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Algebraic properties of subspace topologies

Propriétés algébriques des sous-espaces de topologies

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ABSTRACT. It is shown that the collection of all topologies on a given set X coincide with the set of subsemirings of the power set $\mathcal{P}(X)$ (equipped with union and intersection) if and only if X is finite. Furthermore, given a topological space $(X, \mathcal{T})$ and a subset A of X, we characterize when the subspace topology $\mathcal{T}_A$ is a maximal (resp., a prime) ideal of the semiring $\mathcal{T}$. As applications, we provide an algebraic characterization of the one-point compactification of a noncompact, Tychonoff space. Moreover, we describe explicitly the semiring homomorphisms from $\mathcal{P}(X)$ into $\mathcal{P}(Y)$ in case X is a finite set and Y is an arbitrary nonempty set.
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1. Introduction and preliminaries

The concept of semirings and their structures has been first introduced by Vandiver [14] in 1934. However, the early examples of semirings had appeared in 1894 with Dedekind's work [6] which investigates the algebra of ideals of a commutative ring. Semirings have been found useful for dealing with problems in several areas of applied mathematics and information sciences, as the semiring structure provides an algebraic framework for modeling and investigating the key factors in these problems. Moreover, the application of semirings to areas such as graph theory, coding theory, formal language theory, theory of discrete event dynamical systems and analysis of computer programs have been widely studied in the literature. Good accounts of semiring theory and its applications can be found in [8, 10].

Throughout this paper, by a *semiring*, we mean a nonempty set S together with two binary operations called *addition* and *multiplication* (denoted in the usual manner) such that $(S, +)$ is a commutative monoid with identity element 0 verifying $s0 = 0s = 0$ for any $s \in S$. Moreover, $(S, \cdot)$ is a monoid such that multiplication distributes over addition both from the left and from the right. If multiplication is commutative, the semiring S is called *commutative*. Any topology $\mathcal{T}$ on a set X, with union as addition and intersection as multiplication, is one of the typical examples of a commutative semiring.

We point out that the theory of semirings is far from being a straightforward generalization of the ring theory. In fact, several results on ring ideals cannot be extended to semiring ideals. For an illustration, the reader can consult Chapter 8 in [9] by Golan.

Recall that a nonempty subset H of a semiring S is said to be a *subsemiring* of S if H itself is a semiring under the operations of S. A subset I of a semiring S is called an *ideal* if $x + y \in I$, $sx \in I$ and $xs \in I$, whenever $x, y \in I$ and $s \in S$ (cf. [5]). An ideal P of the semiring S is said to be *prime* provided that $P \neq S$ and if I and J are ideals in S such that $IJ \subseteq P$, then $I \subseteq P$ or $J \subseteq P$, where

$IJ = \{xy : x \in I, y \in J\}$. It is worth noticing that if S is a commutative semiring and P is a proper ideal of S, then P is prime if and only if $x, y \in S$ and $xy \in P$ imply either $x \in P$ or $y \in P$ (see [9, Corollary 7.6] and [11]). Recall also that an ideal M in the semiring S is said to be *maximal* provided that $M \neq S$ and if I is an ideal in S such that $M \subsetneq I$, then $I = S$. It follows from [1, Theorem 11] that if S is a commutative semiring, then any maximal ideal of S is prime.

Throughout this paper, by a *space* is meant a topological space. Our main purpose here, is to deal with some algebraic properties of topologies defined on a given set. The paper is organized as follows: In Section 2, we let $\mathcal{T}(X)$ denote the collection of all topologies on a set X. Let S denote the power set $\mathcal{P}(X)$ of X and put $R = \{\emptyset, X\}$. Notice that S, with union as addition and intersection as multiplication, is a semiring and any topology $\mathcal{T}$ on X is a subsemiring of S. We use the symbol $[R, S]_{\text{sring}}$ to denote the set of all subsemirings of S containing R. The main result of this section is Theorem 2.1 which states that $\mathcal{T}(X) = [R, S]_{\text{sring}}$ if and only if X is finite.

