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Long time behavior of a class of nonlocal parabolic equations without uniqueness

Comportement à long terme d'une classe d'équations paraboliques non-locales sans unicité

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ABSTRACT. In this paper we consider a class of nonlocal parabolic equations without uniqueness using a new framework developed by Cheskidov and Lu which called evolutionary system. We first prove the existence of weak solutions by using the compactness method. However, the Cauchy problem can be non-unique and we also give a sufficient condition for uniqueness. Then we use the theory of evolutionary system to investigate the asymptotic behavior of weak solutions via attractors and its properties. The novelty is that our results extend and improve the previous results and it seems to be the first results for this kind of system via using evolutionary systems.
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1. Introduction

Recently, the problem how to describe the limit behavior of solutions of evolution equations for which the Cauchy problem can have non-unique solution arouses much interest (see [7, 12, 14, 17, 24, 28]). In such a situation one cannot use directly the classical scheme of construction of a dynamical system in the phase space of initial conditions of the Cauchy problem and find a global attractor of this dynamical system. There are several abstract frameworks for studying dynamical systems without uniqueness. In recent years, a new framework work was developed in [16, A. Cheskidov and C. Foias], [17, A. Cheskidov], [18, 19, A. Cheskidov and S. Lu], [28, S. Lu] and was called evolutionary system. It was first introduced in [16, A. Cheskidov and C. Foias] to study a weak global attractor and a trajectory attractor for the autonomous 3D Navier Stokes equation, and the theory was developed further in [17, A. Cheskidov], [18, 19, A. Cheskidov and S. Lu], [28, S. Lu] to make it applicable to arbitrary autonomous and nonautonomous dissipative partial differential equation without uniqueness. The avantage of this framework lies in a simultaneous use of weak and strong metrics, which makes it applicable to any partial differential equation for which the uniqueness of solutions may be in limbo and avoids the necessity of constructing a symbol space (see [16, 17, 18, 19, 28] for more details).

Before to start, let us denote the spaces by $H := L^2(\Omega)$ and $V := H_0^1(\Omega)$, and denote by $(\cdot,\cdot)$ and $|\cdot|_2$ the H-inner product and the corresponding H-norm. The inner product in V is presented by $((\cdot,\cdot))$ and by $\|\cdot\|$ its associated norm. Let V' be the dual of V and denote by $\langle\cdot,\cdot\rangle$ the duality product between spaces V and V'. We identify H with its dual, and so, we have a chain of compact and dense embeddings $V \subset\subset H \subset V'$. This allows us to make an abuse of the notation considering $\ell \in H$ and denoting (ℓ, u) like $\ell(u)$. We sometime use the letter C denote a constant which may be different in each occasion throughout this paper.

Let $\Omega \subset \mathbb{R}^N (N \geq 1)$ be a bounded open set with smooth boundary which satisfies the Poincaré inequality, i.e., there exists a constant $\lambda_\Omega > 0$ such that

$$\lambda_\Omega \int_\Omega u^2(x)dx \leq \int_\Omega |\nabla u(x)|^2 dx, \ \forall u \in V. \tag{1.1}$$

In this paper we will use the theory of the evolutionary system to consider the following nonlinear nonlocal parabolic problem with zero Dirichlet boundary condition:

$$\begin{cases} \dfrac{\partial u}{\partial t} + Au = \mathcal{F}(u) + f(x,t), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ u(.,\tau) = u_\tau, & \tau \in \mathbb{R}, \end{cases} \tag{1.2}$$

where $\mathcal{F}$ is a nonlocal and possibly nonlinear operator and f is an external source. The conditions of $\mathcal{F}$ and f are specified latter. The operator A is a uniformly parabolic operator defined by

$$Au(x) := -\sum_{i,j=1}^N \partial_{x_j}(a^{ij}(\ell(u(x)))\partial_{x_i}u(x)),$$

and $a^{ij}(s) = a^{ji}(s)$, $1 \leq i,j \leq N$, are bounded continuous from $\mathbb{R}$ into $\mathbb{R}$, i.e., there exists a positive constants M such that

$$|a^{ji}(t)| \leq M, \ \forall t \in \mathbb{R}. \tag{1.3}$$

Moreover, there exist positive constants λ_A, Λ_A such that

$$\lambda_A|\eta|^2 \leq \sum_{i,j=1}^N a^{ij}(\nu)\eta_i\eta_j \leq \Lambda_A|\eta|^2, \ \forall \eta \in \mathbb{R}^N, \forall \nu \in \mathbb{R}. \tag{1.4}$$

In special case, $a^{ij}(s)$, $1 \leq i,j \leq N$, are will also be assumed to satisfy a local Lipschitz condition:

$$|a^{ij}(\xi) - a^{ij}(\eta)| \leq L_a|\xi - \eta|, \ \forall \xi, \eta \in \mathbb{R}. \tag{1.5}$$

Actually, the function ℓ should be ℓ_g. However, for simplicity and no affect on our results, we could use the notation ℓ. If there is any necessity, we will specify it. The continuous functional $\ell : H \to \mathbb{R}$ is defined by

$$\ell(u(t)) := \int_\Omega g(x)u(x,t)dx. \tag{1.6}$$

The functions g, u_τ are such that

$$g, u_\tau \in H.$$

Of course, we can have many different forms of ℓ in these kinds of problems. For instance, in the case of population dynamics (see [23]), several obvious candidates come in mind such as

$$\ell(u) = \int_\Omega u(x,t)dx,$$

for $g \equiv 1$. If $g = \chi_{\Omega'}$ where Ω' is a subdomain of Ω, then

$$\ell(u) = \int_{\Omega'} u(x,t)dx.$$

In [22] ℓ is no more a linear form on H but represents some elastic energy given by

$$\ell(u) = \int_{\Omega} |\nabla u(x,t)|^2 dx.$$

It is also possible - depending on the applications that we have in mind - to have different ℓ's in the coefficient and to have coefficients depending on several terms (see [21]).

Remark 1.1. *The operator induced by A can be interpreted as $A \in \mathcal{L}(V, V')$ and is symmetric with*

$$\langle Au, u\rangle \geq \lambda_A \big|\big|\nabla u\big|\big|_2^2,\ \forall u \in V.$$

Since V is included in H with compact injection, as a consequence of the Hilbert-Schmidt Theorem there exists a nondecreasing sequence of positive real numbers,

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots,$$

with $\lim_{n\to\infty} \lambda_n = +\infty$ and there exists an orthonormal basis $\{e_j : j \geq 1\}$ of H. Moreover, $\{e_j : j \geq 1\}$ is an orthogonal basis of V.

The operator $\mathcal{F} : H \to H$ fulfills the following assumptions:

(H1) $\mathcal{F}$ is continuous with respect to the H-norm.

(H2) There exists $\beta \in (0, \lambda_A\lambda_\Omega)$ and $c_\beta > 0$ such that

$$(u, \mathcal{F}(u)) \leq \beta|u|_2^2 + c_\beta, \tag{1.7}$$

for all $u \in H$.

(H3) There exists some nondecreasing continuous function $\rho : [0,\infty) \to \mathbb{R}$ such that

$$|\mathcal{F}(u)|_2^2 \leq \rho(|u|_2).$$

In special cases $\mathcal{F}$ will also be assumed to satisfy a local Lipschitz condition:

(H4) For all $R > 0$ there exists L_R such that if $v, w \in H$ with $|v|_2 \leq R$; $|w|_2 \leq R$ then

$$|\mathcal{F}(v) - \mathcal{F}(w)|_2 \leq L_R|v - w|_2.$$

A motivation for the operator $\mathcal{F}$ is that the nonlocal effect can appear in nonlinear term. For example, in population dynamics, it could depend on the population size on some subdomain of Ω. So one can take $\mathcal{F}(u)(x) = r(u(x))\bar{u}(1 - \bar{u})$ where

$$\bar{u}(x) = \frac{1}{|B(x;\delta)|} \int_{B(x;\delta)\cap\Omega} u(y)dy,$$

for some small $\delta > 0$, and $r : \mathbb{R} \to \mathbb{R}$, r is continuous and there exists a constant $C > 0$ such that r satisfies $|r(s)| \le C$ and $sr(s) \ge 0$ for all $s \in \mathbb{R}$ (see [5]).

Assume also that the external force $f(s) = f(., s)$ is translation bounded in $L^2_{loc}(\mathbb{R}; V')$, i.e.,

$$\|f\|^2_{tb} := \sup_{t\in\mathbb{R}} \int_t^{t+1} \|f(s)\|^2_{V'} ds < \infty.$$

In special cases, $f(s)$ is also a normal function, i.e., for any $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\sup_{t\in\mathbb{R}} \int_t^{t+\eta} \|f(s)\|^2_{V'} ds \le \varepsilon.$$

Denote by $L^2_b(\mathbb{R}; V')$ and $L^2_n(\mathbb{R}; V')$ the set of all translation bounded functions and the set of all normal functions in $L^2_{loc}(\mathbb{R}; V')$, respectively. Following [25, Theorem 4.1], we have $L^2_n(\mathbb{R}; V') \subset L^2_b(\mathbb{R}; V')$ (see [15, 25] for more details).

The problem studied is nonlocal in view of the structure of the diffusion coefficient and the nonlinearity which are determined by some global quantities. One of the justifications of such models lies in the fact that in reality the measurements are not made pointwise but through some local average. This leads to a number of mathematical difficulties which make the analysis of the problem particularly interesting. In the last decade, a lot of attention has been devoted to nonlocal parabolic problems. For more details and motivation in physics, engineering and population dynamics of nonlinear nonlocal parabolic equations of type (1.2), see [8, 29, 20] and references therein. We also refer the interested reader to [1, 2, 22, 32] for other kinds of the nonlocal terms.

To our knowledge, some recent results which are related to our situation were studied in [4, 5, 7, 10, 11]. In [4], they have proved the existence of global attractor with more general class of nonlinear terms, but the uniqueness was still guaranteed and the nonlinearity was local. In [5], they investigated the global attractor for multivalued semiflow with nonlocal nonlinear term, but the diffusive operator was local. Recently, in [7], they use the theory of multivalued semiflow to study attractors for a nonlocal reaction-diffusion equation with an energy functional. In [10, 11], they studied a p-Laplacian parabolic equation with nonlocal diffusion and local nonlinearity via the global attractor for multivalued semiflow and the pullback attractor for multivalued process.

In this paper, we consider the nonlocal effects in both the diffusion and the nonlinearity. The problem has global solutions, but the uniqueness may be not guaranteed. In our work, we will use the new abstract framework developed recently developed by Cheskidov and Lu in [16, 17, 18, 19, 28]. All results obtained here improve and extend all previous results for nonlocal parabolic equations in [4, 5, 7, 10, 11]. To our knowledge, this is the first result of long time behavior of nonlocal parabolic equation using the abstract framework of the evolutionary systems.

The structure of the paper is as follows. In Section 2, we study the existence and uniqueness of weak solutions for (1.2). In Section 3, we use theory of evolutionary systems which are presented in [16, 17, 18, 19, 28] to investigate attractors of these evolutionary systems. We also briefly recall the main results on the evolutionary systems in appendix.

