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On the index of regularity of additive decompositions of forms

Sur l'index de régularité des décompositions additives des formes

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ABSTRACT. Let f be a degree d form in $n+1$ variables $x_0,\dots,x_n$. Any additive decomposition of f is associated to a finite set $A\subset\mathbb{P}^n$ with $\#A$ the number of non-proportional addenda. We study the index of regularity $\rho(A)$ of A, i.e. the first integer t such that $h^1(\mathcal{I}_A(t))=0$, of the finite subset $A\subset\mathbb{P}^n$ associated to the additive decompositions of degree d forms in $n+1$ variables. Obviously $\rho(A)\le d$. We prove that $\rho(A)\ge d-k$ if A spans $\mathbb{P}^n$ and k is the maximal integer such that x_0^k divides at least one monomial of f. If f essentially depends on less variables, but A spans $\mathbb{P}^n$, then $\rho(A)=d$. We give examples (but with $\#A$ bigger that the rank of f) in which we have $\rho(A)=d$.
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1. Introduction

Fix integers $n\ge 1$ and $d\ge 1$. Take an algebraically closed field $\mathbb{K}$ with characteristic 0. For any degree d form $f\ne 0$, i.e. for any $f\in\mathbb{K}[x_0,\dots,x_n]_d$, $f\ne 0$, an additive decomposition of f is a finite sum $f=\sum_i \ell_i^d$ with ℓ_i a linear form. The rank $r_{n,d}(f)$ of f is the minimal number of addenda in an additive decomposition of f. The integer $r_{n,d}(f)$ is a very important measure of the complexity of f. In this paper we explain another discrete invariant of additive decompositions of f, the index of regularity of the finite subsets A of $\mathbb{P}^n$ or a zero-dimensional subschemes Z of $\mathbb{P}^n$. In an additive decomposition $f=\sum_i \ell_i^d$ in which no two linear forms are proportional the set A is the union of all $[\ell_i]\in\mathbb{P}^n$. The reader may consider only finite sets instead of zero-dimensional schemes. We pose Theorem 1.2 in a more general setting (allowing zero-dimensional schemes) because is some examples with low border rank they may give very different answers (see Remark 1.4), but not with our restriction for $k=0$.

Let $Z\subset\mathbb{P}^n$, $n\ge 1$, be a zero-dimensional scheme, $Z\ne\emptyset$. Let $\mathcal{I}_Z\subset\mathcal{O}_{\mathbb{P}^n}$ be the ideal sheaf of Z. The *index of regularity* $\rho(Z)$ of Z is the first integer $t\ge 0$ such that $h^1(\mathcal{I}_Z(t))=0$. Remember that $H^0(\mathcal{O}_{\mathbb{P}^n}(d))=\mathbb{K}[x_0,\dots,x_n]_d$. For each integer t $H^0(\mathcal{O}_Z(t))$ is a vector space of dimension $\deg(Z)$. The integer $\rho(Z)$ is the first $t\in\mathbb{N}$ such that the restriction map $u_{Z,t}: H^0(\mathcal{O}_{\mathbb{P}^n}(t))\to H^0(\mathcal{O}_Z(t))$ is surjective. Thus $\rho(Z)$ is the first integer t such that Z gives $\deg(Z)$ independent conditions to $\mathbb{K}[x_0,\dots,x_n]_t$. For $0\le t<\rho(Z)$ the integers $\dim\operatorname{Im}(u_{Z,t})$ are increasing, while $\dim\operatorname{Im}(u_{Z,t})=\deg(Z)$ for $t\ge\rho(Z)$. Indeed, it is well-known that $h^1(\mathcal{I}_Z(t))=0$ for all $t\ge\rho(Z)$ and that $\rho(Z)\le\deg(Z)-1$, with equality if and only if Z is contained in a line ([3, Remark 22 and Lemma 34]). The index of regularity of Z is an important information to get a finer invariant of Z, its Castelnuovo-Mumford regularity, which is either $\rho(Z)$ or $\rho(Z)+1$. The latter integer, $\rho(Z)+1$, is also an upper bound for the degrees of the minimal generators of the homogeneous ideal of Z.