In Section 3, we characterize, under a certain separation condition on the given topology $\mathcal{T}$, the subspace topologies that are maximal (respectively, prime) ideals of $\mathcal{T}$. In particular, we demonstrate in Theorem 3.2 that if $(X, \mathcal{T})$ is a T_1-space, then $\mathcal{T}_A$ is a maximal ideal of $\mathcal{T}$ if and only if $A = X \setminus \{x\}$ for some $x \in X$. If moreover, $(X, \mathcal{T})$ is assumed to be sober, then the later conditions are equivalent to $X \setminus A$ is an irreducible closed subset of X. Theorem 3.10 provides an "algebraic" characterization of the one-point compactification of a space $(X, \mathcal{T})$. More precisely, we show that if $(X, \mathcal{T})$ is a noncompact, Tychonoff space, and if $(cX, c\mathcal{T})$ is a (Hausdorff) compactification of $(X, \mathcal{T})$, then $(cX, c\mathcal{T})$ is the one-point compactification of $(X, \mathcal{T})$ if and only if $\mathcal{T}$ is a maximal ideal of $c\mathcal{T}$. Section 4 is devoted to the study of semiring homomorphisms of powers sets (see Definition 4.1). More precisely, we consider a finite set X and we provide an explicit description of semiring homomorphisms $\varphi : \mathcal{P}(X) \to \mathcal{P}(Y)$, where Y is an arbitrary (nonempty) set (see Theorem 4.2).

2. Topologies as semirings

For any set X, we let $\mathcal{T}(X)$ denote the collection of all topologies on X. Let $S = \mathcal{P}(X)$ be the *power set semiring* with union as addition and intersection as multiplication. In this case, $R = \{\emptyset, X\}$ is the smallest subsemiring of S. We use the symbol $[R, S]_{\text{sring}}$ to denote the set of all subsemirings of S containing R. It is obvious that for any set X, we have $\mathcal{T}(X) \subseteq [R, S]_{\text{sring}}$. It seems to be natural therefore to ask when does the last inclusion relation become an equality? A complete answer is given next.

Theorem 2.1. *$\mathcal{T}(X) = [R, S]_{\text{sring}}$ if and only if X is finite.*

Proof. The "if" part is immediate. For the "only if" part, assume by way of contradiction that X is infinite. Define T to be the collection of all finite subsets of X together with theirs complements. That is, if $O \in \mathcal{P}(X)$ then $O \in T$ if and only if either O or $X \setminus O$ is finite. It is easy to see that T is an element of $[R, S]_{\text{sring}}$. Now, we claim that T is not a topology. To this end, notice that as X is infinite, there exists $Y \subset X$ such that Y and $X \setminus Y$ are infinite. So $Y \notin T$. However, $Y = \bigcup_{y \in Y} \{y\}$ and $\{y\} \in T$ for any y. This shows that T is not closed under arbitrary union and so $T \notin \mathcal{T}(X)$. The proof is complete. □

3. Maximal ideals arising from Subspace topologies

Recall that if $(X, \mathcal{T})$ is a space and A is a subset of X, then the *subspace topology* on A is defined by $\mathcal{T}_A = \{A \cap O : O \in \mathcal{T}\}$. If A is equipped with the subspace topology then it is a space in its own right, and is called a *subspace* of $(X, \mathcal{T})$. It is easily seen that if $A \in \mathcal{T}$, then the ideal of $\mathcal{T}$ generated by A (in a usual sense) is given by:

$$\langle A \rangle = A\mathcal{T} := \{A \cap O : O \in \mathcal{T}\} = \mathcal{T}_A.$$

For any element $x \in X$, we set $m_x := \mathcal{T}_{X \setminus \{x\}}$. Note that in the particular case where $\mathcal{T} = \mathcal{P}(X)$, we have $m_x = \mathcal{P}(X \setminus \{x\})$. We start our investigation with the following result.

Lemma 3.1. *Let $(X, \mathcal{T})$ be a space, A a subset of X and $x \in X$. Then the following hold:*

1. *$\mathcal{T}_A$ is an ideal of $\mathcal{T}$ if and only if $A \in \mathcal{T}$.*
2. *The following are equivalent:*
   - *(a) m_x is a maximal ideal of $\mathcal{T}$.*
   - *(b) m_x is a prime ideal of $\mathcal{T}$.*
   - *(c) m_x is an ideal of $\mathcal{T}$.*
   - *(d) $\{x\}$ is closed.*

Proof. (1) The "only if" part is immediate since $A = X \cap A \in \mathcal{T}_A \subseteq \mathcal{T}$. For the "if" part, as $A \in \mathcal{T}$, it follows from the above comments that $\mathcal{T}_A$ is exactly the ideal of $\mathcal{T}$ generated by A.