2. Existence and uniqueness of weak solutions

In this section, we are going to investigate the existence and uniqueness of weak solutions to problem (1.2). First, we give the definition of weak solutions.

Definition 2.1. *A weak solution to* (1.2) *on the interval* $[\tau,\infty)$ *(or* $(-\infty,\infty)$*, if* $\tau=-\infty$*) is a function* $u(x,t)$ *such that*

$$u\in L^2_{loc}(\tau,\infty;V),\ \frac{du}{dt}\in L^2_{loc}(\tau,\infty;V')$$

and satisfies (1.2) *in the distribution sense of the space* $\mathcal{D}'(\tau,\infty;V')$.

We now prove the following theorem.

Theorem 2.1. *Let* $a^{ij}=a^{ji}$, $1\le i,j\le N$, *be bounded continuous from* $\mathbb{R}$ *into* $\mathbb{R}$ *satisfying* (1.3), (1.4) *and* $f\in L^2_{loc}(\tau,\infty;V')$ *which is translation bounded. Assume that hypotheses* **(H1), (H2)**, *and* **(H3)** *hold for* $\mathcal{F}$. *Then, for every* $u_\tau\in H$, *there exists a weak solution* u *of* (1.2) *satisfying*

$$u\in L^2_{loc}(\tau,\infty;V)\cap C([\tau,\infty);H),\ \frac{du}{dt}\in L^2_{loc}(\tau,\infty;V').$$

Moreover, the function $|u(t)|^2$ *is absolutely continuous on* $[\tau,\infty)$ *and*

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2+\sum_{i,j=1}^{N}a^{ij}(\ell(u(t)))\int_\Omega\partial_{x_i}u(t)\partial_{x_j}u(t)dx=(\mathcal{F}(u(t)),u(t))+\langle f(t),u(t)\rangle,$$

for a.e. $t\in[\tau,\infty)$.

Furthermore, if a^{ij} *satisfy* (1.5) *and* $\mathcal{F}$ *is locally Lipschitz as in hypothesis* **(H4)**, *then the weak solution is unique.*

Proof. **i) Existence.** We implement the Galerkin approximation method with a complete system $\{e_j : j\ge 1\}$ of functions in V as mentioned in Remark 1.1. The proof is analogous to [5, Proposition 8]. We outline the main points of the method. Let $u_n(s)=\sum_{j=1}^{n}u_{nj}(s)e_j$ be a Galerkin approximation of (1.2) satisfying the finite dimensional system of ordinary differential equations in sense that

$$\begin{cases}\dfrac{d}{ds}(u_n(s),e_j)+\sum\limits_{i,j=1}^{N}a^{ij}(\ell(u_n(s)))\int_\Omega\partial_{x_i}u_n(s)\partial_{x_j}e_jdx=(\mathcal{F}(u_n(s)),e_j)+\langle f(s),e_j\rangle,\\(u_n(\tau),e_j)=(u_\tau,e_j).\end{cases}$$

This implies that

$$\frac{1}{2}\frac{d}{ds}|u_n(s)|_2^2+\sum_{i,j=1}^{N}a^{ij}(\ell(u_n(s)))\int_\Omega\partial_{x_i}u_n(s)\partial_{x_j}u_n(s)dx=(\mathcal{F}(u_n(s)),u_n(s))+\langle f(s),u_n(s)\rangle. \tag{2.1}$$

Using (1.1), (1.4), (1.7) implies from (2.1) that

$$\frac{1}{2}\frac{d}{ds}|u_n(s)|_2^2+\lambda_A\big||\nabla u_n|\big|_2^2\le\beta|u_n|_2^2+c_\beta+\langle f(s),u_n(s)\rangle\le\frac{\beta}{\lambda_\Omega}\big||\nabla u_n|\big|_2^2+c_\beta+\|f(s)\|_{V'}\|u_n\|. \tag{2.2}$$

Applying in the last term of the right hand side of (2.2) and using (1.1), we deduce from (2.2) that

$$\frac{d}{ds}|u_n(s)|_2^2 + \frac{\theta}{\lambda_\Omega}\big||\nabla u_n|\big|_2^2 \leq 2c_\beta + \frac{C\lambda_\Omega}{\theta}\|f(s)\|_{V'}^2, \tag{2.3}$$

where $\theta = \lambda_A\lambda_\Omega - \beta$.

Using (1.1) again, it follows from (2.3) that

$$\frac{d}{ds}|u_n(s)|_2^2 + \theta|u_n(s)|_2^2 \leq 2c_\beta + \frac{C\lambda_\Omega}{\theta}\|f(s)\|_{V'}^2,$$

and hence

$$\frac{d}{ds}(|u_n(s)|_2^2 e^{\theta s}) \leq 2c_\beta e^{\theta s} + \frac{C\lambda_\Omega}{\theta}\|f(s)\|_{V'}^2 e^{\theta s}.$$

Integrating in s from τ to t, we obtain

$$|u_n(t)|_2^2 e^{\theta t} - |u_\tau|_2^2 e^{\theta\tau} \leq \frac{2c_\beta}{\theta}(e^{\theta t} - e^{\theta\tau}) + \frac{C\lambda_\Omega}{\theta}\int_\tau^t \|f(s)\|_{V'}^2 e^{\theta s} ds \tag{2.4}$$

for all $t \geq \tau$.

Estimating the last integral

$$\begin{aligned}
\int_\tau^t \|f(s)\|_{V'}^2 e^{\theta s} ds &\leq \int_{t-1}^t \|f(s)\|_{V'}^2 e^{\theta s} ds + \int_{t-2}^{t-1} \|f(s)\|_{V'}^2 e^{\theta s} ds + \cdots \\
&\leq e^{\theta t}\int_{t-1}^t \|f(s)\|_{V'}^2 ds + e^{\theta(t-1)}\int_{t-2}^{t-1} \|f(s)\|_{V'}^2 ds + \cdots \\
&\leq \|f\|_{tb}^2(1 + e^{-\theta} + e^{-2\theta} + \cdots)\, e^{\theta t} \\
&\leq \frac{\|f\|_{tb}^2 e^\theta}{e^\theta - 1} e^{\theta t}.
\end{aligned} \tag{2.5}$$

Combining (2.4) and (2.5) leads

$$|u_n(t)|_2^2 \leq |u_\tau|_2^2 e^{-\theta(t-\tau)} + \frac{2c_\beta}{\theta} + \frac{C\lambda_\Omega\|f\|_{tb}^2 e^\theta}{\theta(e^\theta - 1)}. \tag{2.6}$$

It follows from (1.1), (2.3), (2.5) and (2.6) that the sequence $\{u_n\}$ remains in a bounded subset of $L^2_{loc}(\tau,\infty;V) \cap L^\infty_{loc}(\tau,\infty;H)$, since $|u_\tau|_2^2$ is bounded. Therefore, there exists $u \in L^2_{loc}(\tau,\infty;V) \cap L^\infty_{loc}(\tau,\infty;H)$ and a subsequence of $\{u_n\}$ (relabeled the same) such that

$u_n \rightharpoonup^* u$ weak star convergence in $L^\infty_{loc}(\tau,\infty;H)$,
$u_n \rightharpoonup u$ weak convergence in $L^2_{loc}(\tau,\infty;V)$.

The hypotheses **(H2)**, and **(H3)** imply that $\{\mathcal{F}(u_n)\}$ is bounded in $L^2_{loc}(\tau,\infty;H)$. Note that

$$\langle Au, w\rangle = \int_\Omega \sum_{i,j=1}^N a^{ij}(\ell(u_n))\partial_{x_i}u_n\partial_{x_j}w dx,$$

for all $w \in V$. Because of boundedness of a^{ij} and (1.4), we deduce that $\{Au_n\}$ is bounded in $L^2_{loc}(\tau,\infty;V')$.

Since

$$\frac{d}{dt}u_n + Au_n = \mathcal{F}(u_n) + f(s),$$

we get $\{\frac{d}{dt}u_n\}$ is bounded in $L^2_{loc}(\tau,\infty;V')$. By the Aubin-Lions-Simon compactness lemma (see [6]), we have that $\{u_n\}$ is compact in $L^2_{loc}(\tau,\infty;H)$. So up to a subsequence that

$$u_n \to u \text{ in } L^2_{loc}(\tau,\infty;H). \tag{2.7}$$

Therefore

$$\ell(u_n) \to \ell(u) \text{ in } L^2_{loc}(\tau,\infty).$$

Since a^{ij} are bounded continuous, we get

$$a^{ij}(\ell(u_n)) \to a^{ij}(\ell(u)) \text{ in } L^2_{loc}(\tau,\infty). \tag{2.8}$$

Combining (1.4), (2.7) and (2.8), we deduce that for all $\varphi \in C^\infty([\tau,\infty);V)$

$$\int_\tau^t \sum_{i,j=1}^N a^{ij}(l(u_n(s))) \int_\Omega \partial_{x_i}u_n\partial_{x_j}\varphi dxds \to \int_\tau^t \sum_{i,j=1}^N a^{ij}(l(u(s))) \int_\Omega \partial_{x_i}u\partial_{x_j}\varphi dxds.$$

Hypothesis **(H1)** on the continuity of $\mathcal{F}$ and (2.7) imply that

$$\mathcal{F}(u_n(t)) \rightharpoonup \mathcal{F}(u(t)) \text{ in } L^2_{loc}(\tau,\infty;H).$$

Taking the limit as $n\to\infty$ and using [13, Lemma 2.2] give that u is a weak solution to problem (1.2).

ii) Uniqueness and continuous dependence on the initial data. Let us denote by u_1 and u_2 two weak solutions of (1.2) with initial data $u_{\tau 1}, u_{\tau 2} \in H$ respectively. By difference we get

$$\begin{aligned}&\frac{d}{dt}(u_1-u_2,v) + \sum_{i,j=1}^N a^{ij}(\ell(u_1(t)))\int_\Omega \partial_{x_i}(u_1-u_2)\partial_{x_j}v dx\\ &\quad = \sum_{i,j=1}^N \left\{a^{ij}(\ell(u_2(t))) - a^{ij}(\ell(u_1(t)))\right\}\int_\Omega \partial_{x_i}u_2\partial_{x_j}v dx + (\mathcal{F}(u_1)-\mathcal{F}(u_2),v),\ \forall v\in V.\end{aligned}$$

It follows from (2.6) that $|u_1|_2^2 \le R_\tau^2$ and $|u_2|_2^2 \le R_\tau^2$ where

$$R_\tau^2 := \max\{|u_{\tau 1}|_2^2; |u_{\tau 2}|_2^2\} + \frac{2c_\beta}{\theta} + \frac{C\lambda_\Omega \|f\|_{tb}^2 e^\theta}{\theta(e^\theta-1)},\ \theta = \lambda_A\lambda_\Omega - \beta.$$

Taking $v = u_1 - u_2$ we derive by (1.5) and **(H4)**

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|u_1-u_2|_2^2 + \lambda_A\big|\big|\nabla(u_1-u_2)\big|\big|_2^2\\ &\qquad \le NL_a|\ell(u_2(t)) - \ell(u_1(t))|\big|\big|\nabla u_2\big|\big|_2\big|\big|\nabla(u_1-u_2)\big|\big|_2 + L_{R_\tau}|u_1-u_2|_2^2.\end{aligned}$$

Recalling (1.6) we get

$$\frac{1}{2}\frac{d}{dt}|u_1-u_2|_2^2+\lambda_A\big|\big|\nabla(u_1-u_2)\big|\big|_2^2 \\ \leq NL_a|g|_2\,|u_1-u_2|_2\,\big|\big|\nabla u_2\big|\big|_2\,\big|\big|\nabla(u_1-u_2)\big|\big|_2+L_{R_\tau}|u_1-u_2|_2^2. \quad (2.9)$$

Applying in the first term of the right hand side of (2.9) the Young inequality, we deduce that

$$\frac{d}{dt}|u_1-u_2|_2^2\leq 2c(t,\tau)|u_1-u_2|_2^2,$$

where

$$c(t,\tau):=L_{R_\tau}+\frac{N^2L_a^2|g|_2^2\big|\big|\nabla u_2\big|\big|_2^2}{2\lambda_A}\in L^1(\tau,\infty).$$

Therefore, the uniqueness follows by the Gronwall inequality. This complete the proof of the theorem. □

Remark 2.1. Theorem 2.1 illustrates how the nonlocal terms effect on the global existence and uniqueness of the weak solutions. These results also improve and extend directly the results in [4, 5, 7].