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The Veronese embedding gives a geometric interpretation of the additive decompositions of forms. Let $\nu_d : \mathbb{P}^n \to \mathbb{P}^r$, $r = \binom{n+d}{n} - 1$, denote the order d Veronese embedding of $\mathbb{P}^n$. Set $X_{n,d} := \nu_d(\mathbb{P}^n)$. Any $f \in \mathbb{K}[x_0, \dots, x_n]_d$, $f \neq 0$, defines a unique $q = [f] \in \mathbb{P}^r$ and the integer $r_{n,d}(f)$ is the $X_{n,d}$-rank $r_{n,d}(q)$ of q, i.e. the minimal cardinality of a finite set $A \subset \mathbb{P}^n$ such that $q \in \langle \nu_d(A) \rangle$, where $\langle\ \ \rangle$ denotes the linear span. Let $\mathcal{S}(q)$ or $\mathcal{S}(f)$ denote the set of all $A \subset \mathbb{P}^n$ such that $q \in \langle \nu_d(A) \rangle$ and $\#A = r_{n,d}(q)$. Often an interesting additive decomposition of a form f has not the minimal number $r_{n,d}(f)$ of addenda or $r_{n,d}(f)$ is not known. We say that a finite set $\nu_d(A) \subset X_{n,d}$ *irredundantly spans* q if $q \in \langle \nu_d(A) \rangle$ and $q \notin \langle \nu_d(A') \rangle$ for any $A' \subsetneq A$. Let $\mathcal{S}(q,t)$ and $\mathcal{S}(f,t)$ denote the set of all $A \subset \mathbb{P}^n$ such that $\#A = t$ and $\nu_d(A)$ irredundantly spans q. Obviously $\mathcal{S}(q,t) = \emptyset$ for all $t < r_{n,d}(q)$ and $\mathcal{S}(q, r_{n,d}(f)) = \mathcal{S}(q) \neq \emptyset$. Sometimes there are integers $t > r_{n,d}(q)$ with $\mathcal{S}(q,t) = \emptyset$. Take q and t such that $\mathcal{S}(q,t) \neq \emptyset$ and fix $A \in \mathcal{S}(q,t)$. We study the regularity index $\rho(A)$ of A. Since $\nu_d(A)$ irredundantly spans q, $\rho(A) \leq d$. If Z is a zero-dimensional scheme we say that $\nu_d(Z)$ irredundantly spans q if $q \in \langle \nu_d(Z) \rangle$ and $q \notin \langle \nu(Z') \rangle$ for any $Z' \subsetneq Z$.

For any $t > 0$ let $\mathcal{Z}(q,t)$ (or $\mathcal{Z}(f,t)$ if $q = [f]$) denote the set of all degree t zero-dimensional schemes $Z \subset \mathbb{P}^n$ such that $\nu_d(Z)$ irredundantly spans q.

We list our main results.

Theorem 1.1. *Fix $f \in \mathbb{K}[x_0, \dots, x_n]_d$, $f \neq 0$, and let k be the maximal degree of a power of x_0 appearing in one of the non-zero terms of f. Fix $A \in \mathcal{S}(f,t)$ with the only restriction that if $k = 0$, i.e. if f does not depend on x_k, assume $\langle A \rangle = \mathbb{P}^n$. Then $\rho(A) \geq d - k$ for all $A \in \mathcal{S}(f,t)$.*

For $k = 1$ we get many forms f such that each additive decomposition of f has index of regularity $d-1$ (just take in the set-up of Theorem 1.1 with $k = 1$ a concise f). See Remark 2.3 for cases which were used to get very high rank ternary forms.

Theorem 1.2. *Take $q \in \mathbb{P}^r$ and $Z, W \in \mathcal{Z}(q,x)$, $W \in \mathcal{Z}(q,y)$. Assume the existence of a linear space $V \subsetneq \mathbb{P}^n$ such that $Z \subset V$ and $W \nsubseteq V$. Then $\rho(W) \geq d$.*

Theorem 1.3. *Fix integers $n \geq 1$, $d \geq 3$ and c such that $1 \leq c \leq d-1$. Then there is $f \in \mathbb{K}[x_0, \dots, x_n]_d$, $f \neq 0$ and f concise, and $A \in \mathcal{S}(f)$ such that $\rho(A) = c$ for all $A \in \mathcal{S}(f)$.*

We give an example of q and $t = r_{n,d}(q) + 1$ such that $\mathcal{S}(q,t) \neq \emptyset$, $\rho(A) = d$ for all $A \in \mathcal{S}(q,t)$, $\rho(B) \in \{d-1, d\}$ for all $B \in \mathcal{S}(q,x)$ and all x such that $\mathcal{S}(q,x) \neq \emptyset$ (Proposition 2.5). If $n = 2$ for any q there is an integer t such that $\mathcal{S}(q,t) \neq \emptyset$ and $\rho(A) \leq d-1$ for some $A \in \mathcal{S}(q,t)$ (Proposition 2.6).

