (2.3)-(2.4) are equal to zero for every fixed $l > 0$, and therefore, we can clarify our second structural result:

Proposition 2.7. *Suppose that $F : \mathbb{R}^n \times X \to Y$ is Stepanov $([-1,1]^n, \mathrm{R}, \mathcal{B}, Z^{\mathcal{P}})$-multi-almost automorphic and for each $l > 0$ there exists a finite $c_l > 0$ such that $\mathbb{F}(l,\mathbf{t})G(l) \le c_l$ for all $\mathbf{t} \in \mathbb{R}^n$. Then $F(\cdot;\cdot)$ is Weyl-$(\mathbb{F}, \mathcal{P}, \mathrm{R}, \mathcal{B})$-multi-almost automorphic of type 1, with $\mathcal{P}$ being defined as above.*

The class of Stepanov-p-almost periodic functions with a general exponent $p > 0$ has been systematically analyzed in the joint paper of the second named author and W.-S. Du [37]. With the help of the metrical Bochner criterion [23, Theorem 3.3], we can simply prove that any Stepanov-p-almost periodic function $F : \mathbb{R}^n \to Y$, where $p > 0$, is Weyl-p-almost automorphic, Weyl-p-almost automorphic of type 1, as well as jointly Weyl-p-almost automorphic in the usual sense, with the limit function $F^* \equiv F$; the converse statement is not true since there exists a jointly Weyl-p-almost automorphic function $f \in L^p_{loc}(\mathbb{R} : \mathbb{R})$ which is not Stepanov-p-almost automorphic ([33]). Furthermore, the joint Weyl-p-almost automorphy of $F : \mathbb{R}^n \times X \to Y$ [$F : \mathbb{Z}^n \times X \to Y$] implies its Weyl-$p$-almost automorphy provided that for each $k \in \mathbb{N}$ the both limits in the equations (2.1)-(2.2) exist as $l \to +\infty$; a similar comment can be given for the Weyl-p-almost automorphy of type 1.

We continue by providing some illustrative examples:

Example 2.8. (i) In [29, Theorem 8.3.8], we have considered the function $f(x) := |x|^\sigma$, $x \in \mathbb{R}$, where $\sigma \in (0,1)$, $p \in [1,\infty)$ and $(1-\sigma)p < 1$. Among many other conclusions, we have deduced there that the function $f(\cdot)$ is Weyl-p-almost automorphic, not Weyl-p-almost automorphic of type 1 nor joint Weyl-p-almost automorphic.

Let us consider now case in which $\sigma > 0$, $p \in (0,1)$ and $a > 1 - (1-\sigma)p > 0$. Concerning the Weyl-p-almost automorphy of $f(\cdot)$, we would like to observe here that the same argumentation as in the proof of the above-mentioned theorem shows that $f(\cdot)$ is Weyl-$(\mathbb{F}, \mathcal{P}, \mathrm{R})$-multi-almost automorphic, where $\mathbb{F}(l,t) \equiv l^{-a}$, $P_l \equiv L^p([-l,l] : Y)$ and R is the collection of all real sequences.

Furthermore, if $\sigma \geq 1$, $\nu \in L^{\infty}[-l,l]$ for all $l > 0$ and $\lim_{l\to+\infty} l^{\sigma-1}\mathbb{F}(l) = 0$, then the function $f(\cdot)$ is Weyl-$(\mathbb{F},\mathcal{P},\mathrm{R})$-multi-almost automorphic, where $\mathbb{F}(l,t) \equiv \mathbb{F}(l)$, $P_l \equiv C_{b,\nu}([-l,l] : Y)$ for all $l > 0$ and R is the collection of all real sequences; in order to see this, we can apply the Lagrange mean value theorem and the following simple computation ($w \in \mathbb{R}$ and $l > 0$ are arbitrary, $f^* \equiv f$):

$$\begin{aligned}
&\mathbb{F}(l) \sup_{x\in[-l,l]} |f(t+w+x) - f(t+x)| \cdot \nu(x) \\
&= \mathbb{F}(l) \sup_{x\in[-l,l]} \Big| \big|t+w+x\big|^{\sigma} - \big|t+x\big|^{\sigma} \Big| \cdot \nu(x) \\
&\leq \sigma|w|\mathbb{F}(l) \sup_{x\in[-l,l]} \sup_{y\in[|t+x|,|t+x+\omega|]\cup[|t+x+\omega|,|t+x|]} y^{\sigma-1} \cdot \nu(x) \\
&\leq \sigma|w|\mathbb{F}(l) \sup_{x\in[-l,l]} \Big[\big|t+x\big|^{\sigma-1} + \big|t+x+w\big|^{\sigma-1} \Big] \cdot \nu(x) \\
&\leq \sigma|w|\mathbb{F}(l) \sup_{x\in[-l,l]} \Big[2\big|t\big|^{\sigma-1} + \big|w\big|^{\sigma-1} + 2\big|x\big|^{\sigma-1} \Big] \cdot \nu(x).
\end{aligned}$$

(ii) Let $p > 0$. Arguing as in the proof of [29, Theorem 8.3.10], we have that the Heaviside function $\chi_{[0,\infty)}(\cdot)$ is not jointly Weyl-p-almost automorphic as well as that $\chi_{[0,\infty)}(\cdot)$ is both Weyl-p-almost automorphic and Weyl-p-almost automorphic of type 1; furthermore, we can similarly prove the following:

(a) The Heaviside function $\chi_{[0,\infty)}(\cdot)$ is not jointly Weyl-$(\mathbb{F},\mathcal{P})$-almost automorphic if $\mathbb{F}(l) \equiv 1/l$ and $P_l \equiv L^p_{\nu}([-l,l] : \mathbb{C})$ for all $l > 0$, where $\nu : \mathbb{R} \to (0,\infty)$ is any Lebesgue measurable function such that $\limsup_{l\to+\infty}(1/l)\int_{-l}^{0} \nu^p(x)\,dx > 0$.

(b) The Heaviside function $\chi_{[0,\infty)}(\cdot)$ is Weyl-$(\mathbb{F},\mathcal{P})$-almost automorphic if $\lim_{l\to+\infty}\mathbb{F}(l) = 0$ and $P_l \equiv L^p_{\nu}([-l,l] : \mathbb{C})$ for all $l > 0$, where $\nu : \mathbb{R} \to (0,\infty)$ is any p-locally integrable function.