3. Attractors for evolutionary systems

3.1. *Evolutionary systems*

In this section, for completeness, we briefly recall here the basic definitions and main results on the evolutionary systems which was developed in recent years in [16, A. Cheskidov and C. Foias], [17, A. Cheskidov], [18, 19, A. Cheskidov and S. Lu], [28, S. Lu] in order to study dynamical systems without uniqueness of solutions. This theory was developed by series of papers and all results can be found in [16, 17, 18, 19, 28].

Phase space endowed with two metrics

Assume that a set X is endowed with two metrics $d_s(\cdot,\cdot)$ and $d_w(\cdot,\cdot)$ respectively, satisfying the following conditions:

(1) X is d_w-compact.

(2) If $d_s(u_n,v_n)\to 0$ as $n\to\infty$ for some $u_n,v_n\in X$, then $d_w(u_n,v_n)\to 0$ as $n\to\infty$.

Due to the property (2), $d_w(\cdot,\cdot)$ will be referred to as a weak metric on X. Denote by $\bar{A}^\bullet$ the closure of a set $A\subset X$ in the topology generated by $d_\bullet$. Here (the same below) $\bullet=s$ or w. Note that any strongly compact (d_s-compact) set is weakly compact (d_w-compact), and any weakly closed set is strongly closed.

Autonomous case

Let

$$\mathcal{T}:=\{I:I=[0,\infty)\subset\mathbb{R},\ \text{or } I=(-\infty,\infty)\},$$

and for each $I \in \mathcal{T}$, let $\mathfrak{F}(I)$ denote the set of all X-valued functions on I. Now we define an evolutionary system $\mathcal{E}$ as follows

Definition 3.1. *[16, Definition 2.1] A map $\mathcal{E}$ that associates to each $I \in \mathcal{T}$ a subset $\mathcal{E}(I) \subset \mathfrak{F}(I)$ will be called an evolutionary system if the following conditions are satisfied:*

(1) $\mathcal{E}([0, \infty)) \neq \emptyset$.

(2) $\mathcal{E}(I + s) = \{u(\cdot) : u(\cdot + s) \in \mathcal{E}(I)\}$ *for all* $s \in \mathbb{R}$.

(3) $\{u(\cdot)|_{I_2} : u(\cdot) \in \mathcal{E}(I_1)\} \subset \mathcal{E}(I_2)$ *for all pairs* $I_1, I_2 \in \mathcal{T}$*, such that* $I_2 \subset I_1$.

(4) $\mathcal{E}((-\infty, \infty)) = \{u(\cdot) : u(\cdot)|_{[\tau,\infty)} \in \mathcal{E}([\tau, \infty)), \forall \tau \in \mathbb{R}\}$.

We will refer to $\mathcal{E}(I)$ as the set of all trajectories on the time interval I. The set $\mathcal{E}((-\infty, \infty))$ is called the kernel of $\mathcal{E}$ and the trajectories in it are called complete.

Let $C([a, b]; X_\bullet)$ be the space of $d_\bullet$-continuous X-valued functions on $[a, b]$ endowed with the metric

$$d_{C([a,b];X_\bullet)}(u, v) := \sup_{t\in[a,b]} d_\bullet(u(t), v(t)).$$

Denote by $C([a, \infty); X_\bullet)$ the space of $d_\bullet$-continuous X-valued functions on $[a, \infty)$ endowed with the metric

$$d_{C([a,\infty);X_\bullet)}(u, v) := \sum_{l\in\mathbb{N}} \frac{1}{2^l} \frac{d_{C([a,a+l];X_\bullet)}(u, v)}{1 + d_{C([a,a+l];X_\bullet)}(u, v)}.$$

Note that the convergence in $C([a, \infty); X_\bullet)$ is equivalent to uniform convergence on compact sets.

Let

$$\bar{\mathcal{E}}([\tau, \infty)) := \overline{\mathcal{E}([\tau, \infty))}^{C([\tau,\infty);X_w)}, \ \forall \tau \in \mathbb{R},$$

and

$$\bar{\mathcal{E}}((-\infty, \infty)) := \{u(\cdot) : u(\cdot)|_{[\tau,\infty)} \in \bar{\mathcal{E}}([\tau, \infty)), \forall \tau \in \mathbb{R}\}.$$

It can be checked that $\bar{\mathcal{E}}$ is also an evolutionary system and it is called the closure of the evolutionary system $\mathcal{E}$. We add for $\bar{\mathcal{E}}$ the top-script $^-$ to the corresponding notations for $\mathcal{E}$.

Let $\mathcal{K} := \mathcal{E}((-\infty, \infty))$ and $\bar{\mathcal{K}} := \bar{\mathcal{E}}((-\infty, \infty))$, which are called the kernel of $\mathcal{E}$ and $\bar{\mathcal{E}}$, respectively. Let also

$$\Pi_+\mathcal{K} := \{u(\cdot)|_{[0,\infty)} : u \in \mathcal{K}\} \ \text{ and } \ \Pi_+\bar{\mathcal{K}} := \{u(\cdot)|_{[0,\infty)} : u \in \bar{\mathcal{K}}\}.$$

We will investigate evolutionary systems $\mathcal{E}$ satisfying the following properties:

(A1) $\mathcal{E}([0, \infty))$ is a precompact set in $C([0, \infty); X_w)$.

(A2) (Energy inequality) Assume that X is a set in some Banach space H satisfying the Radon-Riesz property (see below) with the norm denoted $|\cdot|$, such that $d_s(x, y) = |x - y|$ for $x, y \in X$ and d_w

induces the weak topology on X. Assume also that for any $\varepsilon > 0$, there exists $\delta > 0$, such that for every $u \in \mathcal{E}([0,\infty))$ and $t > 0$,

$$|u(t)| \leq |u(t_0)| + \varepsilon,$$

for t_0 a.e. in $(t-\delta, t)$.

(A3) (Strong convergence a.e.) Let $u_n \in \mathcal{E}([0,\infty))$ be such that, u_n is $d_{C([0,T;X_w)}$-Cauchy sequence in $C([0,T;X_w)$ for some $T > 0$. Then $u_n(t)$ is d_s-Cauchy sequence a.e. in $[0,T]$.

We also recall stronger properties (see [16, 17, 18, 19, 28]) as follows

(B1) $\mathcal{E}([0,\infty))$ is a compact set in $C([0,\infty); X_w)$.

(B2) (Energy inequality) Assume that X is a set in some Banach space H satisfying the Radon-Riesz property (see below) with the norm denoted $|\cdot|$, such that $d_s(x,y) = |x-y|$ for $x, y \in X$ and d_w induces the weak topology on X. Assume also that for any $\varepsilon > 0$, there exists $\delta > 0$, such that for every $u \in \mathcal{E}([0,\infty))$ and $t > 0$,

$$|u(t)| \leq |u(t_0)| + \varepsilon,$$

for t_0 a.e. in $(t-\delta, t)$.

(B3) (Strong convergence a.e.) Let $u, u_n \in \mathcal{E}([0,\infty))$ be such that $u_n \to u$ in $C([0,T;X_w)$ for some $T > 0$. Then $u_n(t) \to u(t)$ strongly a.e. in $[0,T]$.

A Banach $\mathcal{B}$ is said to satisfy the Radon-Riesz property if for any sequence $\{x_n\} \subset \mathcal{B}$,

$$x_n \to x \text{ strongly in } \mathcal{B} \Leftrightarrow \begin{cases} x_n \to x \text{ weakly in } \mathcal{B}, \\ \|x_n\|_{\mathcal{B}} \to \|x\|_{\mathcal{B}}, \end{cases} \quad as\ n \to \infty.$$

In many applications X is bounded closed set in a uniformly convex separable Banach space H. Then the weak topology of H is metrizable on X, and X is compact with respect to such a metric d_w. Moreover, the Radon-Riesz property is automatically satisfied.

If $\mathcal{E}$ satisfies the conditions **(A1)-(A3)**, then $\bar{\mathcal{E}}$ satisfies **(B1)-(B3)** (see [19]).

Let $P(X)$ be the set of all subsets of X. For every $t \geq 0$, define a set-valued map

$$R(t) : P(X) \to P(X),$$
$$R(t)A := \{u(t) : u(0) \in A, u(\cdot) \in \mathcal{E}([0,\infty))\},\ A \subset X.$$

Note that the assumptions on $\mathcal{E}$ implies that $R(t)$ enjoys the following property:

$$R(t+s)A \subset R(t)R(s)A,\ A \subset X,\ t, s \geq 0.$$

Consider an arbitrary evolutionary system $\mathcal{E}$. For a set $A \subset X$ and $r > 0$, denote

$$B_\bullet(A, r) = \{u \in X : d_\bullet(u, A) < r\},$$

where

$$d_\bullet(u, A) := \inf_{x \in A} d_\bullet(u, x),\ \bullet = s, w.$$

Definition 3.2. *[16, Definition 2.2]*

(1) A set $A \subset X$ uniformly attracts a set $B \subset X$ in $d_\bullet$-metric ($\bullet = s, w$) if for any $\varepsilon > 0$, there exists t_0, such that

$$R(t)B \subset B_\bullet(A, \varepsilon),\ \forall t \geq t_0.$$

(2) A set $A \subset X$ is a $d_\bullet$-attracting set ($\bullet = s, w$) if it uniformly attracts X in $d_\bullet$-metric.