Remark 1.4. Take $n = 1$, i.e. look at degree d binary forms and assume $d \geq 3$. Fix any $f \in \mathbb{K}[x_0, x_1]_d$, $f \neq 0$. Set $x := r_{1,d}(f)$. A theorem of Sylvester say that either $x \leq \lfloor (r+1)/2 \rfloor$ and in this case $\#\mathcal{S}(q) = 1$ or $\lfloor (d+1)/2 \rfloor < x \leq d$ and $\dim \mathcal{S}(q) = 2x - 1 - d$ ([6, Theorem 11 and eq. (9)]). Moreover, if $x > \lfloor (d+1)/2 \rfloor$, then f has border rank $r + 2 - x$ and there is a zero-dimensional scheme Z such that $\deg(Z) = r + 2 - x$ and $[f] \in \langle \nu_d(Z) \rangle$. Thus if $x > \lfloor (r+1)/2 \rfloor$ every $S \in \mathcal{S}(f)$ has $\rho(S) = x - 1 > (r + 2 - x) - 1 = \rho(Z)$. Thus for all $f \neq 0$, we may find a scheme Z evincing the border rank of f and with $\rho(Z) \leq d - 2$, while $\rho(S) = d - 1$ for all $S \in \mathcal{S}(f)$ if $r_{1,d}(f) = d$. Theorem 1.1 shows that if we embed $\mathbb{P}^1$ in $\mathbb{P}^n$, $n > 1$, as a line and see f as a non-concise form in $n+1$ variables even allowing zero-dimensional schemes we cannot get an index of regularity $< d$. By [3, Theorem 32] all $[f]$ associated to the tangential variety of $X_{n,d}$, $n > 1$, are not concise and come from an embedding of a line in $\mathbb{P}^n$.

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We raise the following question (we guess that the answer is NO).

Question 1.5. *Is there a form f such that $\rho(A) = d$ for some $A \in \mathcal{S}(q)$ or for all $A \in \mathcal{S}(q)$?*

A referee raised the following interesting question: what occurs if $p := \mathrm{char}(\mathbb{K}) > 0$? For the interested reader we point out that there are two obstructions to use the proofs given in this note. First of all, if $p \leq d$ all first derivatives of some degree d forms may vanish. Moreover, many of the quoted references assume $\mathrm{char}(\mathbb{K}) = 0$ and the interested reader must check if the quoted results are true at least for $p > d$. For instance, for the case $n = 1$ of Theorem 1.3 we quoted a theorem of Sylvester.

2. The proofs

If $W \in \mathcal{Z}(q, t)$, then Z is Gorenstein ([4, Proposition 2.2]).

Let $Z \subset \mathbb{P}^n$ be a zero-dimensional scheme. Fix an integer $e > 0$ and a degree e hypersurface $D \subset \mathbb{P}^n$. We allow the case in which D has multiple components (for the proof of Theorem 1.2 we will take $e = k+1$ and $D = \{x_0^{k+1} = 0\}$, i.e. a multiple of the hyperplane $\{x_0 = 0\}$). The residual scheme $\mathrm{Res}_D(Z)$ of Z with respect to D is the closed subscheme of $\mathbb{P}^n$ with $(\mathcal{I}_Z : \mathcal{I}_D)$ as its ideal sheaf. If Z is a finite set, then $\mathrm{Res}_D(Z) = Z \setminus Z \cap D$. The scheme $\mathrm{Res}_D(Z)$ is zero-dimensional, $\mathrm{Res}_D(Z) \subseteq Z$ and $\deg(Z) = \deg(\mathrm{Res}_D(Z)) + \deg(Z \cap D)$. For any integer t the following sequence of coherent sheaves

$$0 \to \mathcal{I}_{\mathrm{Res}_D(Z)}(t-e) \to \mathcal{I}_Z(t) \to \mathcal{I}_{Z\cap D,D}(t) \to 0 \tag{1}$$

is exact. It is called the residual exact sequence of Z with respect to D.

Remark 2.1. Take any $q \in \langle \nu_d(\mathbb{P}^n)\rangle$ and any $A \in \mathcal{S}(q,t)$. Since $\nu_d(A)$ irredundantly spans q, $\#(R \cap A) \leq d+1$ for all lines R. If $q \in \mathcal{S}(q)$ we have $\#(R \cap A) \leq d$ by a theorem of Sylvester on binary forms ([6]).