(2) The implication (a)⇒(b) follows from [1, Theorem 11] since in any commutative semiring, any maximal ideal is prime. The implication (b)⇒(c) is trivial. The equivalence (c)⇔(d) is an immediate consequence of assertion (1). It remains to show that (c) implies (a). Assume (c). We need to show that m_x is maximal. Let I be an ideal of $\mathcal{T}$ such that $m_x \subsetneq I \subseteq \mathcal{T}$. Pick $O \in I \setminus m_x$. Clearly, $x \in O$. As $X = O \cup (X \setminus \{x\})$ is the union of two elements of I and I is an ideal of $\mathcal{T}$, it follows that $I = \mathcal{T}$. This completes the proof. □

We now record two definitions. A space $(X, \mathcal{T})$ is called:

• *Irreducible* if $X \neq \emptyset$ and whenever $X = F_1 \cup F_2$ with F_i closed, we have $X = F_1$ or $X = F_2$. This is equivalent to the fact that, every pair of non-empty open sets in X intersect.
• *Sober* if every irreducible closed subset of X is the closure of a unique point.

The following theorem provides, under an extra assumption on the space X, a necessary and sufficient condition on a subset A of X for $\mathcal{T}_A$ to be a maximal ideal of the topology of X.

Theorem 3.2. *Let $(X, \mathcal{T})$ be a T_1-space and let A be a subset of X. Then the following are equivalent:*

1. *$\mathcal{T}_A$ is a maximal ideal of $\mathcal{T}$.*
2. *$A = X \setminus \{x\}$ for some $x \in X$.*
   If moreover X is a sober space, then the (1) (and so (2)) is equivalent to
3. *$X \setminus A$ is an irreducible closed set in X.*

Proof. Let us prove the implication (1)⇒(2). Assume (1). Since $\mathcal{T}_A \neq \mathcal{T}$, there exists $O \in \mathcal{T} \setminus \mathcal{T}_A$. Thus, O is not contained in A. Pick $x \in O \setminus A$. As X is a T_1-space, then $\{x\}$ is a closed subset of X. Thus, m_x is a maximal ideal of $\mathcal{T}$ by virtue of Lemma 3.1 (2). We claim that $\mathcal{T}_A \subseteq m_x$. Indeed, from (1) in Lemma 3.1 it follows that $A \in \mathcal{T}$ and so $\mathcal{T}_A \subseteq \mathcal{T}$. But then $\mathcal{T}_A \subseteq \mathcal{T}_{X\setminus\{x\}} = m_x$ as $x \notin A$. This proves our claim. By maximality, $\mathcal{T}_A = m_x = \mathcal{T}_{X\setminus\{x\}}$ and so $A = X \setminus \{x\}$. The implication (2)⇒(1) follows from Lemma 3.1 (2) and from the fact that in any T_1-space, the singletons are closed.

For the "moreover" statement, it is clear that (2)⇒(3). The implication (3)⇒(2) follows from the fact that if X is a sober T_1-space, then the unique closed irreducible subsets are the singletons. The proof is complete. □

Example 3.3. Here, we provide an example showing that the assumption "$(X, \mathcal{T})$ is T_1" is essential in Theorem 3.2.

Let $X = \{0, 1, 2\}$ and consider the following topology $\mathcal{T} = \{\emptyset, X, \{0\}\}$ on X. Clearly, X is not T_1. The ideals of $\mathcal{T}$ are: $\mathcal{T}_\emptyset = \{\emptyset\} \subset \mathcal{T}_{\{0\}} = \{\emptyset, \{0\}\} \subset \mathcal{T}$. Thus, $\mathcal{T}_{\{0\}}$ is a maximal ideal of $\mathcal{T}$, but $X \setminus \{0\} = \{1, 2\}$ is not a singleton. This shows that the implication (1)⇒(2) in Theorem 3.2 does not hold. On the other hand, let $A = \{0, 1\} = X \setminus \{2\}$. Obviously, $\mathcal{T}_A = \{\emptyset, A, \{0\}\}$ is not even an ideal of $\mathcal{T}$ since $A \notin \mathcal{T}$. This shows that the implication (2)⇒(1) in Theorem 3.2 is false.

Example 3.4. We present an example showing that if we leave out the assumption "sober" in the "moreover" statement of Theorem 3.2, the implication (3)⇒(2) fails.

Let X be an infinite set equipped with the cofinite topology $\mathcal{T}$. As the singletons are closed, $(X, \mathcal{T})$ is a T_1-space. However, X is not sober because the whole space is an irreducible closed subset with no generic point (recall that a generic point of the space X is a point a whose closure is all of X). Now $A = \emptyset$ satisfies assertion (3) in Theorem 3.2 but fails to satisfy assertion (2).

The next theorem describes prime ideals of the form $\mathcal{T}_A$.