Definition 3.3. *[16, Definition 2.3] A set $\mathcal{A}_\bullet$ is a $d_\bullet$-global attractor ($\bullet = s, w$) if $\mathcal{A}_\bullet$ is a minimal $d_\bullet$-closed $d_\bullet$-attracting set.*

Note that the empty set is never an attracting set. Note also that since X is not strongly compact, the intersection of two d_s-closed d_s-attracting sets might not be d_s-attracting. Nevertheless, the global attractor $\mathcal{A}_\bullet$ is unique if it exists.

Definition 3.4. *[16, Definition 2.8] The $\omega_\bullet$-limit ($\bullet = s, w$) of a set $A \subset X$ is*

$$\omega_\bullet(A) := \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} R(t)A}^{\bullet}.$$

An equivalent definition of the $\omega_\bullet$-limit set is given by

$$\omega_\bullet(A) = \{x \in X : \text{ there exist sequences } t_n \to \infty \text{ as } n \to \infty \text{ and } x_n \in R(t_n)A, \\ \text{such that } x_n \to x \text{ in } d_\bullet\text{-metric as } n \to \infty\}.$$

Definition 3.5. *[17, Definition 4.1] An evolutionary system $\mathcal{E}$ is asymptotically compact if for any $t_n \to +\infty$ and any $x_n \in R(t_n)X$, the sequence $\{x_n\}$ is relatively strongly compact.*

Theorem 3.1. *[19, Theorem 3.7] Let $\mathcal{E}$ be an evolutionary system satisfying* **(A1)**, **(A2)**, *and* **(A3)**, *and assume that its closure $\bar{\mathcal{E}}$ satisfies $\bar{\mathcal{E}}((-\infty, \infty)) \subset C((-\infty, \infty); X_s)$. Then $\mathcal{E}$ is asymptotically compact.*

Definition 3.6. *[19, Definition 3.8] Let $\mathcal{E}$ be an evolutionary system. If an map $\mathcal{E}^1$ that associates to each $I \in \mathcal{T}$ a subset $\mathcal{E}^1(I) \subset \mathcal{E}(I)$ is also an evolutionary system, we will call it an evolutionary subsystem of $\mathcal{E}$, and denote by $\mathcal{E}^1 \subset \mathcal{E}$.*

We define the following mapping:

$$\widetilde{R}(t)A := \{u(t) : u(0) \in A, u \in \mathcal{K}\},\ A \subset X, t \in \mathbb{R}.$$

Definition 3.7. *[17, Definition 5.1] A set $A \subset X$ is positively invariant if*

$$\widetilde{R}(t)A \subset A,\ \forall t \geq 0.$$

A is invariant if

$$\widetilde{R}(t)A = A,\ \forall t \geq 0.$$

A is quasi-invariant if for every $a \in A$ there exists a complete trajectory $u \in \mathcal{K}$ with $u(0) = a$ and $u(t) \in A$ for all $t \in \mathbb{R}$.

We now reconsider the evolutionary systems $\mathcal{E}$ satisfying $\mathcal{E}([0,\infty)) \subset C([0,\infty);X_w)$. Note that $\mathcal{E}([0,\infty))$ may not be closed in $C([0,\infty);X_w)$. Define the family of translation operators $\{T(s)\}_{s\geq 0}$,

$$(T(s)u)(\cdot) := u(\cdot + s)|_{[0,\infty)},\ u \in C([0,\infty);X_w).$$

We consider the dynamics of the translation semigroup $\{T(s)\}_{s\geq 0}$ acting on the phase space $C([0,\infty);X_w)$. Due to the property (3) of the evolutionary system, we see that $T(s)\mathcal{E}([0,\infty)) \subset \mathcal{E}([0,\infty)), \forall s \geq 0$.

Definition 3.8. *[28, Definition 2.9, Section 2.4]*

(1) A set $P \subset C([0,\infty);X_w)$ weakly uniformly attracts a set $Q \subset \mathcal{E}([0,\infty))$ if for any $\varepsilon > 0$, there exists t_0, such that

$$T(t)Q \subset \{v \in C([0,\infty);X_w) : \inf_{u\in P} d_{C([0,\infty);X_w)}(u,v) < \varepsilon\}, \forall t \geq t_0.$$

(2) A set $P \subset C([0,\infty);X_w)$ is a weak trajectory attracting set for an evolutionary system $\mathcal{E}$ if it weakly uniformly attracts $\mathcal{E}([0,\infty))$.

Definition 3.9. *[28, Definition 2.10, Section 2.4] A set $\mathfrak{A}_w \subset C([0,\infty);X_w)$ is a weak trajectory attractor for an evolutionary system $\mathcal{E}$ if $\mathfrak{A}_w$ is a minimal weak trajectory attracting set that is*

(i) Closed in $C([0,\infty);X_w)$.

(ii) Invariant: $T(t)\mathfrak{A}_w = \mathfrak{A}_w$, $\forall t \geq 0$.

Definition 3.10. *[28, Definition 3.1] A set $P \subset C([0,\infty);X_w)$ satisfies the weak uniform tracking property for an evolutionary system $\mathcal{E}$ if for any $\varepsilon > 0$, there exists t_0, such that for any $t^* > t_0$, every trajectory $u \in \mathcal{E}([0,\infty))$ satisfies*

$$d_{C([t^*,\infty);X_w)}(u(\cdot), v(\cdot - t^*)) < \varepsilon,$$

for some trajectory $v \in P$.

Definition 3.11. *[28, Definition 3.5] A set $P \subset C([0,\infty);X_w)$ satisfies the finite weak uniform tracking property for an evolutionary system $\mathcal{E}$ if for any $\varepsilon > 0$, there exist t_0 and a finite subset $P^f \subset P$, such that for any $t^* > t_0$, every trajectory $u \in \mathcal{E}([0,\infty))$ satisfies*

$$d_{C([t^*,\infty);X_w)}(u(\cdot), v(\cdot - t^*)) < \varepsilon,$$

for some trajectory $v \in P^f$.

Theorem 3.2. *[28, Theorem 3.6] Let $\mathcal{E}$ be an evolutionary system. Then*

1. The weak global attractor $\mathcal{A}_w$ exists, and $\mathcal{A}_w = \omega_w(X)$.

Furthermore, assume that $\mathcal{E}$ satisfies **(A1)**. *Let $\bar{\mathcal{E}}$ be the closure of $\mathcal{E}$. Then*

2. $\mathcal{A}_w = \omega_w(X) = \bar{\omega}_w(X) = \bar{\omega}_s(X) = \bar{\mathcal{A}}_w$.

3. *$\mathcal{A}_w$ is the maximal invariant and maximal quasi-invariant set w.r.t. $\bar{\mathcal{E}}$:*

$$\mathcal{A}_w := \{u_0 \in X : u_0 := u(0) \text{ for some } u \in \bar{\mathcal{K}}\}.$$

4. *The weak trajectory attractor $\mathfrak{A}_w$ exists, it is weakly compact, and $\mathfrak{A}_w = \Pi_+\bar{\mathcal{K}}$. Hence, $\mathfrak{A}_w$ satisfies the finite weak uniform tracking property for $\mathcal{E}$ and is weakly equicontinuous on $[0, \infty)$.*

5. *$\mathcal{A}_w$ is a section of $\mathfrak{A}_w$:*

$$\mathcal{A}_w = \mathfrak{A}_w(t) := \{u(t) : u \in \mathfrak{A}_w\}, \ \forall t \geq 0.$$

Definition 3.12. *[28, Definition 2.11, Section 2.5]*

(1) A set $P \subset C([0, \infty); X_w)$ strongly uniformly attracts a set $Q \subset \mathcal{E}([0, \infty))$ if for any $\varepsilon > 0$ and $T > 0$, there exists t_0, such that

$$T(t)Q \subset \{v \in C([0, \infty); X_w) : \inf_{u \in P} \sup_{\tau \in [0,T]} d_s(u(\tau), v(\tau)) < \varepsilon\}, \forall t \geq t_0.$$

(2) A set $P \subset C([0, \infty); X_w)$ is a strong trajectory attracting set for an evolutionary system $\mathcal{E}$ if it strongly uniformly attracts $\mathcal{E}([0, \infty))$.

Note that a strong trajectory attracting set for an evolutionary system $\mathcal{E}$ is a weak trajectory attracting set for $\mathcal{E}$.

Definition 3.13. *[28, Definition 3.8] A set $\mathfrak{A}_s \subset C([0, \infty); X_w)$ is a strong trajectory attractor for an evolutionary system $\mathcal{E}$ if $\mathfrak{A}_s$ is a minimal strong trajectory attracting set that is*

(1) Closed in $C([0, \infty); X_w)$.

(2) Invariant: $T(t)\mathfrak{A}_s = \mathfrak{A}_s, \ \forall t \geq 0$.

It is said that $\mathfrak{A}_s$ is strongly compact if it is compact in $C([0, \infty); X_s)$.

Definition 3.14. *[28, Definition 3.9] A set $P \subset C([0, \infty); X_w)$ satisfies the strong uniform tracking property for an evolutionary system $\mathcal{E}$ if for any $\varepsilon > 0$ and $T > 0$, there exists t_0, such that for any $t^* > t_0$, every trajectory $u \in \mathcal{E}([0, \infty))$ satisfies*

$$d_s(u(t), v(t - t^*)) < \varepsilon, \ \forall t \in [t^*, t^* + T],$$

for some T-time length piece $v \in P_T$. Here $P_T := \{v(\cdot)|_{[0,T]} : v \in P\}$.

Definition 3.15. *[28, Theorem 3.12] A set $P \subset C([0, \infty); X_w)$ satisfies the finite strong uniform tracking property for an evolutionary system $\mathcal{E}$ if for any $\varepsilon > 0$ and $T > 0$, there exist t_0 and a finite subset $P_T^f \subset \mathfrak{A}_s|_{[0,T]}$, such that for any $t^* > t_0$, every trajectory $u \in \mathcal{E}([0, \infty))$ satisfies*

$$d_s(u(t), v(t - t^*)) < \varepsilon, \ \forall t \in [t^*, t^* + T],$$

for some T-time length piece $v \in P_T^f$.

Theorem 3.3. *[28, Theorem 3.12] Let $\mathcal{E}$ be an asymptotically compact evolutionary system. Then*

1. *The strong global attractor $\mathcal{A}_s$ exists, it is strongly compact, and $\mathcal{A}_s = \mathcal{A}_w$.*

Furthermore, assume that $\mathcal{E}$ satisfies **(A1)**. *Let $\bar{\mathcal{E}}$ be the closure of $\mathcal{E}$. Then*

2. *The strong trajectory attractor $\mathfrak{A}_s$ exists and $\mathfrak{A}_s = \mathfrak{A}_w = \Pi_+\bar{\mathcal{K}}$, it is strongly compact.*
3. *$\mathfrak{A}_s$ satisfies the finite strong uniform tracking property for $\mathcal{E}$.*
4. *$\mathfrak{A}_s = \Pi_+\bar{\mathcal{K}}$ is strongly equicontinuous on $[0,\infty)$, i.e.,*

$$d_s(v(t_1), v(t_2)) \leq \theta(|t_1 - t_2|),\ \forall t_1, t_2 \geq 0, \forall v \in \mathfrak{A}_s,$$

where $\theta(s)$ is a positive function tending to 0 as $s \to 0^+$.