Lemma 2.2. *Let $Z \subset \mathbb{P}^n$ be any zero-dimensional scheme. Assume that Z is contained in a linear subspace V of $\mathbb{P}^n$. Then the index of regularity of Z is the same as a subscheme of $\mathbb{P}^n$ and as a subscheme of V.*

Proof. Use that the restriction map $H^0(\mathcal{O}_{\mathbb{P}^n}(t)) \to H^0(\mathcal{O}_V(t))$ is surjective for all $t \in \mathbb{N}$. □

Proof of Theorem 1.2: Take a general hyperplane H of $\mathbb{P}^n$ containing V. Thus $Z \subset H$. Since H is general, $W \nsubseteq H$. Thus by induction on the codimension of the subspace it is sufficient to prove the case $V = H$. It is sufficient to prove that $h^1(\mathcal{I}_W(d-1)) \neq 0$. It is sufficient to find a scheme $W' \subseteq W$ such that $h^1(\mathcal{I}_{W'}(d-1)) > 0$. Consider the residual exact sequence of $Z \cup W$ with respect to H:

$$0 \to \mathcal{I}_{\mathrm{Res}_H(Z\cup W)}(d-1) \to \mathcal{I}_{Z\cup W}(d) \to \mathcal{I}_{H\cap(Z\cup W),H}(d) \to 0 \tag{2}$$

Since $Z \subset H$, $\mathrm{Res}_H(Z) = \emptyset$ and $\mathrm{Res}_H(Z \cup W) = \mathrm{Res}_H(W)$. Since $W \nsubseteq H$, $\mathrm{Res}_H(W) \neq \emptyset$. We have $h^1(\mathcal{I}_{\mathrm{Res}_H(W)}(d-1)) > 0$ by [2, Lemma 5.1]. Take $W' := \mathrm{Res}_H(W) \subseteq W$. □

Remark 2.3. For $n = 2$ and $d = 2k+2$ even A. De Paris used the form $f = x_0x_1^{k-1}x_2^{k+2} + x_1^{2k}x_2^2$ to get a form with very high rank, $r_{2,d}(f) \geq k^2 + 3k + 3$ ([8]), which for large d is near the upper bound $r_{n,d}(g) \leq \lfloor \frac{d^2+6d+5}{4} \rfloor$ proved for all $g \in \mathbb{K}[x_0, x_1, x_2]_d$ ([7]). For odd d, say $d = 2k+1$, it is easy to check (using [5] as it was used in [8]) that $r_{2,d}(g) \geq k^2 + 2k$, where $g = x_0x_1^{k-1}x_2^{k+1} + x_1^{2k-1}x_2^2$.

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Proof of Theorem 1.1: Set $q := [f] \in \mathbb{P}^r$. If $k = 0$, then q is not concise and we may apply Theorem 1.2. Thus we may assume $k > 0$. Set $H := \{x_0 = 0\}$ and let $D = (k+1)H \subset \mathbb{P}^n$ be the degree $k+1$ hypersurface with x_0^{k+1} as one of its equations.

Claim 1: $q \in \langle \nu_d(D)\rangle$ and there is a zero-dimensional scheme $W \subset D$ such that $\langle \nu_d(W)\rangle$.

Proof of Claim 1: $q \in \langle \nu_d(D)\rangle$, because every monomial appearing with a non-zero coefficient in the expansion of f is associated to a point of $\langle \nu_d(D)\rangle$. To get W it is sufficient to take the intersection of $\nu_d(D)$ with $n-1$ general hyperplanes of $\mathbb{P}^r$ containing q.

Since $q \notin H$ and A is a finite set, $A \nsubseteq D$. Note that $\mathrm{Res}_D(W) = \emptyset$ and $\mathrm{Res}_D(W \cup A) = A \setminus A \cap H \neq \emptyset$. By [2, Lemma 5.1] we have $h^1(\mathcal{I}_{\mathrm{Res}_H(A)}(d-k-1)) > 0$ and hence $\rho(A) \geq \rho(\mathrm{Res}_H(A)) \geq d-k$. □

Proof of Theorem 1.3: The case $n = 1$ is true by Sylvester's theorem (all ranks between 1 and d are realized by some binary form) and the fact that $\rho(A) = \#A - 1$ for any non-empty finite set $A \subset \mathbb{P}^1$. Without the assumption that f is concise this would be the end of the proof.

Now assume $n \geq 2$. If $c = 1$ we take any $A \subset \mathbb{P}^n$ such that $\langle A\rangle = \mathbb{P}^n$ and $\#A = n+1$. If $c = 2$ we take a general $A \subset \mathbb{P}^n$ such that $\#A = n+2$. Thus from now on we assume $c \geq 3$ and hence $d \geq 4$.

Fix a line $L \subset \mathbb{P}^n$, a set $F \subset L$ such that $\#F = c+1$ and $\nu_{1,d}(F)$ evinces a rank. Fix a set $E \subset \mathbb{P}^n \setminus L$ such that $\#E = n-1$ and $\langle E \cup L\rangle = \mathbb{P}^n$. Since $\#F > 1$, $G := E \cup F$ spans $\mathbb{P}^n$.