Theorem 3.5. *Let $(X, \mathcal{T})$ be a space and let A be a subset of X. Then the following are equivalent:*

1. *$\mathcal{T}_A$ is a prime ideal of $\mathcal{T}$.*
2. *$X \setminus A$ is an irreducible closed subset of X.*
   If moreover X is a sober T_1-space, then the (1) (and so (2)) is equivalent to
3. *$\mathcal{T}_A$ is a maximal ideal of $\mathcal{T}$.*

Proof. (1)⇒(2) As $\mathcal{T}_A$ is an ideal of $\mathcal{T}$, $A \in \mathcal{T}$ according to Lemma 3.1 (1). Thus, $X \setminus A$ is a closed subset of X. Moreover, $X \setminus A \neq \emptyset$ because $\mathcal{T}_A \neq \mathcal{T}$. It remains to show that $X \setminus A$ is irreducible. To this end, let F_1 and F_2 be two closed subsets of X such that $X \setminus A \subseteq F_1 \cup F_2$. Thus, $(X \setminus F_1) \cap (X \setminus F_2) \subseteq A$ and thus, $(X \setminus F_1) \cap (X \setminus F_2) \in \mathcal{T}_A$. Since $\mathcal{T}_A$ is a prime ideal of $\mathcal{T}$, it follows that either $X \setminus F_1 \in \mathcal{T}_A$ or $X \setminus F_2 \in \mathcal{T}_A$. Therefore, either $X \setminus A \subseteq F_1$ or $X \setminus A \subseteq F_2$.

(2)⇒(1) Since $X \setminus A$ is a closed subset of X, A is open. Thus, Lemma 3.1(1) entails that $\mathcal{T}_A$ is an ideal of $\mathcal{T}$. Moreover, $\mathcal{T}_A \neq \mathcal{T}$ because $X \setminus A$ is irreductible and so $A \neq X$. Let $O_1, O_2 \in \mathcal{T}$ such that $O_1 \cap O_2 \in \mathcal{T}_A$. Thus, $O_1 \cap O_2 \subseteq A$ and thus, $X \setminus A \subseteq (X \setminus O_1) \cup (X \setminus O_2)$. Now, the irreducibility of

$X \setminus A$ implies that either $X \setminus A \subseteq X \setminus O_1$ or $X \setminus A \subseteq X \setminus O_2$. Hence, either $O_1 \in \mathcal{T}_A$ or $O_2 \in \mathcal{T}_A$. This proves that $\mathcal{T}_A$ is a prime ideal of $\mathcal{T}$.

For the "moreover" statement, it is clear that (3)⇒(1). The implication (2)⇒(3) follows from Theorem 3.2. □

Example 3.6. Using Example 3.3, we show that the assumption "$(X, \mathcal{T})$ is T_1" is essential in Theorem 3.5. Actually, if $A = \emptyset$, then $\mathcal{T}_A = \{\emptyset\}$ is a prime ideal of $\mathcal{T}$ (since $X \setminus \emptyset = X$ is a closed irreducible subset of X), but it is not a maximal ideal because $\mathcal{T}_A \subset \mathcal{T}_{\{0\}} \subset \mathcal{T}$. Note that the space X is sober. Indeed, the closed irreducible subsets of X are exactly X and $\{1, 2\}$ and clearly, $X = \overline{\{0\}}$ and $\{1, 2\} = \overline{\{1\}} = \overline{\{2\}}$.

Example 3.7. Taking $A = \emptyset$ in Example 3.4 shows that "sober" is essential in proving (3)⇒(2) in Theorem 3.5.

We derive the following consequence from Theorems 3.2 and 3.5 by noting that a T_2-space is T_1 and sober.

Corollary 3.8. *Let $(X, \mathcal{T})$ be a T_2-space and let A be a subset of X. Then the following are equivalent:*

1. *$\mathcal{T}_A$ is a maximal ideal of $\mathcal{T}$.*
2. *$\mathcal{T}_A$ is a prime ideal of $\mathcal{T}$.*
3. *$A = X \setminus \{x\}$ for some $x \in X$.*
4. *$X \setminus A$ is an irreducible closed subset of X.*

Remark 3.9. There exist spaces that are simultaneously T_1 and sober, but not T_2. As an example, let X be the set of real numbers with a new point p adjoined; the open sets being all real open sets, and all cofinite sets containing p.

Recall from [12] that a space X is called *locally compact* if every point $x \in X$ has a compact neighborhood, i.e., there exists $U \in \mathcal{T}$ and a compact set K, such that $x \in U \subseteq K$.