Theorem 3.3 gives us the results that indicate how the dynamics on the global attractor determine the long-time dynamics of all trajectories of an evolutionary system (see [28, Corollary 3.13; Corollary 3.14]). Comparing with Theorem 3.2, Theorem 3.3 implies that the strong compactness of both the strong global attractor and the strong trajectory attractor follow simultaneously once we obtain the asymptotical compactness of an evolutionary system. Moreover, the global attractor is a section of the trajectory attractor and the trajectory attractor consists of the restriction of all the complete trajectories on the global attractor on time semiaxis $[0,\infty)$; the notion of a global attractor stresses the property of attracting trajectories starting from sets in phase space X while the notion of a trajectory attractor emphasizes the uniform tracking property.

The following theorem is an important result for the asymptotical compactness of $\mathcal{E}$.

Theorem 3.4. *[28, Theorem 3.17] An evolutionary system $\mathcal{E}$ is asymptotically compact if and only if its strongly compact strong global attractor $\mathcal{A}_s$ exists*

Corollary 3.1. *[28, Corollary 3.18] Let $\mathcal{E}$ be an evolutionary system satisfying* **(A1)** *and let $\bar{\mathcal{E}}$ be the closure of $\mathcal{E}$. If the strongly compact strong global attractor $\mathcal{A}_s$ for $\mathcal{E}$ exists, then the strongly compact strong trajectory attractor $\mathfrak{A}_s$ for $\mathcal{E}$ exists. Hence*

(1) $\mathfrak{A}_s = \Pi_+\bar{\mathcal{K}}$ satisfies the finite strong uniform tracking property for $\mathcal{E}$, i.e., for any $\varepsilon > 0$ and $T > 0$, there exist t_0 and a finite subset $P_T^f \subset \mathfrak{A}_s|_{[0,T]}$, such that for any $t^* > t_0$, every trajectory $u \in \mathcal{E}([0,\infty))$ satisfies

$$d_s(u(t), v(t - t^*)) < \varepsilon,\ \forall t \in [t^*, t^* + T],$$

for some T-time length piece $v \in P_T^f$.

(2) $\mathfrak{A}_s = \Pi_+\bar{\mathcal{K}}$ is strongly equicontinuous on $[0,\infty)$, i.e.,

$$d_s(v(t_1), v(t_2)) \leq \theta(|t_1 - t_2|),\ \forall t_1, t_2 \geq 0, \forall v \in \mathfrak{A}_s,$$

where $\theta(s)$ is a positive function tending to 0 as $s \to 0^+$.

Nonautonomous case and reducing to autonomous case

Let Σ be a parameter set and $\{T(h)|h \geq 0\}$ be a family of operators acting on Σ satisfying $T(h)\Sigma = \Sigma$, $\forall h \geq 0$. Any element $\sigma \in \Sigma$ is called (time) symbol and Σ is called (time) symbol space.

Definition 3.16. *[28, Definition 2.7] A family of maps $\mathcal{E}_\sigma$, $\sigma \in \Sigma$ that for every $\sigma \in \Sigma$ associates to each $I \in \mathcal{T}$ a subset $\mathcal{E}_\sigma(I) \subset \mathfrak{F}(I)$ will be called a nonautonomous evolutionary system if the following conditions are satisfied:*

(1) $\mathcal{E}_\sigma([\tau, \infty)) \neq \emptyset, \ \forall \tau \in \mathbb{R}$.

(2) $\mathcal{E}_\sigma(I + s) = \{u(\cdot) : u(\cdot + s) \in \mathcal{E}_{T(s)\sigma}(I)\}, \ \forall s \geq 0$.

(3) $\{u(\cdot)|_{I_2} : u(\cdot) \in \mathcal{E}_\sigma(I_1)\} \subset \mathcal{E}_\sigma(I_2)$ *for all pairs* $I_1, I_2 \in \mathcal{T}$, *such that* $I_2 \subset I_1$.

(4) $\mathcal{E}_\sigma((-\infty, \infty)) = \{u(\cdot) : u(\cdot)|_{[\tau,\infty)} \in \mathcal{E}_\sigma([\tau, \infty)), \forall \tau \in \mathbb{R}\}$.

Define

$$\mathcal{E}_\Sigma(I) := \bigcup_{\sigma \in \Sigma} \mathcal{E}_\sigma(I), \ \forall I \in \mathcal{T} \backslash \{(-\infty, \infty)\},$$

and

$$\mathcal{E}_\Sigma((-\infty, \infty)) := \{u(\cdot) : u(\cdot)|_{[\tau,\infty)} \in \mathcal{E}_\Sigma([\tau, \infty)), \forall \tau \in \mathbb{R}\}.$$

Therefore, the nonautonomous evolutionary system can be viewed as an (autonomous) evolutionary system in the following way

$$\mathcal{E}(I) := \mathcal{E}_\Sigma(I), \ \forall I \in \mathcal{T}.$$

Consequently, the above notions of invariance, quasi-invariance, and a global attractor for $\mathcal{E}$ can be extended to the nonautonomous evolutionary system $\{\mathcal{E}_\sigma\}_{\sigma \in \Sigma}$. The global attractor in the nonautonomous case will be conventionally called a uniform global attractor (or simply a global attractor). Thus, we will not distinguish between autonomous and nonautonomous evolutionary systems. If it is necessary, we denote an evolutionary system with a symbol space Σ by $\mathcal{E}_\Sigma$ and its global attractor by $\mathcal{A}^\Sigma$, trajectory attractor by $\mathfrak{A}^\Sigma$.

Definition 3.17. *[28, Definition 2.7] An evolutionary system $\mathcal{E}_\Sigma$ is a system with uniqueness if for every $u_0 \in X$ and $\sigma \in \Sigma$, there is a unique trajectory $u \in \mathcal{E}_\sigma([0, \infty))$ such that $u(0) = u_0$.*

Definition 3.18. *[28, Definition 3.20] An evolutionary system $\mathcal{E}_\Sigma$ is (weakly) closed if for any $\tau \in \mathbb{R}$, $u_n \in \mathcal{E}_{\sigma_n}([\tau, \infty))$, the convergences $u_n \to u$ in $C([\tau, \infty), X_w)$ and $\sigma_n \to \sigma$ in some topological space $\mathfrak{T}$ as $n \to \infty$ imply $u \in \mathcal{E}_\sigma([\tau, \infty))$.*

Lemma 3.1. *[28, Lemma 3.21-3.23] Let $\mathfrak{T}$ be some topological space and $\Sigma \subset \mathfrak{T}$ be sequentially compact in itself. Let $\mathcal{E}_\Sigma$ be a closed evolutionary system satisfying* **(A1)**. *Then, $\mathcal{E}_\sigma((-\infty, \infty))$ is nonempty for any $\sigma \in \Sigma$, and*

$$\mathcal{E}_\Sigma((-\infty, \infty)) = \bigcup_{\sigma \in \Sigma} \mathcal{E}_\sigma((-\infty, \infty)),$$

and

$$\mathcal{E}_\Sigma([\tau,\infty)) = \bigcup_{\sigma\in\Sigma} \mathcal{E}_\sigma([\tau,\infty)),$$

is closed in $C([\tau,\infty);X_w)$.

Suppose that $\bar{\Sigma}$ is the sequential closure of Σ in some topological space $\mathfrak{T}$. Let $\mathcal{E}_{\bar{\Sigma}}$ be an evolutionary system with symbol space $\bar{\Sigma}$.

Theorem 3.5. *[28, Theorem 3.24] Let $\mathcal{E}_\Sigma$ be an evolutionary system with uniqueness and with symbol space Σ satisfying* **(A1)** *and let $\bar{\mathcal{E}}_\Sigma$ be the closure of $\mathcal{E}_\Sigma$. Let $\bar{\Sigma}$ be the sequential closure of Σ in some topological space $\mathfrak{T}$ and $\mathcal{E}_{\bar{\Sigma}} \supset \mathcal{E}_\Sigma$ be a closed evolutionary system with uniqueness and with symbol space $\bar{\Sigma}$. Then, $\mathcal{E}_{\bar{\Sigma}} \subset \bar{\mathcal{E}}_\Sigma$. Hence,*

1. *The three weak uniform global attractors $\mathcal{A}_w^\Sigma$, $\bar{\mathcal{A}}_w^\Sigma$ and $\mathcal{A}_w^{\bar{\Sigma}}$ for evolutionary systems $\mathcal{E}_\Sigma$, $\bar{\mathcal{E}}_\Sigma$ and $\mathcal{E}_{\bar{\Sigma}}$, respectively, exist.*
2. *$\mathcal{A}_w^\Sigma$, $\bar{\mathcal{A}}_w^\Sigma$ and $\mathcal{A}_w^{\bar{\Sigma}}$ are the maximal invariant and maximal quasi-invariant set with respect to $\bar{\mathcal{E}}_\Sigma$ and satisfy the following*

   $$\mathcal{A}_w^\Sigma = \bar{\mathcal{A}}_w^\Sigma = \mathcal{A}_w^{\bar{\Sigma}} = \{u_0 : u_0 = u(0) \text{ for some } u \in \bar{\mathcal{E}}_\Sigma((-\infty,\infty))\}.$$

3. *The three weak trajectory attractors $\mathfrak{A}_w^\Sigma$, $\bar{\mathfrak{A}}_w^\Sigma$ and $\mathfrak{A}_w^{\bar{\Sigma}}$ for evolutionary systems $\mathcal{E}_\Sigma$, $\bar{\mathcal{E}}_\Sigma$ and $\mathcal{E}_{\bar{\Sigma}}$, respectively, exist and satisfy the following*

   $$\mathfrak{A}_w^\Sigma = \bar{\mathfrak{A}}_w^\Sigma = \mathfrak{A}_w^{\bar{\Sigma}} = \Pi_+ \bar{\mathcal{E}}_\Sigma((-\infty,\infty)).$$

   Hence, the three weak trajectory attractors satisfy the finite weak uniform tracking property for all the three evolutionary systems and are weakly equicontinuous on $[0,\infty)$.

4. *$\mathcal{A}_w^\Sigma$, $\bar{\mathcal{A}}_w^\Sigma$ and $\mathcal{A}_w^{\bar{\Sigma}}$ are sections of $\mathfrak{A}_w^\Sigma$, $\bar{\mathfrak{A}}_w^\Sigma$ and $\mathfrak{A}_w^{\bar{\Sigma}}$:*

   $$\mathcal{A}_w^\Sigma = \bar{\mathcal{A}}_w^\Sigma = \mathcal{A}_w^{\bar{\Sigma}} = \mathfrak{A}_w^\Sigma(t) = \bar{\mathfrak{A}}_w^\Sigma(t) = \mathfrak{A}_w^{\bar{\Sigma}}(t), \ \forall t \geq 0.$$

Furthermore, assume that $\bar{\Sigma} \subset \mathfrak{T}$ is sequentially compact in itself. Then, $\mathcal{E}_{\bar{\Sigma}} = \bar{\mathcal{E}}_\Sigma$. Hence,

5. *The following relationships on kernels hold:*

   $$\bar{\mathcal{E}}_\Sigma((-\infty,\infty)) = \mathcal{E}_{\bar{\Sigma}}((-\infty,\infty)) = \bigcup_{\sigma\in\bar{\Sigma}} \mathcal{E}_\sigma((-\infty,\infty)),$$

and $\mathcal{E}_\sigma((-\infty,\infty))$ is nonempty for any $\sigma \in \bar{\Sigma}$.