Claim 1: $\rho(G) = c$.

Proof of Claim 1: Since $G \supset F$, $\rho(G) \geq \rho(F) = c$. Fix $E' \subset E$ such that $\#E' = n-2$ (hence $E' = \emptyset$ if $n = 2$). We prove Claim 1 by induction on n, starting the induction with the case $n = 1$. Set $H := \langle E' \cup L\rangle$ and $G' := E' \cup F$. Note that $\langle G'\rangle = H$, $G \cap H = G'$ and $\#(G \setminus G') = 1$.

Take $q \in \mathbb{P}^r$ such that $q \in \langle \nu_d(G)\rangle$ and $q \notin \langle \nu_d(G')\rangle$ for any $G' \subsetneq G$ (it exists, because $h^1(\mathcal{I}_G(d)) = 0$).

Take $A \in \mathcal{S}(q)$. To show that $\rho(A) = c$ it is sufficient to prove that $A = E \cup F_1$ with $F_1 \subset L$ and $\#F_1 = \#F$. By Claim 1 we may assume $A \neq G$. Note that $\nu_d(A)$ and $\nu_d(G)$ irredundantly span q. Thus $h^1(\mathcal{I}_{A\cup G}(d)) > 0$. Fix $E' \subset E$ such that $\#E' = n-2$ and set $\{o\} := E \setminus E'$ and $H := \langle E' \cup L\rangle$. We have $G \setminus G \cap H = \{o\}$. Consider the residual exact sequence of $G \cup B$ with respect to H:

$$0 \to \mathcal{I}_{(A\setminus A\cap H)\cup\{o\}}(d-1) \to \mathcal{I}_{A\cup G} \to \mathcal{I}_{H\cap((A\cap H)\cup G'}(d) \to 0 \tag{3}$$

(a) Assume $h^1(\mathcal{I}_{(A\setminus A\cap H)\cup\{o\}}(d-1)) = 0$. We get $A \setminus A \cap H = \{o\}$ ([2, Lemma 5.1]). Since $q \notin \langle \nu_d(H)\rangle$ and $q \in \langle \nu_d(o) \cup H\rangle$, there is a unique $q' \in \langle \nu_d(H)\rangle$ such that $q \in \langle \{q', \nu_d(o)\}\rangle$. Since $\{o\} = G \setminus G \cap H = A \setminus A \cap H$, we get $A \cap H \in \mathcal{S}(q')$ and $G \cap H \in \mathcal{S}(q', c+n-1)$. First assume $n = 2$, i.e. $H = L$ and $G' = F$. The rank of the binary form q' is evinced by the minimal subset F'' of F such that $q' \in \langle \nu_d(F'')\rangle$, Since $q \notin \langle \nu_d(E \cup F_2)\rangle$, for any $F_2 \subsetneq F$, we get $F'' = F$, concluding the proof for $n = 2$. Now assume $n \geq 3$. By induction on n we get $\#(A \cap H) = c+n-2$ and $A \cap H = E' \cup F_1$ with $F_1 \subset L$ and $\#F_1 = c+1$. Since $a \in A$, we conclude the proof in this case.

(b) Assume $h^1(\mathcal{I}_{(A\setminus A\cap H)\cup\{o\}}(d-1)) > 0$. Thus $h^1(\mathcal{I}_{A\cup\{o\}}(d-1)) > 0$. Note that $\#(A \cup \{o\}) \leq n+c+1$, $c \leq d-1$, A spans $\mathbb{P}^n$. Let $W \subseteq A \cup \{o\}$ be a minimal subset of $A \cup \{o\}$ such that $h^1(\mathcal{I}_W(d-1)) > 0$. Take any plane $M \subseteq \mathbb{P}^n$. Since A spans $\mathbb{P}^n$, $\#(A \cap M) \leq c+3-n$ and hence $\#((A \cup \{o\}) \cap M) \leq c+4-n \leq d+3 \leq 4(d-1)-5$ for all $d \geq 4$. We have $\#(A \cup \{o\}) \leq n+d \leq$

$4(d-1)+n-5$. By [1, Theorem 1] W is in a short list in which either $\#W = d+1$ and W is contained in a line or $\#W + n - \dim\langle W\rangle \geq 2d-2+n$. Since $d+n < 2d-2+w$, we get that $\#W = d+1$ and that W is contained in a line R. Since A evinces a rank, $\#A \cap R) \leq d$. Thus $o \in R$ and $\#(R\cap A) = d$. Since A spans $\mathbb{P}^n$, we get $\#A \geq d+n-1$. Thus $c = d-1$. The line R shows that $\rho(A) = d-1$. □