We can similarly consider the multi-dimensional analogue of this example and provide the basic information about the metrical Weyl-p-almost automorphic properties of the function $\chi_K(\cdot)$, where K is a non-empty compact subset of $\mathbb{R}^n$; cf. [29, Example 8.3.21] for more details in this direction.

(iii) In [29, Example 8.3.20], we have reconsidered the well-known example proposed by J. Stryja [41, pp. 42–47]; see also [10, Example 4.28]: Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) := 0$ for $x \leq 0$, $f(x) := \sqrt{n/2}$ if $x \in (n-2, n-1]$ for some $n \in 2\mathbb{N}$ and $f(x) := -\sqrt{n/2}$ if $x \in (n-1, n]$ for some $n \in 2\mathbb{N}$. We have shown that the function $f(\cdot)$ is not (jointly) Weyl 1-almost automorphic (of type 1; of type 2). On the other hand, we have $|f(x)| \leq 2^{-1/2}\sqrt{|x|}$, $x \in \mathbb{R}$ so that a simple computation shows that the function $f(\cdot)$ is $(\mathbb{F},\mathcal{P})$-almost automorphic, provided that for each $l > 0$ we have $P_l = L^p_{\nu}([-l,l])$ with some $p > 0$ and a Lebesgue measurable function $\nu(\cdot)$ such that

$$\lim_{l\to+\infty} \mathbb{F}(l) \int_{-l}^{l} \Big(1 + |x|^{p/2}\Big) \nu^p(x)\,dx = 0.$$

Furthermore, we can simply prove that the function $f(\cdot)$ is $(\mathbb{F},\mathcal{P},\mathrm{R})$-almost automorphic, provided that R is the collection of all real sequences (a_m) satisfying that $a_m \in 2\mathbb{N}$ for all $m \in \mathbb{N}$ as well as that for each $l > 0$ we have $P_l = L^p_{\nu}([-l,l])$ with some $p > 0$ and a Lebesgue measurable function $\nu(\cdot)$ such that $\lim_{l\to+\infty}\mathbb{F}(l)\int_{-l}^{l} \nu^p(x)\,dx = 0$.

(iv) In [32, Example 2.3] (cf. also [30, Example 3.4.4]), we have analyzed the Weyl-p-almost automorphic properties of the function $f : \mathbb{R} \to l_\infty$, given by $f(t) := (e^{-|t|/k})_{k\in\mathbb{N}}$, $t \in \mathbb{R}$. Since this function is slowly oscillating, it can be simply shown that the function $f(\cdot)$ is Weyl-$(\mathbb{F}, \mathcal{P})$-almost automorphic provided that $\lim_{l\to+\infty} \mathbb{F}(l) = 0$, there exist $l_0 > 0$ and $M > 0$ such that, for every $l \geq l_0$, we have $\mathbb{F}(l) \int_{-l}^{l} \nu^p(x)\, dx \leq M$ and $P_l = L^p_\nu([-l, l])$ for some $p > 0$ $(l > 0)$ and a p-locally integrable function $\nu : \mathbb{R} \to (0,\infty)$. Moreover, we can simply prove that the function $f(\cdot)$ is Weyl-$(\mathbb{F}, \mathcal{P})$-almost automorphic of type 1 provided that $\nu : \mathbb{R} \to (0,\infty)$ is a p-locally integrable function and $P_l = L^p_\nu([-l, l])$ for some $p > 0$ $(l > 0)$. In connection with this example, we would like to stress the following:

(a) Clearly, $\|f(t)\| = 1$ for all $t \in \mathbb{R}$ so that $\lim_{t\to+\infty} t^{-1} \int_0^t \|f(s)\|\, ds = 1 \neq 0$, as it has been mistakenly written.

(b) Concerning the Weyl-p-almost automorphy of the function $f(\cdot)$, we should use the limit function $f^* \equiv f$, if (b_k) is a sequence tending to plus infinity or minus infinity (not $f^* \equiv 0$).

(c) The corresponding statement for the joint Weyl-p-almost automorphy of the function $f(\cdot)$ is not true.

We continue by providing the following useful observations:

Remark 2.9. (i) The result of [29, Proposition 8.3.9] remains true in the higher-dimensional setting so that any Weyl-p-almost automorphic function of type 1 (jointly Weyl-p-almost automorphic function) $F : \mathbb{R}^n \to Y$ must be Stepanov-p-bounded, i.e., $\sup_{\mathbf{t}\in\mathbb{R}^n} \int_{\mathbf{t}+[0,1]^n} \|F(\mathbf{t})\|^p\, d\mathbf{t} < +\infty$ $(p > 0$; cf. also Proposition 2.5(i)). Furthermore, in the discrete setting, a similar argumentation shows that any Weyl-p-almost automorphic sequence of type 1 (jointly Weyl-p-almost automorphic sequence) $F : \mathbb{Z}^n \to Y$, where $p > 0$, must be bounded.

(ii) The statement of [29, Proposition 8.3.13] can be also formulated in the higher-dimensional setting, with the usage of spaces $L^p_\nu(l[-1, 1]^n : Y)$, where $p > 0$ and $\nu : \mathbb{R}^n \to (0,\infty)$ is any Lebesgue measurable function. Details can be left to the interested readers (cf. also [29, Question 8.3.14, Question 8.3.15]).

Concerning [29, Example 8.3.16], we would like to recall that A. Haraux and P. Souplet have considered, in [27, Theorem 1.1], the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(t) := \sum_{m=1}^{\infty} \frac{1}{m} \sin^2\Big(\frac{t}{2^m}\Big), \quad t \in \mathbb{R}.$$

We already know that this function is (Besicovitch) unbounded and Weyl p-almost automorphic for any finite exponent $p \geq 1$ as well as that for each $\tau \in \mathbb{R}$ the function $f(\cdot + \tau) - f(\cdot)$ is almost anti-periodic (see [28] for the notion). Due to Proposition 2.5(i), this clearly implies that the function $f(\cdot)$ is Weyl-p-almost automorphic for any finite exponent $p > 0$. Here we would like to note that the function $f(\cdot)$ has bounded differences, i.e., for each $\tau \in \mathbb{R}$ we have that the function $f(\cdot + \tau) - f(\cdot)$ is bounded. Every such a Lebesgue measurable function has to be Weyl-$(\mathbb{F}, \mathcal{P})$-almost automorphic, where $p > 0$ and $P_l = L^p_\nu([-l, l])$ for some p-locally integrable function $\nu(\cdot)$ satisfying that

$$\lim_{l\to+\infty} \mathbb{F}(l, t) \int_{-l}^{l} \nu^p(x)\, dx = 0, \quad t \in \mathbb{R};$$

cf. also [29, p. 63] for more details about functions with bounded differences.