Theorem 3.6. *[28, Theorem 3.25] Assume that all conditions of Theorem 3.5 hold and one of the followings is valid:*

1. *$\bar{\mathcal{E}}_\Sigma$ is asymptotically compact.*
2. *$\mathcal{E}_\Sigma$ satisfies* **(A1)**, **(A2)** *and* **(A3)**, *and $\bar{\mathcal{E}}_\Sigma((-\infty,\infty)) \subset C((-\infty,\infty);X_s)$.*
3. *$\bar{\mathcal{E}}_\Sigma$ possesses a strongly compact strong global attractor.*

Then the three weak uniform global attractors in Theorem 3.5 are strongly compact strong uniform global attractors and the three weak trajectory attractors are strongly compact strong trajectory attractors. Moreover, the three trajectory attractors satisfy the finite strong uniform tracking property for all the three evolutionary systems and are strongly equicontinuous on $[0,\infty)$.

3.2. *Attractors*

In this section we use theory of evolutionary systems to investigate attractors of our problem. We now consider a fixed driving force $f_0(t) \in L^2_b(\mathbb{R}, V')$. Let

$$\Sigma := \{f_0(\cdot + h) : h \in \mathbb{R}\}.$$

Note that for every $f \in \Sigma$, we have

$$\|f\|_{tb} \le \|f_0\|_{tb}.$$

Let $u(t)$, $t \in [\tau, \infty)$, be a weak solution of (1.2) with $f \in \Sigma$ guaranteed by Theorem 2.1. Repeating the same arguments in the proof of Theorem 2.1, we receive

$$\frac{d}{dt}|u(t)|_2^2 + \theta |u(t)|_2^2 \le 2c_\beta + \frac{C\lambda_\Omega}{\theta}\|f_0(t)\|_{V'}^2, \tag{3.1}$$

with $\theta = \lambda_A \lambda_\Omega - \beta$ and hence

$$|u(t)|_2^2 \le |u_\tau|_2^2 e^{-\theta(t-\tau)} + \frac{2c_\beta}{\theta} + \frac{C\lambda_\Omega \|f_0\|_{tb}^2 e^\theta}{\theta(e^\theta - 1)}.$$

Therefore, there exists a uniformly (w.r.t. $\tau \in \mathbb{R}$ and f) absorbing ball $B_s(0, R) \subset H$, where the radius R depends on $\lambda_\Omega, \lambda_A, \beta, c_\beta$ and $\|f_0\|_{tb}$. We denote by X a closed absorbing ball

$$X = \{u \in H : |u|_2 \le R\}.$$

That is, for any bounded set $A \subset H$, there exists a time $t_A \ge 0$ independent of the initial time τ, such that

$$u(t) \in X, \;\; \forall t \ge \tau + t_A,$$

for every weak solution $u(t)$ with $f \in \Sigma$ and the initial data $u_\tau \in A$. It is known X is weakly compact in H and d_w-metrizable.

We investigate the following evolutionary system:

$$\mathcal{E}([\tau,\infty)) := \{u(\cdot) : u(\cdot) \text{ is a weak solution on } [\tau,\infty) \text{ with } f \in \Sigma \text{ and } u(t) \in X, \; \forall t \in [\tau,\infty)\}, \tau \in \mathbb{R},$$

$$\mathcal{E}((-\infty,\infty)) := \{u(\cdot) : u(\cdot) \text{ is a weak solution on } (-\infty,\infty) \text{ with } f \in \Sigma \text{ and } u(t) \in X, \; \forall t \in (-\infty,\infty)\}.$$

Clearly, all conditions in Definition 3.1 hold for the above evolutionary system $\mathcal{E}$ because of the translation identity, i.e., a weak solution of (1.2) with $f \in \Sigma$ initiating at time $\tau + h$ is also a weak solution of (1.2) with $f(\cdot + h) \in \Sigma$ initiating at time τ.

Lemma 3.2. *Let $u_k(t)$ be a sequence of weak solutions of* (1.2) *with $f_k \in \Sigma$, such that $u_k(t) \in X$ for all $t \geq t_1$. Then*

$$u_k \text{ is bounded in } L^2(t_1, t_2; V),$$
$$\frac{\partial u_k}{\partial t} \text{ is bounded in } L^2(t_1, t_2; V'),$$

for all $t_2 > t_1$. Moreover, there exists a subsequence u_{k_j} converges in $C([t_1, t_2]; H_w)$ to some $\upsilon \in C([t_1, t_2]; H)$, i.e.,

$$(u_{k_j}, \varphi) \to (\upsilon, \varphi) \text{ uniformly on } [t_1, t_2],$$

as k_j tends to infinity and all $\varphi \in H$.

Proof. It follows from the proof of Theorem 2.1 that, for all $t_2 > t_1$,

$$\{u_k\} \text{ is bounded in } L^2(t_1, t_2; V) \cap L^\infty(t_1, t_2; H), \tag{3.2}$$

and

$$\{\frac{\partial u_k}{\partial t}\} \text{ is bounded in } L^2(t_1, t_2; V'), \tag{3.3}$$

$$\{\mathcal{F}(u_k)\} \text{ is bounded in } L^2(t_1, t_2; H). \tag{3.4}$$

By the compactness theorem (see [15, Theorem II.1.4], [30, Theorem 8.1]), we obtain that

$$\{u_k\} \text{ is precompact in } L^2(t_1, t_2; H). \tag{3.5}$$

Passing to a subsequence and dropping a subindex, we infer from (3.2), (3.3), (3.4) and (3.5) that

$$u_k \rightharpoonup^* \upsilon \text{ in } L^\infty(t_1, t_2; H), \tag{3.6}$$
$$u_k \rightharpoonup \upsilon \text{ in } L^2(t_1, t_2; V), \tag{3.7}$$
$$u_k \to \upsilon \text{ in } L^2(t_1, t_2; H). \tag{3.8}$$

We also deduce from hypotheses of our problem that

$$\begin{aligned} \ell(u_k) &\to \ell(\upsilon) \text{ in } L^2(t_1, t_2), \\ a^{ij}(\ell(u_k)) &\to a^{ij}(\ell(\upsilon)) \text{ in } L^2(t_1, t_2), \\ Au_k &\rightharpoonup A\upsilon \text{ in } L^2(t_1, t_2; V'), \\ \frac{\partial u_k}{\partial t} &\rightharpoonup \frac{\partial \upsilon}{\partial t} \text{ in } L^2(t_1, t_2; V'), \\ \mathcal{F}(u_k) &\to \mathcal{F}(\upsilon) \text{ in } L^2(t_1, t_2; H), \end{aligned}$$

for some

$$\upsilon \in L^\infty(t_1, t_2; H) \cap L^2(t_1, t_2; V).$$

Note that f_0 is translation compact in $L^{2,w}_{loc}(\mathbb{R}, V')$ (see [15]). Thus, passing to a subsequence and dropping a subindex again, we also have

$$f_k \rightharpoonup f \text{ in } L^2(t_1, t_2; V'),$$

with some $f \in L^2(t_1,t_2;V')$. Passing the limits yields the following equality

$$\frac{\partial v}{\partial t} + Av = \mathcal{F}(v) + f$$

in the distribution sense of the space $\mathcal{D}'(t_1,t_2;V')$. Thanks to [13, Lemma 2.2], we infer that $v \in C([t_1,t_2];H)$. Now we need to prove that $u_k \to v$ in $C([t_1,t_2];H_w)$. Using (3.6), (3.7) and (3.8) implies that

$$u_k(t) \to v(t) \text{ in } H, \text{ a.e. } t \in [t_1,t_2].$$

Thus, for any test function $\varphi \in C_0^\infty(\Omega)$

$$(u_k(t),\varphi) \to (v(t),\varphi) \text{ a.e. } t \geq t_1.$$

Combining with (3.2), we obtain that $\{(u_k(t),\varphi)\}$ is uniformly bounded on $[t_1,t_2]$. On the other hand, using (3.3), let $0 < \delta < 1$ and $t_1 \leq t \leq t+\delta \leq t_2$, we have

$$\begin{aligned} |(u_k(t+\delta) - u_k(t),\varphi)| &= |\int_t^{t+\delta} \langle \frac{\partial u_k(s)}{\partial t},\varphi\rangle ds| \\ &\leq \delta^{\frac{1}{2}} \|\varphi\| \, \|\frac{\partial u_k}{\partial t}\|_{L^2(t,t+\delta;V')} \\ &\leq C\delta^{\frac{1}{2}}\|\varphi\|, \end{aligned}$$

for every $\varphi \in C_0^\infty(\Omega)$. That is, the sequence $\{(u_k(t),\varphi)\}$ is equicontinuous on $[t_1,t_2]$. Therefore, in view of Arzelà-Ascoli theorem, we get

$$(u_k(t),\varphi) \to (v(t),\varphi) \text{ uniformly on } [t_1,t_2], \ \forall \varphi \in C_0^\infty(\Omega).$$

Thanks to density of $C_0^\infty(\Omega)$ in H, we get

$$(u_k(t),\varphi) \to (v(t),\varphi) \text{ uniformly on } [t_1,t_2], \ \forall \varphi \in H,$$

and so the proof is complete. The readers also can consult [28, Lemma 5.3], [27, Lemma 3.2] and [31, Lemma 2.1] for more details and analogous proof. □

Lemma 3.3. *The evolutionary system $\mathcal{E}$ of* (1.2) *with the fixed f_0 satisfies* **(A1)** *and* **(A3)**. *Moreover, if f_0 is normal in $L^2_{loc}(\mathbb{R};V')$ then $\mathcal{E}$ of* (1.2) *also satisfies* **(A2)**.

Proof. Using Lemma 3.2 and (3.1). First, we verify that **(A1)** holds. Indeed, by Theorem 2.1, $\mathcal{E}([0,\infty)) \subset C([0,\infty);X_s)$. Let $\{u_k\}$ be a sequence in $\mathcal{E}([0,\infty))$. It follows from Lemma 3.2 that there exists a subsequence, still denoted by $\{u_k\}$, which converges in $C([0,1];X_w)$ to some $v^1 \in C([0,1];X_s)$ as $k \to \infty$. Passing to a subsequence and dropping a subindex once more, we have that this subsequence converges in $C([0,2];X_w)$ to some $v^2 \in C([0,2];X_s)$ as $k \to \infty$. Note that $v^1(t) = v^2(t)$ on $[0,1]$. Continuing this diagonalization process, we obtain a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ that converges in $C([0,\infty);X_w)$ to some $v \in C([0,\infty);X_s)$ as $k_j \to \infty$.