Remark 2.4. It is easy to find a concise f and an integer $t > r_{n,d}(f)$ such that there is $A \in \mathcal{S}(f,t)$ with $\rho(A) = d$. A very stupid example which works for any fixed concise f is to take $\binom{n+d}{n}$ general points of $\mathbb{P}^n$. For $n = 1$ this is the only way to get some $A \in \mathcal{S}(f, d+1)$ with $\rho(A) = d$. For $n \geq 2$ we get the following example for all $t \geq d+n$. For $t = d+n$ up to isomorphisms the following is the unique example. Fix a line $L \subset \mathbb{P}^n$, a set $F \subset L$ such that $\#F = d+1$. Fix a set $E \subset \mathbb{P}^n \setminus L$ such that $\#E = n-1$ and $\langle E \cup L\rangle$. Since $\#F > 1$, $G := E \cup F$ spans $\mathbb{P}^n$. Take $q \in \mathbb{P}^r$ such that $q \in \langle \nu_d(A)\rangle$ and $q \notin \langle \nu_d(A')\rangle$ for any $A' \subsetneq A$ (it exists, because $h^1(\mathcal{I}_A(d)) = 0$). Since $F \subset L$, we have $\rho(F) = \#F - 1 = d$. Obviously for any q and any $\binom{d+n-1}{n} < t \leq \binom{d+n}{n}$ any $A \in \mathcal{S}(q,t)$ has $\rho(A) = d$.

The following example is a particular case of the one used for the proof of Theorem 1.3, but the thesis is stronger, since it says something about the regularity of all $A \in \mathcal{S}(q,t)$ even if $t > r_{n,d}(q)$.

Proposition 2.5. *Fix integers $n \geq 2$ and $d \geq 3$. Fix a line $L \subset \mathbb{P}^n$, a degree 2 connected zero-dimensional scheme $v \subset L$ and a set $E \subset \mathbb{P}^n \setminus L$ such that $\#E = n-1$ and $\langle E \cup L\rangle = \mathbb{P}^n$. Fix any $q \in \langle \nu_d(E\cup v)\rangle$ such that $q \notin \langle \nu_d(G)\rangle$ for any scheme $G \subsetneq E \cup v$. Then:*

(i) $r_{n,d}(q) = d+n-1$ and $\rho(A) = d-1$ for all $A \in \mathcal{S}(q)$.

(ii) Take any t such that $\mathcal{S}(q,t) \neq \emptyset$ and take $A \in \mathcal{S}(q,t)$. Then $\rho(A) \in \{d-1, d\}$ and if $\rho(A) = d-1$, then E is contained in the base locus of $|\mathcal{I}_A(d-1)|$.

(iii) Assume $d \geq 6$. Then $\mathcal{S}(q, n+d) \neq \emptyset$ and every $A \in \mathcal{S}(q, n+d)$ has $\rho(A) = d$ and it is the union of E and $d+1$ points of L.

Proof. Since $q \in \langle \nu_d(E\cup v)\rangle$ and $q \notin \langle \nu_d(G)\rangle$ for any scheme $G \subsetneq E\cup v$, there is a unique $q' \in \langle \nu_d(v)\rangle$ such that $q \in \langle \{q', \nu_d(E)\}\rangle$. Since $q \notin \langle \nu_d(G)\rangle$ for any scheme $G \subsetneq E \cup v$, $q' \neq \nu_d(o)$, where $\{o\} = v_{\mathrm{red}}$. Note that $q' \in \langle \nu_d(L)\rangle$. By a theorem of Sylvester q' has rank d with respect to the degree d rational normal curve $\nu_d(L)$ ([6]). Take $F \subset L$ such that $\#F = d$ and $q' \in \langle \nu_d(F)\rangle$. Note that $B := E \cup F \in \mathcal{S}(q, d+n-1)$. Hence $r_{n,d}(q) \leq n+d-1$. Since $B \in \mathcal{S}(q, d+n-1)$, $\rho(B) \leq d$. Since B contains d collinear points, $\rho(B) \geq d-1$. By induction on n taking hyperplanes containing L and a subset of E cardinality $\#E - 1$ (as in the proof of Theorem 1.3) we get $\rho(G) = d-1$, $G \in \mathcal{S}(q)$ and that every element of $\mathcal{S}(q)$ has index of regularity $d-1$. Thus (i) holds.