Before proceeding to the next section, we would like to observe that the statements of [29, Proposition 8.3.23, Proposition 8.3.24] can be simply formulated for the function spaces introduced in this paper. The interested reader may also try to extend the statement of [29, Proposition 8.3.30] concerning the pointwise products of metrically Weyl almost automorphic type functions.

3 Extensions of metrically Weyl almost automorphic sequences

This section aims to provide the basic information about the extensions of metrically Weyl almost automorphic sequences. At the very beginning, we would like to emphasize that the statements of [38, Theorem 2.3, Theorem 2.5] cannot be so simply formulated for the general classes of metrically Weyl almost automorphic sequences and functions. Discretization of Weyl almost automorphic type functions is a completely new topic and here we will first present some positive results in this direction, which can be approved similarly as in [33] (we will also consider non-continuous functions; for more details about discretization of generalized almost periodic type functions and generalized almost automorphic type functions, we refer the reader to [11, 16, 20, 38]):

Example 3.1. Suppose that $p \geq 1$.

(i) If K is any non-empty compact subset of $\mathbb{R}^n$, then the function $\chi_K(\cdot)$ is (jointly) Weyl-p-almost automorphic (of type 1). The same holds for the sequence $(\chi_K(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^n}$.

(ii) We already know that the function $\chi_{[0,\infty)^n}(\cdot)$ is not jointly Weyl-p-almost automorphic, not Weyl-p-almost automorphic of type 1 as well as that the function $\chi_{[0,\infty)^n}(\cdot)$ is Weyl-p-almost automorphic. The same holds for the sequence $(\chi_{[0,\infty)^n}(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^n}$.

(iii) All conclusions established for the function $x \mapsto f(x) \equiv |x|^\sigma$, $x \in \mathbb{R}$, where $\sigma \in (0,1)$ and $(1-\sigma)p < 1$, remain true for the sequence $(f(k))_{k\in\mathbb{Z}}$; cf. [29, Theorem 8.3.8] for more details.

Concerning some negative results in this direction, we will only observe that J. Andres and D. Pennequin have constructed, in [11, Example 4], an infinitely differentiable Stepanov-1-almost periodic function $f : \mathbb{R} \to \mathbb{R}$ such that the sequence $(f(k))_{k\in\mathbb{Z}}$ is not bounded. Therefore, the function $f(\cdot)$ is (jointly) Weyl-1-almost automorphic (of type 1), but the sequence $(f(k))_{k\in\mathbb{Z}}$ is not bounded and therefore not (jointly) Weyl-1-almost automorphic (of type 1); see also Remark 2.9(i) and [36, Remark 4].

In connection with [38, Theorem 2.5], we will state and prove the following result (cf. Definition 2.4 for the notion):

Theorem 3.2. *Suppose that $p > 0$ and $F : \mathbb{Z}^n \to Y$ is bounded, jointly Weyl-(p, R)-multi-almost automorphic sequence, where R is any collection of sequences in $\mathbb{Z}^n$ such that the assumption $(b_k) \in \mathrm{R}$ implies that any subsequence of (b_k) also belongs to R. Let R' be the collection of all sequences (a_k) in $\mathbb{R}^n$ satisfying that there exists a sequence $(b_k) \in \mathrm{R}$ such that $\sup_{k\in\mathbb{N}} |a_k - b_k| < +\infty$. Then there exists a bounded, uniformly continuous, jointly Weyl-(p, R')-multi-almost automorphic function $\tilde{F} : \mathbb{R}^n \to Y$ such that $R(\tilde{F}(\cdot)) \subseteq CH(\overline{R(F)})$, $\|\tilde{F}\|_\infty = \|F\|_\infty$ and $\tilde{F}(k) = F(k)$ for all $k \in \mathbb{Z}^n$.*

Proof. We will consider the one-dimensional setting for the sake of brevity (in general case, the result follows from a similar argumentation but the proof is much more technically complicated). If $t \in [k, k+1)$ for some $k \in \mathbb{Z}$, then we define $\tilde{F}(t) := F(k) + (t-k) \cdot [F(k+1) - F(k)]$. Since $F(\cdot)$ is bounded, it readily follows that $\tilde{F}(\cdot)$ is bounded, uniformly continuous as well as that $R(\tilde{F}(\cdot)) \subseteq CH(\overline{R(F)})$, $\|\tilde{F}\|_\infty = \|F\|_\infty$ and $\tilde{F}(k) = F(k)$ for all $k \in \mathbb{Z}$. It remains to be proved that $\tilde{F}(\cdot)$ is jointly Weyl-(p, R')-almost automorphic. Let $(a_k) \in \mathrm{R}'$ be fixed. Then there exists a sequence $(b_k) \in \mathrm{R}$ such that $\sup_{k \in \mathbb{N}} |a_k - b_k| < +\infty$. Due to our assumptions, for every $l > 0$, we can find a subsequence $(b_{k_m}) \in \mathrm{R}$ of (b_k) and a function $F^* : \mathbb{Z} \to L^p([-l, l] \cap \mathbb{Z})$ such that, for every $t \in \mathbb{Z}$, we have:

$$\lim_{(m,l) \to +\infty} l^{-1} \sum_{j \in [-l,l] \cap \mathbb{Z}} \Big\| F\big(t + b_{k_m} + j\big) - \big[F^*(t)\big](j) \Big\|_Y^p = 0 \tag{3.1}$$

and

$$\lim_{(m,l) \to +\infty} l^{-1} \sum_{j \in [-l,l] \cap \mathbb{Z}} \Big\| \big[F^*\big(t - b_{k_m}\big)\big](j) - F\big(t + j\big) \Big\|_Y^p = 0. \tag{3.2}$$

Since $F(\cdot)$ is bounded, it simply follows that for each $\epsilon > 0$ there exist real numbers $c_p > 0$ and $l_0 > 0$ such that

$$\sum_{j \in [-l,l] \cap \mathbb{Z}} \Big\| \big[F^*(0)\big](j) \Big\|_Y^p \leq c_p \Big((2l+2)\|F\|_\infty^p + \epsilon l \Big), \quad l \geq l_0.$$