Next, we prove that **(A3)** is valid. Take a sequence $\{u_k\} \subset \mathcal{E}([0,\infty))$ be such that it is a $d_{C([0,T];X_w)}$-Cauchy sequence in $C([0,T];X_w)$ for some $T > 0$. Thanks to Lemma 3.2 again, the sequence $\{u_k\}$ is bounded in $L^2(0,T;V)$. Hence, there exists some $v(t) \in C([0,T];X_w)$, such that

$$\int_0^T |u_k(s) - v(s)|_2^2 ds \to 0, \text{ as } k \to \infty.$$

In particular, $|u_k(s)|_2 \to |v(t)|_2$ as $k \to \infty$ a.e. on $[0,T]$, which means that $\{u_k(t)\}$ is a d_s-Cauchy sequence a.e. on $[0,T]$. Thus, **(A3)** is valid.

Finally, for any $u \in \mathcal{E}([0,\infty))$ and $t > 0$, it follows from (3.1) and the absolute continuity of $|u(\cdot)|_2^2$ that

$$|u(t)|_2^2 \leq |u(t_0)|_2^2 + 2c_\beta(t-t_0) + \frac{C\lambda_\Omega}{\theta}\int_{t_0}^{t} \|f_0(s)\|_{V'}^2 ds, \tag{3.9}$$

for all $0 \leq t_0 < t$. Assume now that f_0 is normal in $L^2_{loc}(\mathbb{R}; V')$. So given $\varepsilon > 0$, there exists $0 < \delta < \frac{\varepsilon}{4c_\beta}$, such that

$$\sup_{t\in\mathbb{R}} \int_{t-\delta}^{t} \|f_0(s)\|_{V'}^2 ds \leq \frac{\theta\varepsilon}{2C\lambda_\Omega}.$$

It follows from (3.9) that

$$|u(t)|_2^2 \leq |u(t_0)|_2^2 + \varepsilon, \ \forall t_0 \in (t-\delta, t),$$

which concludes that **(A2)** holds. The readers also can find more details in [18, Lemma 3.4], [28, Lemma 5.4]. □

As a direct consequence of Theorem 3.1, Theorem 3.2, Theorem 3.3 and Lemma 3.3, we get the following result.

Theorem 3.7.

(i) Let $a^{ij} = a^{ji}$, $1 \leq i,j \leq N$, be bounded continuous from $\mathbb{R}$ into $\mathbb{R}$ satisfying (1.3)*,* (1.4) *and f_0 is translation bounded in $L^2_{loc}(\mathbb{R}; V')$. Assume that the hypotheses* **(H1), (H2)** *and* **(H3)** *hold for $\mathcal{F}$. Then the weak uniform global attractor $\mathcal{A}_w$ and the weak trajectory attractor $\mathfrak{A}_w$ for* (1.2) *with the fixed f_0 exist, $\mathcal{A}_w$ is the maximal invariant and maximal quasi-invariant set w.r.t. the closure $\bar{\mathcal{E}}$ of the corresponding evolutionary system $\mathcal{E}$ and*

$$\mathcal{A}_w = \omega_w(X) = \omega_s(X) = \{u(0) : u \in \bar{\mathcal{K}}\},$$
$$\mathfrak{A}_w = \Pi_+\bar{\mathcal{K}} = \{u(\cdot)|_{[0,\infty)} : u \in \bar{\mathcal{K}}\},$$
$$\mathcal{A}_w = \mathfrak{A}_w(t) = \{u(t) : u \in \mathfrak{A}_w\}, \ \forall t \geq 0.$$

Moreover, $\mathfrak{A}_w$ satisfies the finite weak uniform tracking property and is weakly equicontinuity on $[0,\infty)$.

(ii) Furthermore, if f_0 is normal in $L^2_{loc}(\mathbb{R}; V')$, then then the weak global attractor $\mathcal{A}_w$ is a strongly compact strong global attractor $\mathcal{A}_s$, and the weak trajectory attractor $\mathfrak{A}_w$ is a strongly compact strong trajectory attractor $\mathfrak{A}_s$. Moreover, $\mathfrak{A}_s = \Pi_+\bar{\mathcal{K}}$ satisfies the finite strong uniform tracking property and is strongly equicontinuous on $[0,\infty)$.

Denote by

$$\bar{\Sigma} := \overline{\{f_0(\cdot + h) : h \in \mathbb{R}\}}^{L^{2,w}_{loc}(\mathbb{R};V')}.$$

Then, $\bar{\Sigma}$ endowed with the topology of $L^{2,w}_{loc}(\mathbb{R}; V')$ is metrizable and the corresponding metric space is compact (see [15]). The Lemma 3.2 can be improved as follows

Lemma 3.4. *Let $u_k(t)$ be a sequence of weak solutions of* (1.2) *with $f_k \in \bar{\Sigma}$, such that $u_k(t) \in X$ for all $t \geq t_1$. Then*

$$u_k \text{ is bounded in } L^2(t_1, t_2; V),$$
$$\frac{\partial u_k}{\partial t} \text{ is bounded in } L^2(t_1, t_2; V'),$$

for all $t_2 > t_1$. Moreover, there exists a subsequence k_j, such that f_{k_j} converges in $L^{2,w}_{loc}(\mathbb{R}; V')$ to some $f \in \bar{\Sigma}$, and u_{k_j} converges in $C([t_1, t_2]; H_w)$ to some weak solution υ of (1.2) *with f, i.e.,*

$$(u_{k_j}, \varphi) \to (\upsilon, \varphi) \text{ uniformly on } [t_1, t_2],$$

as k_j tends to infinity and all $\varphi \in H$.

The proof is similar to the proof of Lemma 3.2, so we omit it here. We can now consider another evolutionary system with $\bar{\Sigma}$ as a symbol. The family of trajectories for this evolutionary system consists of all weak solutions of the family of (1.2) with $f \in \bar{\Sigma}$ in X:

$$\mathcal{E}_{\bar{\Sigma}}([\tau, \infty)) := \{u(\cdot) : u(\cdot) \text{ is a weak solution on } [\tau, \infty) \text{ with } f \in \bar{\Sigma} \text{ and } u(t) \in X,\ \forall t \in [\tau, \infty)\}, \tau \in \mathbb{R},$$

$$\mathcal{E}_{\bar{\Sigma}}((-\infty, \infty)) := \{u(\cdot) : u(\cdot) \text{ is a weak solution on } (-\infty, \infty) \text{ with } f \in \bar{\Sigma} \text{ and } u(t) \in X,\ \forall t \in (-\infty, \infty)\}.$$

We carry out the arguments step by step as in [18, Lemma 3.4], [28, Lemma 5.10; Lemma 5.11; Lemma 5.12], we will have proved the following results.

Lemma 3.5.

(i) The evolutionary system $\mathcal{E}_{\bar{\Sigma}}$ of the family of (1.2) *with $f \in \bar{\Sigma}$ satisfies* **(B1)** *and* **(B3)**. *Moreover, if f_0 is normal in $L^2_{loc}(\mathbb{R}; V')$, then* **(B2)** *holds.*

(ii) The evolutionary system $\mathcal{E}_{\bar{\Sigma}}$ of the family of (1.2) *with $f \in \bar{\Sigma}$ is closed.*

(iii) $\bar{\mathcal{E}}_{\bar{\Sigma}} = \mathcal{E}_{\bar{\Sigma}}$.

It follows from Lemma 3.5 that $\mathcal{E} \subset \bar{\mathcal{E}} \subset \mathcal{E}_{\bar{\Sigma}}$. We get the following results for the evolutionary system $\mathcal{E}_{\bar{\Sigma}}$.

Theorem 3.8.

(i) Let $a^{ij} = a^{ji}$, $1 \leq i, j \leq N$, be bounded continuous from $\mathbb{R}$ into $\mathbb{R}$ satisfying (1.3), (1.4) *and f_0 is translation bounded in $L^2_{loc}(\mathbb{R}; V')$. Assume that the hypotheses* **(H1), (H2)** *and* **(H3)** *hold for $\mathcal{F}$. Then the weak uniform global attractor $\mathcal{A}_w^{\bar{\Sigma}}$ and the weak trajectory attractor $\mathfrak{A}_w^{\bar{\Sigma}}$ for* (1.2) *with $f \in \bar{\Sigma}$ exist, $\mathcal{A}_w^{\bar{\Sigma}}$ is the maximal invariant and maximal quasi-invariant set w.r.t. the corresponding evolutionary system $\mathcal{E}_{\bar{\Sigma}}$ and*

$$\mathcal{A}_w^{\bar{\Sigma}} = \{u(0) : u \in \mathcal{E}_{\bar{\Sigma}}((-\infty, \infty))\} = \{u(0) : u \in \bigcup_{f \in \bar{\Sigma}} \mathcal{E}_f((-\infty, \infty))\},$$

$$\mathfrak{A}_w^{\bar{\Sigma}} = \Pi_+ \bigcup_{f\in\bar{\Sigma}} \mathcal{E}_f((-\infty,\infty)),$$
$$\mathcal{A}_w^{\bar{\Sigma}} = \mathfrak{A}_w^{\bar{\Sigma}}(t) = \{u(t) : u \in \mathfrak{A}_w^{\bar{\Sigma}}\},\ \forall t \geq 0,$$

where $\mathcal{E}_f((-\infty,\infty))$ is nonempty for any $f \in \bar{\Sigma}$. Moreover, $\mathfrak{A}_w^{\bar{\Sigma}}$ satisfies the finite weak uniform tracking property and is weakly equicontinuity on $[0,\infty)$.

(ii) Furthermore, if f_0 is normal in $L^2_{loc}(\mathbb{R};V')$, then then the weak global attractor $\mathcal{A}_w^{\bar{\Sigma}}$ is a strongly compact strong global attractor $\mathcal{A}_s^{\bar{\Sigma}}$, and the weak trajectory attractor $\mathfrak{A}_w^{\bar{\Sigma}}$ is a strongly compact strong trajectory attractor $\mathfrak{A}_s^{\bar{\Sigma}}$. Moreover, $\mathfrak{A}_s^{\bar{\Sigma}}$ satisfies the finite strong uniform tracking property and is strongly equicontinuous on $[0,\infty)$.

Proof. Using Lemma 3.5, $\mathcal{E}_{\bar{\Sigma}}$ is equal to its closure $\bar{\mathcal{E}}_{\bar{\Sigma}}$. Moreover, $\mathcal{E}_{\bar{\Sigma}}((-\infty,\infty)) = \bar{\mathcal{E}}_{\bar{\Sigma}}((-\infty,\infty))$. Using Lemma 3.1 implies that $\mathcal{E}_{\bar{\Sigma}}((-\infty,\infty)) = \bigcup_{f\in\bar{\Sigma}} \mathcal{E}_f((-\infty,\infty))$. It follows from Lemma 3.5 that $\mathcal{E}_{\bar{\Sigma}}$ satisfies **(B1)**. Due to Lemma 3.1, $\mathcal{E}_f((-\infty,\infty))$ is nonempty. By applying Theorem 3.2 and Theorem 3.3, we get the rest part of the conclusions (see also [28, Theorem 4.14; Theorem 5.15] for analogous proof). □

We now have an interesting problem as follows: Are the attractors $\mathcal{A}_\bullet$, $\mathfrak{A}_\bullet$ and $\mathcal{A}_\bullet^{\bar{\Sigma}}$, $\mathfrak{A}_\bullet^{\bar{\Sigma}}$ in Theorem 3.7 and Theorem 3.8 are identical ?