Now we prove part (ii). Fix $t \geq n+d$ such that $\mathcal{S}(q,t) \neq \emptyset$ and fix $A \in \mathcal{S}(q,t)$.Note that $\rho(v\cup E) = 1$. Set $Z := v \cup E \cup A$ and $H := \langle L \cup \{e_1, \dots, e_{n-2}\}\rangle$ with the convention $H := L$ if $n = 2$. Since q is concise, $\langle A\rangle = \mathbb{P}^n$.

Consider the residual exact sequence of Z with respect to H:

$$0 \to \mathcal{I}_{\mathrm{Res}_H(Z)}(d-1) \to \mathcal{I}_Z(d) \to \mathcal{I}_{H\cap Z,H}(d) \to 0 \tag{4}$$

For $i = 1, \dots, n-1$ let H_i be the hyperplane spanned by $L \cup (\{e_1, \dots, e_{n-1}\} \setminus \{e_i\})$. Note that $H = H_{n-1}$.

(a) First assume $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(d-1)) > 0$. Since $\mathrm{Res}_H(Z) = \mathrm{Res}_H(A) \cup \{e_{n-1}\}$, we easily get $h^1(\mathcal{I}_{\mathrm{Res}_H(A)}(d-2)) > 0$. Hence $\rho(A) \ge \rho(\mathrm{Res}_H(A)) \ge d-1$. Assume $h^1(\mathcal{I}_A(d-1)) = 0$. Thus $h^1(\mathcal{I}_{\mathrm{Res}_H(A)}(d-1)) = 0$ and $e_{n-1} \notin A$. Since $h^1(\mathcal{I}_{\mathrm{Res}_H(A)\cup\{e_{n-1}\}}(d-1)) > 0$, e_{n-1} is in the base locus of $|\mathcal{I}_A(d-1)|$. Taking a different $e_i \in \{e_1, \dots, e_{n-1}\}$ and using H_i we get $E \cap A = \emptyset$ and that E is in the base locus of $|\mathcal{I}_A(d-1)|$.

(b) Now assume $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(d-1)) = 0$. By [2, Lemma 5.1] we get $\mathrm{Res}_H(A) = \mathrm{Res}_H(E \cup v) = \{e_{n-1}\}$. Thus $e_{n-1} \in A$ and $A \setminus A \cap H = \{e_{n-1}\}$. Using H_i instead of H we get $A \setminus A \cap H_i = \{e_i\}$. Hence $E \subset A$ and $A \setminus E \subset L$. Since $t > r_{n,d}(q) = d+n-1$, we get $t = n+d$ (Remark 2.1) and get part (iii) in this case. Thus part (ii) is proved.

(c) To conclude the proof of (iii) and hence to conclude the proof of Proposition 2.5 we may assume $t = d+n$ and $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(d-1)) > 0$. Thus $h^1(\mathcal{I}_{A\cup\{e_{n-1}\}}(d-1)) > 0$.

We first will try to get a line D such that $\#(D \cap A) \ge d+1$. Note that if such a line exists, Remark 2.1 gives $\#(A \cap D) = d+1$ and hence $\rho(A) = d+1$. Then we will prove that $D = L$ and that $A \setminus A \cap L = E$. Note that we work under the assumption of (a) and hence the aim is to find a contradiction, although in some steps we stop when we first get the statement of (iii)

Since A spans $\mathbb{P}^n$ any plane contains at most $d+3$ points of $A \cup \{e_{n-1}\}$. Since $d+3 \le 4(d-1)+n-5$, [1, Theorem 1] gives the existence of a line R such that $\#(R \cap (A \cup \{e_{n-1}\}) \ge d+1$. Hence $\#(R \cap A) \ge d$. Since $\langle A \rangle = \mathbb{P}^n$, $\#A = d+n$ and $d \ge 6$, R is the unique line containing at least d points of A. Assume for the moment $n \ge 4$. Taking each e_i and H_i instead of e_{n-1} and H we get $\#(R \cap A) \ge d+1$. Take $D := R$.

Now assume $n \in \{2, 3\}$. Thus $\deg(Z) \le 2n+d+1 \le 2d+1$. Since $h^1(\mathcal{I}_Z(d)) > 0$, [3, Lemma 34] gives the existence of a line D such that $\deg(D \cap Z) \ge d+2$. Set $e := \deg(D \cap (v \cup E))$. Since $v \cup E$ is linearly independent, $e \le 2$. Take a general hyperplane M containing D. Since $\deg(\mathrm{Res}_M(Z)) \le d-1$, $h^1(\mathcal{I}_{\mathrm{Res}_M(Z)}(d-1)) = 0$. Thus $\mathrm{Res}_M(A) = \mathrm{Res}_M(v \cup E)$. We get $\#(A \cap M) - \deg((v \cup E) \cap M) = d-1$. Since Z is curvilinear, $D \cap Z = M \cap Z$ even in the case $n = 3$. Thus $\#(A \cap D) + e \ge d+2$ and $\#(A \cap D) - e = d-1$. Since $e \le 2$, we get $\#(A \cap D) = d+1$ and $e = 2$.