Keeping this in mind, it is not difficult to prove that (3.1)-(3.2) imply

$$\lim_{(m,l) \to +\infty} l^{-1} \sum_{j \in [-l,l] \cap \mathbb{Z}} \Big\| F\big(t + b_{k_m} + j\big) - \big[F^*(0)\big](t+j) \Big\|_Y^p = 0 \tag{3.3}$$

and

$$\lim_{(m,l) \to +\infty} l^{-1} \sum_{j \in [-l,l] \cap \mathbb{Z}} \Big\| \big[F^*\big(0\big)\big](t - b_{k_m} + j) - F\big(t + j\big) \Big\|_Y^p = 0. \tag{3.4}$$

Furthermore, we may assume without loss of generality that $\lim_{k \to \infty}(a_k - b_k) = c \in \mathbb{R}$. In order to complete the proof, it suffices to prove that, for every fixed number $t \in \mathbb{R}$, we have:

$$\lim_{(m,l) \to +\infty} l^{-1} \int_{-l}^{l} \Big\| \tilde{F}\big(t + a_{k_m} + x\big) - G(t + x + c) \Big\|_Y^p \, dx = 0 \tag{3.5}$$

and

$$\lim_{(m,l) \to +\infty} l^{-1} \int_{-l}^{l} \Big\| G\big(t - a_{k_m} + x + c\big) - F\big(t + x\big) \Big\|_Y^p \, dx = 0, \tag{3.6}$$

where $G(t) := [F^*(0)](k) + (t-k) \cdot \{[F^*(0)](k+1) - [F^*(0)](k)\}$ if $t \in [k, k+1)$ for some $k \in \mathbb{Z}$. We will only prove the limit equation (3.5) since the limit equation (3.6) can be proved analogously, with the help of the limit equation (3.4). Let $\epsilon > 0$ be fixed. Since $\tilde{F}(\cdot)$ is uniformly continuous, (3.5) follows automatically if we prove that there exists a sufficiently large real number $s > 0$ such that, for every $l \geq s$ and $m \geq s$, we have:

$$l^{-1} \int_{-l}^{l} \Big\| \tilde{F}\big(t + x + c + b_{k_m}\big) - G(t + x + c) \Big\|_Y^p \, dx \leq \epsilon. \tag{3.7}$$

To deduce the validity of (3.7), we divide the segment $[-l, l]$ into disjoint intervals depending on the belonging of the number $t+c+x$ to some interval of the form $[k, k+1)$, where $k \in \mathbb{Z}$. Keeping in mind the definitions of $\tilde{F}(\cdot)$, $G(\cdot)$ and such a division of the interval $[-l, l]$, we easily get that there exists a finite real number $c_p > 0$ such that

$$\int_{-l}^{l} \Big\| \tilde{F}\big(t+x+c+b_{k_m}\big) - G(t+x+c) \Big\|_Y^p \, dx \leq c_p \sum_{j=-\lceil l \rceil -1}^{\lceil l \rceil +1} \Big\| F\big(\lfloor t \rfloor + j + b_{k_m}\big) - G(\lfloor t \rfloor + j) \Big\|_Y^p,$$

for all sufficiently large numbers $l > 0$ and $m \in \mathbb{N}$, which simply implies the required. $\square$

We can similarly prove the following result for jointly Weyl-$(\mathbb{F}, \mathcal{P}, \mathrm{R}, W)$-multi-almost automorphic sequences of type 2:

Theorem 3.3. *Suppose that $p > 0$ and $F : \mathbb{Z}^n \to Y$ is a bounded, jointly Weyl-$(\mathbb{F}, \mathcal{P}, \mathrm{R}, W)$-multi-almost automorphic sequence of type 2, where $\mathbb{F}(l, \mathbf{t}) \equiv l^{-n/p}$ if $p \geq 1$ [$\mathbb{F}(l, \mathbf{t}) \equiv l^{-1}$ if $p \in (0,1)$], $\emptyset \neq W \subseteq \mathbb{Z}^n$, for each $l > 0$ and $f \in P_l$ we have $\|f\|_l \equiv [\sum_{j \in [-l,l]^n \cap \mathbb{Z}^n} \|f(j)\|_Y^p]^{1/p}$ if $p \geq 1$ [$\|f\|_l \equiv \sum_{j \in [-l,l]^n \cap \mathbb{Z}^n} \|f(j)\|_Y^p$ if $p \in (0,1)$], R is any collection of sequences in $\mathbb{Z}^n$ such that the assumption $(b_k) \in \mathrm{R}$ implies that any subsequence of (b_k) also belongs to R. Let R' be the collection of all sequences (a_k) in $\mathbb{R}^n$ satisfying that there exists a sequence $(b_k) \in \mathrm{R}$ such that $\sup_{k \in \mathbb{N}} |a_k - b_k| < +\infty$. Then there exists a bounded, uniformly continuous, Weyl-$(\mathbb{F}, \mathcal{P}', \mathrm{R}', W)$-multi-almost automorphic function $\tilde{F} : \mathbb{R}^n \to Y$ of type 2, where for each $l > 0$ we have $P_l' = L^p([-l, l]^n : Y)$; furthermore, we have $R(\tilde{F}(\cdot)) \subseteq CH(\overline{R(F)})$, $\|\tilde{F}\|_\infty = \|F\|_\infty$ and $\tilde{F}(k) = F(k)$ for all $k \in \mathbb{Z}^n$.*

It is worthwhile to mention that Theorem 3.3 can be simply formulated for the class of jointly Weyl-$(\mathbb{F}, \mathcal{P}, \mathrm{R}, W)$-multi-almost automorphic sequences, as well. We close this section with the observation that we have recently introduced and analyzed, in [37, Section 4], various classes of Stepanov-p-almost periodic functions in norm and Stepanov-p-almost automorphic functions in norm ($p > 0$). We will not consider here the corresponding classes of (metrically) Weyl-p-almost automorphic functions in norm ($p > 0$).

4 Applications to the abstract fractional integro-differential equations and the abstract fractional difference equations

In this section, we will provide several applications of the established results in the qualitative analysis of solutions for various classes of the abstract fractional integro-differential equations and the abstract fractional difference equations.