By Theorem 2.1, Lemma 3.3, Lemma 3.5, Theorem 3.5 and Theorem 3.6, the answer of the above problem is positive. We get the following theorem

Theorem 3.9.

(i) Let $a^{ij} = a^{ji}$, $1 \leq i,j \leq N$, be bounded continuous from $\mathbb{R}$ into $\mathbb{R}$ satisfying (1.3)*,* (1.4)*,* (1.5) *and f_0 is translation bounded in $L^2_{loc}(\mathbb{R};V')$. Assume that the hypotheses* **(H1), (H2)**, **(H3)** *and* **(H4)** *hold for $\mathcal{F}$. Then the two weak uniform global attractors $\mathcal{A}_w$, $\mathcal{A}_w^{\bar{\Sigma}}$ and the two weak trajectory attractors $\mathfrak{A}_w$, $\mathfrak{A}_w^{\bar{\Sigma}}$ for* (1.2) *with the fixed f_0 and for* (1.2) *with $f \in \bar{\Sigma}$, respectively, exist, $\mathcal{A}_w$ and $\mathcal{A}_w^{\bar{\Sigma}}$ are the maximal invariant and maximal quasi-invariant set w.r.t. the closure $\bar{\mathcal{E}} = \mathcal{E}_{\bar{\Sigma}}$ of the corresponding evolutionary system $\mathcal{E}$ and*

$$\mathcal{A}_w = \mathcal{A}_w^{\bar{\Sigma}} = \{u(0) : u \in \mathcal{E}_{\bar{\Sigma}}((-\infty,\infty))\} = \{u(0) : u \in \bigcup_{f\in\bar{\Sigma}} \mathcal{E}_f((-\infty,\infty))\},$$
$$\mathfrak{A}_w = \mathfrak{A}_w^{\bar{\Sigma}} = \Pi_+ \bigcup_{f\in\bar{\Sigma}} \mathcal{E}_f((-\infty,\infty)),$$
$$\mathcal{A}_w = \mathcal{A}_w^{\bar{\Sigma}} = \mathfrak{A}_w^{\bar{\Sigma}}(t) = \{u(t) : u \in \mathfrak{A}_w^{\bar{\Sigma}}\},\ \forall t \geq 0,$$

where $\mathcal{E}_f((-\infty,\infty))$ is nonempty for any $f \in \bar{\Sigma}$. Moreover, $\mathfrak{A}_w = \mathfrak{A}_w^{\bar{\Sigma}}$ satisfies the finite weak uniform tracking property for $\mathcal{E}_{\bar{\Sigma}}$ and is weakly equicontinuity on $[0,\infty)$.

(ii) Furthermore, if f_0 is normal in $L^2_{loc}(\mathbb{R};V')$, then then the two weak global attractors $\mathcal{A}_w$ and $\mathcal{A}_w^{\bar{\Sigma}}$ are strongly compact strong global attractors $\mathcal{A}_s$ and $\mathcal{A}_s^{\bar{\Sigma}}$, respectively, and the two weak trajectory attractors $\mathfrak{A}_w$ and $\mathfrak{A}_w^{\bar{\Sigma}}$ are strongly compact strong trajectory attractors $\mathfrak{A}_s$ and $\mathfrak{A}_s^{\bar{\Sigma}}$, respectively. Moreover, $\mathfrak{A}_s = \mathfrak{A}_s^{\bar{\Sigma}}$ satisfies the finite strong uniform tracking property and is strongly equicontinuous on $[0,\infty)$.

Proof. Thanks to Theorem 2.1, the evolutionary systems $\mathcal{E}$ and $\mathcal{E}_{\bar{\Sigma}}$ are unique. Lemma 3.3 indicates that $\mathcal{E}$ satisfies **(A1)** and Lemma 3.5 indicates that $\mathcal{E}_{\bar{\Sigma}}$ is closed. By applying Theorem 3.5 and Theorem 3.6, we receive the rest part of the conclusions (see also [28, Theorem 4.20; Theorem 5.17; Theorem 5.18] for analogous proof). □

Remark 3.1. Our results in this paper extend and improve the previous results such as in [4, 5, 7] because we cannot use directly the classical scheme of the dynamical system of the Cauchy problem which has non-unique solution to find attractors.

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Declarations

I confirm that the manuscript has been read and approved that there are no other persons who satisfied the criteria for authorship but are not listed.

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References

[1] A.S. Ackleh and L. Ke, Existence-uniqueness and long time behavior for a class of nonlocal nonlinear parabolic evolution equations, *Proc. Amer. Math. Soc.* 128 (2000), 3483-3492.

[2] R.M.P. Almeida, S.N. Antontsev, J.C.M. Duque, On a nonlocal degenerate parabolic problem, *Nonlinear Anal. Real World Appl.* 27 (2016), 146-157.

[3] R.M.P. Almeida, S.N. Antontsev, J.C.M. Duque and J. Ferreira, A reaction-diffusion model for the non-local coupled system: existence, uniqueness, long-time behaviour and localization properties of solutions, *IMA J. Appl. Math.* 81 (2016), 44-364.

[4] C.T. Anh, L.T. Tinh and V.M. Toi, Global attractors for nonlocal parabolic equations with a new class of nonlinearities, *J. Korean Math. Soc.* 55 (2018), no. 3, 531-551.

[5] M. Anguiano, P.E. Kloeden and T. Lorenz, Asymptotic behaviour of nonlocal reaction-diffusion equations, *Nonlinear Anal.* 73 (2010), 3044-3057.

[6] F. Boyer and P. Fabrie, Mathematical Tools for the Study of the Incompressible Navier Stokes Equations and Related Models, *Applied Mathematical Sciences 183, Springer, New York*, 2013.

[7] R. Caballero, P. Marín-Rubio and José Valero, Existence and characterization of attractors for a nonlocal reaction-diffusion equation with an energy functional, *J. Dynam. Differential Equations* 34 (2022), no. 1, 443-480.

[8] T. Caraballo, M. Herrera-Cobos and P. Marín-Rubio, Long-time behavior of a non-autonomous parabolic equation with nonlocal diffusion and sublinear terms, *Nonlinear Anal.* 121 (2015), 3-18.

[9] T. Caraballo, Marta Herrera-Cobos and P. Marín-Rubio, Robustness of nonautonomous attractors for a family of nonlocal reaction diffusion equations without uniqueness, *Nonlinear Dynam.* 84 (2016), 35-50.

[10] T. Caraballo, M. Herrera-Cobos and P. Marín-Rubio, Global attractor for a nonlocal p-Laplacian equation without uniqueness of solution, *Discrete Contin. Dyn. Syst. Ser. B* 22 (2017), 1801-1816.

[11] T. Caraballo, M. Herrera-Cobos and P. Marín-Rubio, Asymptotic behaviour of nonlocal p-Laplacian reaction-diffusion problems, *J. Math. Anal. Appl.* 459 (2018), 997-1015.

[12] V. V. Chepyzhov and M. I. Vishik, Trajectory attractors for evolution equations, *C. R. Acad. Sci. Paris Sér. I Math.* 321 (1995), no. 10, 1309-1314.

[13] V. V. Chepyzhov and M. I. Vishik, Trajectory attractors for reaction-diffusion systems, *Topological Methods in Nonlinear Analysis, ournal of the Juliusz Schauder Center*, Volume 7, 1996, 49-76

[14] V.V. Chepyzhov and M.I.Vishik, Evolution equations and their trajectory attractors, *J. Math. Pures Appl.* (9) 76 (1997), no. 10, 913-964.

[15] V. Chepyzhov and M. Vishik, *Attractors for equations of mathematical physics, volume 49 of American Mathematical Society Colloquium Publications*, American Mathematical Society, Providence, RI 2002.

[16] A. Cheskidov and C. Foias, On global attractors of the 3D Navier-Stokes equations, *J. Differential Equations* 231 (2006) 714-754.

[17] A. Cheskidov, Global attractors of evolutionary systems, *J. Dynam. Differential Equations* 21 (2009), no. 2, 249-268.

[18] A. Cheskidov and S. Lu, The existence and the structure of uniform global attractors for nonautonomous reaction - diffusion systems without uniqueness, *Discrete Contin. Dyn. Syst. Ser. S* 2 (2009) 55-66.

[19] A. Cheskidov and S. Lu, Uniform global attractor of the 3D Navier-Stokes equations, *Adv. Math.* 267 (2014), 277-306.

[20] M. Chipot and B. Lovat, Some remarks on nonlocal elliptic and parabolic problems, *Nonlinear Anal.* 30 (1997), 4619-4627.

[21] M. Chipot and B. Lovat, Existence and uniqueness results for a class of nonlocal elliptic and parabolic problems, *DCDIS*, Series A, 8, (2001) 1, 35-51.

[22] M. Chipot, V. Valente and G. Vergara Caffarelli, Remarks on a nonlocal problem involving the Dirichlet energy, *Rend. Sem. Mat. Univ. Padova* 110 (2003), 199-220.

[23] N.H. Chang and M. Chipot, Nonlinear nonlocal evolution problems, *Racsam*, 97(3), 2003, pp. 423-445.

[24] O.V. Kapustyan and J. Valero, Comparison between trajectory and global attractors for evolution systems without uniqueness of solutions, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* 20 (2010), no. 9, 2723-2734.

[25] S. Lu, H. Wu and C. Zhong, Attractors for nonautonomous 2D Navier-Stokes equations with normal external forces, *Discrete Contin. Dyn. Syst.* 13 (2005), 701-719.

[26] S. Lu, Attractors for nonautonomous 2D Navier-Stokes equationswith less regular normal forces, *J. Differential Equations* 230 (2006), 196-212.

[27] S. Lu, Attractors for nonautonomous reaction-diffusion systems with symbols without strong translation compactness, *Asymptot. Anal.* 54 (2007), 197-210. Erratum: *Asymptot. Anal.* 58 (2008), 189-190.

[28] S. Lu, Strongly compact strong trajectory attractors for evolutionary systems and their applications, *Asymptot. Anal.* 133 (2023), no. 1-2, 13-75.

[29] B. Lovat, *Etudes de quelques problèmes paraboliques non locaux. Mathématiques générales*, Université Paul Verlaine-Metz, 1995. Français.

[30] J.C. Robinson, *Infinite-Dimensional Dynamical Systems*, Cambridge University Press, 2001.

[31] R. Rosa,The global attractor for the 2D Navier-Stokes flow on some unbounded domains, *Nonlinear Anal.* 32 (1998), 71-85.

[32] S. Zheng and M. Chipot, Asymptotic behavior of solutions to nonlinear parabolic equations with nonlocal terms, *Asymptot. Anal.* 45 (2005), 301-312.