Now we take again any n to prove that $D = L$ and that $A \setminus A \cap L = E$. Let M be a general hyperplane containing D. Since Z is curvilinear, $D \cap A = D \cap M$ and $(v \cup E) \cap M = (v \cup E) \cap D$. Hence $\mathrm{Res}_M(A)$ and $\mathrm{Res}_M(v \cup E)$ span a linear subspace of $\mathbb{P}^n$ of dimension at least $n-2$. We proved that $\deg(\mathrm{Res}_M(Z)) \le 2n+d+1-d-1 = 2n+d$ if $n \ge 4$ and that $\deg(\mathrm{Res}_M(Z)) \le 2n+d+1-d-3 = 2n+d-2$ if $n = 2, 3$. Set $\{o\} := v_{\mathrm{red}}$.

(c1) Assume $h^1(\mathcal{I}_{\mathrm{Res}_M(Z)}(d-1)) = 0$. Thus [2, Lemma 5.1] gives $A \setminus A \cap D = \mathrm{Res}_M(Z)$. If $o \notin M$, then $v \subset \mathrm{Res}_M(Z)$, while $v \nsubseteq A$, a contradiction. Now assume $o \in M$, but $v \nsubseteq M$. We get that $\{o\}$ is a connected component of $\mathrm{Res}_M(z \cup E)$. Since A is a finite set and $o \in M$, $o \notin (A \setminus A \cap M)$, a contradiction. Thus $v \subset M$. Since $(v \cup E) \cap M = (v \cup E) \cap D$, we get $D = \langle v \rangle = L$. In this case we proves that $A \setminus A \cap L = E$.

(c2) Assume $h^1(\mathcal{I}_{\mathrm{Res}_M(Z)}(d-1)) > 0$. If $n = 2, 3$ we proved that $\deg(\mathrm{Res}_M(Z)) \le 2n-2$ and hence $h^1(\mathcal{I}_{\mathrm{Res}_M(Z)}(d-1)) = 0$, a contradiction. Now assume $n \ge 4$. We proved that $\deg(\mathrm{Res}_M(Z)) \le 2n$ and that $\mathrm{Res}_M(Z)$ spans a linear space of dimension at least $n-2$. Set $U' := \langle A \setminus A \cap M \rangle$. Since $\#(A \setminus A \cap D) \le n-1$, $\dim U' \le n-2$. Let U be a general hyperplane containing U'. Since Z is curvilinear, $Z \cap U = Z \cap U'$. Since $v \cup E$ is linearly independent, $h^1(\mathcal{I}_{\mathrm{Res}_{M+U}(v+E)}(d-2)) = 0$. Since $A \subset M \cup U$, [2, Lemma 5.1] gives $v \cup E \subset U' \cup D$. Since $v \cup E$ is linearly independent, we get $\deg(D \cap (v \cup E)) = 2$. In the case $n \geq 4$ with the assumption of (a) we got $D = R$ with $R \cap E = \emptyset$. Thus $R = \langle v \rangle = L$. We also found that $\langle E \rangle = U'$. Then by induction on n we get $A \setminus A \cap L = E$. □

Proposition 2.6. *Assume $n = 2$ and $d \geq 4$. Then any tensor q has some $A \in \mathcal{S}(q,t)$ such that $\rho(A) \leq d-1$.*

Proof. If q is not concise it is sufficient to use Sylvester's theorem of binary forms ([6]). Now assume that q is concise. The statement is a consequence of the proofs in [7]. Fix q. If q is not concise we may take $t = r_{1,d}(q)$. If q is not concise A. De Paris proved the existence of d distinct lines $L_1, \dots, L_d$, t and $A \in \mathcal{S}(q,t)$ such that $t \leq \lfloor (d^2+6d+1)/4 \rfloor$ and $\#(A \cap L_i) \leq \lfloor (i+2)/2 \rfloor$. Use d residual exact sequences to prove that $h^1(\mathcal{I}_A(d-1)) = 0$. □

Remark 2.7. Proposition 2.5 shows that Proposition 2.6 is sharp, at least for some concise q, i.e. $\rho(A) > d-2$ even for large d. Part (iii) of Proposition 2.5 shows the existence of a concise q and an integer t such that $\mathcal{S}(q,t) \neq \emptyset$ and $\rho(A) = d$ for all $A \in \mathcal{S}(q,t)$.

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