1. **Convolution invariance of joint Weyl-$(\mathbb{F}, \mathcal{P}, \mathrm{R}, \mathcal{B}, \omega)$-multi-almost automorphy and Weyl-$(\mathbb{F}, \mathcal{P}, \mathrm{R}, \mathcal{B}, W)$-multi-almost automorphy of type 2.** The introduced classes of metrically Weyl almost automorphic type functions, especially the class of Weyl-$(\mathbb{F}, \mathcal{P}, \mathrm{R}, \mathcal{B})$-multi-almost automorphic functions (of type 1), behave very badly with the respect to the invariance under the actions of infinite convolution products. The main problem is the existence of the second limits in the equations (2.1)-(2.2), resp. (2.3)-(2.4). Concerning this question, we will only emphasize that the corresponding classes of metrically Besicovitch almost automorphic functions and sequences, which will be considered in our forthcoming paper [34], behave much better with the respect to this matter as well as that the certain

results can be obtained provided that the second limits in these equations are equal to zero or that the limit function $F^*(\cdot;\cdot)$ satisfies $F^* \equiv F$.

Concerning [29, Proposition 8.3.6, Question 8.3.7], we would like to present the following result which can be deduced by means of the argumentation contained in the proof of [32, Proposition 3.2] (observe only that the function $F(\cdot)$ appearing on [32, l. -2, p. 47] is bounded in the newly arisen situation; taking into account the corresponding definition of joint Weyl-$(\mathbb{F}, \mathcal{P}, \mathrm{R}, \mathcal{B}, \omega)$-multi-almost automorphy, the remainder of proof can be simply copied):

Proposition 4.1. *Suppose that the operator family $(R(t))_{t>0} \subseteq L(X, Y)$ satisfies that there exist finite real constants $M > 0$, $\beta \in (0, 1]$ and $\gamma > \beta$ such that*

$$\|R(t)\|_{L(X,Y)} \leq M \frac{t^{\beta-1}}{1+t^{\gamma}}, \quad t > 0.$$

Suppose, further, that $a > 0$, $\alpha > 0$, $1 \leq p < +\infty$, $\alpha p \geq 1$, $ap \geq 1$, $\alpha p(\beta - 1)/(\alpha p - 1) > -1$ if $\alpha p > 1$, and $\beta = 1$ if $\alpha p = 1$. If $b \in [0, \gamma - \beta)$, $w(t) := (1 + |t|)^b$, $t \in \mathbb{R}$ and the function $f : \mathbb{R} \to X$ is both essentially bounded and jointly Weyl-$(\mathbb{F}, \mathcal{P}, \mathrm{R}, \omega)$-multi-almost automorphic with $\mathbb{F}(l) \equiv l^{-a/\alpha}$ and $P_l = L^{\alpha p}([-l, l])$ for all $l > 0$, then the function $F(\cdot)$, given by

$$t \mapsto F(t) := \int_{-\infty}^{t} R(t - s) f(s)\, ds, \quad t \in \mathbb{R}, \tag{4.1}$$

is continuous, bounded and jointly Weyl-$(\mathbb{F}, \mathcal{P}, \mathrm{R}, \omega)$-multi-almost automorphic.

It is clear that Proposition 4.1 can be applied in the analysis of the existence and uniqueness of jointly Weyl-$(\mathbb{F}, \mathcal{P}, \mathrm{R}, \omega)$-multi-almost automorphic type solutions for a large class of the abstract fractional integro-differential inclusions without initial conditions. For example, we can apply this result in the study of the fractional Poisson heat equation $D^{\gamma}_{t,+}[m(x)v(t, x)] = (\Delta - b)v(t, x) + f(t, x)$, $t \in \mathbb{R}$, $x \in \Omega$; $v(t, x) = 0$, $v(t, x) \in [0, \infty) \times \partial\Omega$ in the space $X := L^p(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$ and some extra assumptions are satisfied. See [28] for more details about applications of this type.

The proof of following result is very similar to the proof of [29, Theorem 8.3.25] and therefore omitted:

Theorem 4.2. *Suppose that $h \in L^1(\mathbb{R}^n)$, $p \in \mathcal{P}(\mathbb{R}^n)$ and $F : \mathbb{R}^n \times X \to Y$ is Weyl-$(\mathbb{F}, \mathcal{P}, \mathrm{R}, \mathcal{B}, (2\mathbb{Z} + 1)^n)$-multi-almost automorphic of type 2, where $P_l = L^{p(\mathbf{u})}_{\nu}(l\Omega : Y)$ for all $l > 0$ and some Lebesgue measurable function $\nu : \mathbb{R}^n \to (0, +\infty)$. Let p_1, $q \in \mathcal{P}(\mathbb{R}^n)$, let $1/p(\mathbf{u}) + 1/q(\mathbf{u}) \equiv 1$, and let $\mathbb{F}_1 : (0, \infty) \times \mathbb{R}^n \to (0, \infty)$. Suppose further that, for every $x \in X$, one has $\sup_{\mathbf{t} \in \mathbb{R}^n} \|F(\mathbf{t}; x)\|_Y < \infty$, as well as that $\emptyset \neq W_2 \subseteq (2\mathbb{Z})^n$ and for every $\mathbf{t} \in \mathbb{R}^n$ there exists $l_1 > 0$ such that, for every $l \geq l_1$ and $w \in lW_2$, we have*

$$\int_{l\Omega} \varphi_{p_1(\mathbf{u})} \left(2\mathbb{F}_1(l, \mathbf{t} + w)\nu_1(\mathbf{u}) \sum_{k \in l(2\mathbb{Z}+1)^n} \frac{\|h(\mathbf{u} + k - \mathbf{v})/\nu(\mathbf{v})\|_{L^{q(\mathbf{v})}(l\Omega)}}{\mathbb{F}(l, \mathbf{t} - k + w)} \right) du \leq 1.$$

Then the function $h * F : \mathbb{R}^n \times X \to Y$, defined by

$$(h * F)(\mathbf{t}; x) := \int_{\mathbb{R}^n} h(\sigma) F(\mathbf{t} - \sigma; x)\, d\sigma, \quad \mathbf{t} \in \mathbb{R}^n,\ x \in X, \tag{4.2}$$

is Weyl $(\mathbb{F}_1, \mathcal{P}_1, \mathrm{R}, \mathcal{B}, W_2)$-multi-almost automorphic of type 2, where $p_1 \in \mathcal{P}(\mathbb{R}^n)$ and $P^1_l = L^{p_1(\mathbf{u})}_{\nu_1}(l\Omega : Y)$ for all $l > 0$ and some Lebesgue measurable function $\nu_1 : \mathbb{R}^n \to (0, +\infty)$.

We can similarly consider the invariance of Weyl-$(\mathbb{F}, \mathcal{P}, \mathrm{R}, \mathcal{B}, W)$-multi-almost automorphy of type 2 under the actions of the infinite convolution product (4.1) as well as the convolution invariance of and joint Weyl-$(\mathbb{F}, \mathcal{P}, \mathrm{R}, \mathcal{B}, W)$-multi-almost automorphy under the actions of convolution product (4.2) and the convlution product (4.1); cf. also [29, Theorem 8.3.27, Theorem 8.3.29]. The possible applications can be given to the heat equation in $\mathbb{R}^n$ and the evolution systems generated by the family of operators $(A(t) \equiv \Delta + a(t)\mathrm{I})_{t\geq 0}$, where Δ denotes the Dirichlet Laplacian on $L^r(\mathbb{R}^n)$ for some $r \geq 1$ and $a \in L^\infty([0,\infty))$; cf. also the second application and the third application given in [29, Subsection 8.3.5].

2. **Some applications to the abstract fractional difference equations.** The class of jointly equi-Weyl-p-normal functions, where $p \geq 1$, have been introduced in the final section of [33]. This class and its metrical generalizations are important because they are stable, in a certain sense, under the actions of convolution products. This enables to study the asymptotically Weyl almost automorphic type solutions for a class of the abstract impulsive first-order differential inclusions (cf. [20, Subsection 4.1, Subsection 4.2] for more details in this direction).

For our next application, we need the following notion:

Definition 4.3. Suppose that $\mathbb{F} : \mathbb{R} \to (0,\infty)$, $\nu : \mathbb{Z} \to (0,\infty)$ and $p \in [1,\infty)$. Then we say that a sequence $f : \mathbb{Z} \to X$ is jointly equi-Weyl-$(\mathbb{F}, p, \nu, p_b)$-almost automorphic if and only if there exist positive real numbers $M \geq 1$ and $s \geq 0$ such that $\|f(k)\| \leq M(1+|k|)^s$, $k \in \mathbb{Z}$ as well as that for any integer sequence (s_r) there exist a subsequence (s_{r_m}) of (s_r), a sequence $f^* : \mathbb{Z} \to X$ and positive real numbers $M' \geq 1$ and $s' \geq 0$ such that $\|f^*(k)\| \leq M'(1+|k|)^{s'}$, $k \in \mathbb{Z}$ and

$$\lim_{(m,l)\to+\infty} \sup_{k\in\mathbb{R}} \mathbb{F}(l) \left[\sum_{j=-l}^{l} \big\| f\big(j+k+s_{r_m}\big) - f^*(k+j) \big\|^p \nu^p(j) \right]^{1/p} = 0.$$

In the following result, we analyze the existence and uniqueness of jointly equi-Weyl-$(\mathbb{F}, p, \nu, p_b)$-almost automorphic solutions of the first-order difference equation

$$u(k+1) = Au(k) + f(k), \quad k \in \mathbb{Z}, \tag{4.3}$$

where $A \in L(X)$ and $(f_k \equiv f(k))_{k\in\mathbb{Z}}$ is a jointly equi-Weyl-$(\mathbb{F}, p, \nu, p_b)$-almost automorphic sequence; cf. also [12, Section 3] and [36, Theorem 7]:

Theorem 4.4. *Suppose that $\|A\| < 1$, $\mathbb{F} : \mathbb{R} \to (0,\infty)$, $\nu : \mathbb{Z} \to (0,\infty)$, $p \in [1,\infty)$ and $f : \mathbb{Z} \to X$ is a jointly equi-Weyl-$(\mathbb{F}, p, \nu, p_b)$-almost automorphic sequence. Then there exists a unique polynomially bounded solution $u(\cdot)$ of* (4.3) *and $u(\cdot)$ is jointly equi-Weyl-$(\mathbb{F}, p, \nu, p_b)$-almost automorphic, provided that there exist a sequence $\psi : \mathbb{Z} \to (0,\infty)$ and a number $\sigma > 0$ such that $\nu(x+y) \leq \nu(x)\psi(y)$ for all $x,\ y \in \mathbb{Z}$ and $\sum_{v=0}^{+\infty} [\psi(v+1)]^p/(1+v^\sigma)^p < +\infty$.*

Proof. It is well known that a unique polynomially bounded solution $u(\cdot)$ of (4.3) is given by

$$u(k) = \sum_{v=0}^{+\infty} A^v f(k-v-1), \quad k \in \mathbb{Z};$$

observe that the above series converges since there exist positive real numbers $M \geq 1$ and $s \geq 0$ such that $\|f(k)\| \leq M(1+|k|)^s$, $k \in \mathbb{Z}$. Let $\epsilon > 0$ be fixed. Then we know that for any integer sequence (s_r) there exist a subsequence (s_{r_m}) of (s_r), a sequence $f^* : \mathbb{Z} \to X$ and positive real numbers $M' \geq 1$ and

$s' \geq 0$ such that $\|f^*(k)\| \leq M'(1+|k|)^{s'}$, $k \in \mathbb{Z}$ as well as that there exists $w > 0$ such that, for every $k \in \mathbb{Z}$, $l \geq w$ and for every $m \in \mathbb{N}$ with $m \geq w$, we have

$$\mathbb{F}(l)\left[\sum_{j=-l}^{l}\left\|f\big(j+k+s_{r_m}\big)-f^*(k+j)\right\|^p \nu^p(j)\right]^{1/p} \leq \epsilon.$$

Set $u^*(k) := \sum_{v=0}^{+\infty} A^v f^*(k-v-1)$, $k \in \mathbb{Z}$; obviously, this series is convergent due to the polynomial boundedness of the sequence $f^*(\cdot)$. Observing that $\nu(j) \leq \nu(j-v-1)\psi(v+1)$ for all $j \in \mathbb{Z}$, $v \in \mathbb{N}_0$ and $\sum_{v=0}^{+\infty}[\psi(v+1)]^p/(1+v^\sigma)^p < +\infty$, we can repeat verbatim the argumentation contained in the proof of [36, Theorem 7] in order to see that there exists an absolute constant $c_A^p > 0$ such that, for every $k \in \mathbb{Z}$, $l \geq w$ and for every $m \in \mathbb{N}$ with $m \geq w$, we have

$$\mathbb{F}(l)\left[\sum_{j=-l}^{l}\left\|u\big(j+k+s_{r_m}\big)-u^*(k+j)\right\|^p \nu^p(j)\right]^{1/p} \leq c_A^p \epsilon.$$

□

It is clear that Theorem 4.4 can be applied to any bounded linear operator of the form $A = B/c$, where $B \in L(X)$, $c \in \mathbb{C}$ and $|c| > \|B\|$. Further on, the statements of [36, Theorem 8, Theorem 9] can be formulated for jointly equi-Weyl-$(\mathbb{F}, p, \nu, p_b)$-almost automorphic sequences, as well. In [36, Subsection 4.2], we have analyzed the abstract fractional difference equation

$$\Delta^\alpha u(k) = Au(k+1) + f(k), \quad k \in \mathbb{Z},$$

where A is a closed linear operator on X, $0 < \alpha < 1$ and $\Delta^\alpha u(k)$ denotes the Caputo fractional difference operator of order α (cf. [9, Definition 2.3] for the notion). We would like to note that we can similarly analyze the existence and uniqueness of jointly equi-Weyl-$(\mathbb{F}, p, \nu, b)$-almost automorphic solutions of this equation (cf. Definition 4.3 with $s = s' = 0$).

The existence and uniqueness of $\mathbb{D}$-asymptotically jointly equi-Weyl-$(\mathbb{F}, p, \nu, p_b)$-almost automorphic type solutions for the difference equation

$$u(k+1) = Au(k) + f(k), \quad k \geq 0; \;\; u(0) = u_0$$

as well as the difference equation

$$u(k,m) = A(k,m)u(k-1,m-1) + f(k,m), \quad k,\ m \in \mathbb{N},$$

subjected with the initial conditions

$$u(k,0) = u_{k,0}; \quad u(0,m) = u_{0,m}, \quad k,\ m \in \mathbb{N}_0,$$

can be considered similarly as in [38, Subsection 3.1, Subsection 3.2]. Details can be left to the interested readers.

5 Appendix: Vectorial Weyl almost automorphic type functions

Concerning the various notions of (metrical) Weyl almost automorphy considered in [2], [33] and this paper, we would like to emphasize that we can also consider the following important classes of functions (here, the assumption $p = 1$ is almost inevitable; similarly we can introduce and analyze several new classes of vectorial Stepanov almost periodic (automorphic) functions and vectorial Weyl almost periodic functions):

Definition 5.1. Let $f \in L^1_{loc}(\mathbb{R} : X)$. Then we say that $f(\cdot)$ is vectorially Weyl almost automorphic, resp. vectorially Weyl almost automorphic of type 1, if and only if for every real sequence (s_k), there exist a subsequence (s_{k_m}) and a function $f^* \in L^1_{loc}(\mathbb{R} : X)$ such that

$$\lim_{m\to\infty} \lim_{l\to+\infty} \frac{1}{2l} \int_{-l}^{l} \Big[f\big(t + s_{k_m} + x\big) - f^*(t + x) \Big]\, dx = 0, \tag{5.1}$$

resp.

$$\lim_{l\to+\infty} \lim_{m\to\infty} \frac{1}{2l} \int_{-l}^{l} \Big[f\big(t + s_{k_m} + x\big) - f^*(t + x) \Big]\, dx = 0, \tag{5.2}$$

and

$$\lim_{m\to\infty} \lim_{l\to+\infty} \frac{1}{2l} \int_{-l}^{l} \Big[f^*\big(t - s_{k_m} + x\big) - f(t + x) \Big]\, dx = 0, \tag{5.3}$$

resp.

$$\lim_{l\to+\infty} \lim_{m\to\infty} \frac{1}{2l} \int_{-l}^{l} \Big[f^*\big(t - s_{k_m} + x\big) - f(t + x) \Big]\, dx = 0, \tag{5.4}$$

for each $t \in \mathbb{R}$. We similarly introduce the class of jointly Weyl almost automorphic functions in the vectorial sense.

It is worth noting that we use the vector-valued integration in the equations (5.1)-(5.4). This can be essentially in some concrete situations, as the following well-known example indicates:

Example 5.2. Let $X := c_0$, the Banach space of all complex sequences vanishing at plus infinity, endowed with the sup-norm. Define $f(t) := ((1/n)\cos(t/n))_n$, $t \in \mathbb{R}$. Then $f : \mathbb{R} \to X$ is almost periodic but its first anti-derivative $F(t) := (\sin(t/n))_n$, $t \in \mathbb{R}$ is bounded, uniformly continuous but not almost automorphic since the range of $F(\cdot)$ is not relatively compact in c_0 (see, e.g., [40, Example 2.7]). The above implies that $F(\cdot)$ cannot be Stepanov-p-almost automorphic for any exponent $p > 0$; see [37] for the notion and more details.

On the other hand, it is clear that $F(\cdot)$ is vectorially Weyl almost automorphic since for every fixed integer $k_m \in \mathbb{N}$ we have that the second limits in (5.1) and (5.3) are equal to zero, with $F^* \equiv F$. In order to see this, notice that we have ($t,\ x \in \mathbb{R}$, $s_{k_m} \in \mathbb{R}$, $k_m \in \mathbb{N}$):

$$F\big(t + x + s_{k_m}\big) - F(t + x) = 2\left(\sin \frac{s_{k_m}}{2n} \cdot \cos \frac{2t + 2x + 2s_{k_m}}{2n} \right)_n,$$

which simply implies that, for every $l > 0$, we have:

$$\left\| \frac{1}{2l} \int_{-l}^{l} \Big[F\big(t + x + s_{k_m}\big) - F(t + x) \Big]\, dx \right\|$$

$$= \left\| \frac{4}{l} \left(n \cdot \sin \frac{s_{k_m}}{2n} \cdot \sin \frac{l}{n} \cdot \cos \frac{2t + s_{k_m}}{2n} \right)_n \right\| \leq \frac{4}{l} \frac{|s_{k_m}|}{2};$$

the second limit equation in (5.3) can be considered similarly. Moreover, $F(\cdot)$ is not vectorially Weyl almost automorphic of type 1, which can be shown as follows. If we suppose the contrary, then a simple computation shows that for each sequence (s_k) there exist a subsequence (s_{k_m}) of (s_k) and a mapping $F^* : \mathbb{R} \to c_0$ such that, for every $t \in \mathbb{R}$, we have:

$$\lim_{l \to +\infty} \lim_{m \to +\infty} l^{-1} \left[\left(n \cdot \sin \frac{t + s_{k_m}}{n} \cdot \sin \frac{l}{n} \right)_n - \frac{1}{2} \int_{-l}^{l} F^*(t + x)\, dx \right] = 0.$$

This, in particular, implies that the second limit $\lim_{m \to +\infty} \cdot$ in the above expression always exists if l is an irrational mutiple of π. After a simple calculation, we get that the sequence $(\sin((t + s_{k_m})/n))_n$ exists in c_0 as $m \to +\infty$, which implies by the proof of [17, Proposition 2.5] that the range of function $F(\cdot)$ is relatively compact in c_0. This is false and yields the contradiction.

For the continuation, let us observe that an application of Proposition 2.5 shows the following:

(HP) For each exponent $p > 0$, the function $F(\cdot)$ is (jointly) Weyl-p-almost automorphic (of type 1) if and only if $F(\cdot)$ is (jointly) Weyl-1-almost automorphic (of type 1).

We will prove that $F(\cdot)$ is neither (jointly) Weyl-p-almost automorphic (of type 1) nor jointly Weyl almost automorphic in the vectorial sense. Due to (HP), it suffices to consider the case $p = 1$ in the sequel.

We will first prove that $F(\cdot)$ is not Weyl-1-almost automorphic of type 1. Let $\epsilon \in (0, 2^{-1} \sin 1)$ be fixed. If we suppose the contrary, then for each sequence (s_k) there exist a subsequence (s_{k_m}) of (s_k), a mapping $F^* : \mathbb{R} \to c_0$ and a number $l_0 > 0$ such that for each $l \geq l_0$ there exists $m_l \in \mathbb{N}$ such that for each $m \geq m_l$ we have (put $t = 0$):

$$\frac{1}{2l} \int_{-l}^{l} \left\| F\big(s_{k_m} + x\big) - F^*(x) \right\| dx \leq \epsilon/2.$$

In particular, for each $l \geq l_0$ there exists $m_l \in \mathbb{N}$ such that for each $m,\ m' \geq m_l$ we have

$$\frac{1}{2l} \left\| \int_{-l}^{l} \Big[F\big(s_{k_m} + x\big) - F\big(x + s_{k_{m'}}\big) \Big]\, dx \right\| \leq \epsilon. \tag{5.5}$$

Applying the Newton-Leibnitz formula and a few elementary transformations, we get for each $l \geq l_0$ there exists $m_l \in \mathbb{N}$ such that for each $m,\ m' \geq m_l$ we have

$$\sup_{n \in \mathbb{N}} \left| \frac{2n}{l} \cdot \cos \frac{s_{k_m} + s_{k_{m'}}}{2n} \cdot \sin \frac{l}{n} \cdot \sin \frac{s_{k_m} - s_{k_{m'}}}{2n} \right| \leq \epsilon.$$

Plug now $s_k \equiv k$, $l = l_0$ and $m' = m_0$. Since $[n/l \sin(l/n)] \to 1$ as $n \to +\infty$, we obtain that there exist integers $n_0,\ m_0 \in \mathbb{N}$ such that, for every $n \geq n_0$ and $m \geq m_0$, we have

$$\sup_{n \geq n_0} \left| \cos \frac{k_m + k_{m_0}}{2n} \cdot \sin \frac{k_m - k_{m_0}}{2n} \right| \leq \epsilon.$$

But, if $k_m \geq k_{m_0} + n_0$, then

$$\sup_{n \geq n_0} \left| \cos \frac{k_m + k_{m_0}}{2n} \cdot \sin \frac{k_m - k_{m_0}}{2n} \right|$$

$$\geq \left| \cos \frac{k_m + k_{m_0}}{2(k_m - k_{m_0})} \cdot \sin \frac{k_m - k_{m_0}}{2(k_m - k_{m_0})} \right| \to 2^{-1} \sin 1, \quad m \to +\infty,$$

which is a contradiction. We can similalrly prove that $F(\cdot)$ is not jointly Weyl almost automorphic in the vectorial sense and therefore not jointly Weyl almost automorphic.

It remains to be proved that $F(\cdot)$ is not Weyl-1-almost automorphic. If we suppose the contrary, then there exists $m_0 \in \mathbb{N}$ such that, for every $m,\ m' \geq m_0$, there exists $l_{m,m'} > 0$ such that for each $l \geq l_{m,m'}$ we have (5.5). Plug now $s_k \equiv k$, $m' = m_0$ and minorize the norm of the sequence in (5.5) in c_0 by the absolute value of its $(k_m - k_{m_0})$-th element. This yields that for each integer $m \geq m_0$ there exists $l_m > 0$ such that for each $l \geq l_m$ we have

$$\frac{1}{2l} \int_{-l}^{l} \left| \cos \frac{2x + k_m + k_{m_0}}{2(k_m - k_{m_0})} \cdot \sin \frac{k_m - k_{m_0}}{2(k_m - k_{m_0})} \right| dx \leq \epsilon. \tag{5.6}$$

Substituting $y = \frac{2x+k_m+k_{m_0}}{2(k_m-k_{m_0})}$ in (5.6), we simply get

$$\frac{k_m - k_{m_0}}{2l} \int_{\frac{-2l+k_m+k_{m_0}}{2(k_m-k_{m_0})}}^{\frac{2l+k_m+k_{m_0}}{2(k_m-k_{m_0})}} |\cos y|\, dy \leq \epsilon / \sin(1/2), \quad l \geq l_m.$$

This is impossible since the term on the left hand side of the above estimate behaves like

$$\frac{k_m - k_{m_0}}{2l} \cdot 2 \cdot \frac{\frac{2l+k_m+k_{m_0}}{2(k_m-k_{m_0})} - \frac{-2l+k_m+k_{m_0}}{2(k_m-k_{m_0})}}{\pi} = \frac{2}{\pi}.$$

More details about the vectorial Weyl almost automorphic type functions will be given somewhere else.

6 Conclusions and final remarks

In this paper, we have reexamined and slightly generalized various notions of Weyl almost automorphy ([29], [33]). We have analyzed various classes of metrically Weyl almost automorphic functions of the form $F : \mathbb{R}^n \times X \to Y$ and metrically Weyl almost automorphic sequences of the form $F : \mathbb{Z}^n \times X \to Y$, where X and Y are complex Banach spaces. Several illustrative examples, applications of established theoretical results and brief analysis of vectorial Weyl almost automorphic type functions are provided.

We close the paper with the observation that the metrically Weyl, Besicovitch and Doss classes of ρ-almost periodic sequences will be considered in [35].

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