A *Hausdorff compactification* of a space $(X, \mathcal{T})$ is a compact Hausdorff space $(cX, c\mathcal{T})$ such that X is (homeomorphic with) a dense set in cX. Recall that a space X is called *completely regular* if for any closed set $F \subseteq X$ and any point $x \in X \setminus F$, there exists a real-valued continuous function $f : X \to [0, 1]$ such that $f(x) = 1$ and $f(y) = 0$ for any $y \in F$. A space is called a *Tychonoff space* if it is a completely regular Hausdorff space. Being a Tychonoff space is both necessary and sufficient for possessing a Hausdorff compactification (cf. [7]).

It is well known that every locally compact noncompact Hausdorff space has a compactification [2, 12], called the *one-point compactification*, denoted by $(\widetilde{X}, \widetilde{\mathcal{T}})$, where $\widetilde{X} = X \cup \{\infty\}$, $\infty \notin X$ and $\widetilde{\mathcal{T}}$ is the family of all open subsets of X and all subsets U of $\widetilde{X}$ such that $\infty \in U$ and $\widetilde{X} \setminus U$ is a compact subset of X.

The following result provides an algebraic characterization of the one-point compactification of a noncompact Tychonoff space.

Theorem 3.10. *Let $(X, \mathcal{T})$ be a noncompact, Tychonoff space, and let $(cX, c\mathcal{T})$ be a Hausdorff compactification of $(X, \mathcal{T})$. Then, the following statements are equivalent:*

1. *$(cX, c\mathcal{T})$ is the one-point compactification of $(X, \mathcal{T})$.*
2. *$\mathcal{T}$ is a maximal ideal of $c\mathcal{T}$.*
3. *$\mathcal{T}$ is a prime ideal of $c\mathcal{T}$.*

Proof. (1)⇒(2) As $(cX, c\mathcal{T})$ is the one-point compactification of $(X, \mathcal{T})$, then $cX = X \cup \{\infty\}$, where $\infty \notin X$. Hence, $X = cX \setminus \{\infty\} \in c\mathcal{T}$. Therefore, $\mathcal{T} = (c\mathcal{T})_{cX\setminus\{\infty\}}$ is a maximal ideal of $c\mathcal{T}$ by virtue of Theorem 3.2.

(2)⇒(3) Trivial.

(3)⇒(1) Observe that $\mathcal{T} = (c\mathcal{T})_X$. Indeed, let $O \in \mathcal{T}$, then $O \in c\mathcal{T}$ since $\mathcal{T} \subseteq c\mathcal{T}$. Thus, $O = O \cap X \in (c\mathcal{T})_X$. Hence, $\mathcal{T} \subseteq (c\mathcal{T})_X$. Conversely, let $O \in c\mathcal{T}$. As $X \in \mathcal{T}$ and $\mathcal{T}$ is an ideal of $c\mathcal{T}$, we get $O \cap X \in \mathcal{T}$. This shows that $(c\mathcal{T})_X \subseteq \mathcal{T}$. It follows that $\mathcal{T} = (c\mathcal{T})_X$. Since cX is a Hausdorff space, it is a sober T_1-space. As $\mathcal{T}$ is a prime ideal of $c\mathcal{T}$, Theorems 3.2 and 3.5 ensure that $cX \setminus X$ is reduced to a singleton, which we denote by $\{\infty\}$. Thus, X is an open subspace of the compact Hausdorff space cX, and so X is locally compact (cf. [12]). On the other hand, one can easily check that, if $O \in c\mathcal{T}$ and $\infty \in O$, then $cX \setminus O$ is a compact subset of X. We derive that $(cX, c\mathcal{T})$ is the one-point compactification of $(X, \mathcal{T})$. This completes the proof. □

4. Semiring homomorphisms from $\mathcal{P}(X)$ into $\mathcal{P}(Y)$

Recall the following definition.

Definition 4.1. Let S_1 and S_2 be two semirings. By a *semiring homomorphism* from S_1 to S_2, we mean a mapping $\varphi : S_1 \to S_2$ satisfying the following conditions:

1. $\varphi(x + y) = \varphi(x) + \varphi(y)$ and $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in S_1$.
2. $\varphi(0) = 0$ and $\varphi(1) = 1$.

In the particular case when $S_1 = S_2$, we say that φ is a *semiring endomorphism*.

In [3, Theorem 1], we have provided an explicit description of the endomorphisms of the ring $(\mathcal{P}(X), \Delta, \cap)$, where Δ denotes the symmetric difference. Our aim, in this section, is to describe explicitly the semiring homomorphisms from $(\mathcal{P}(X), \cup, \cap)$ to $(\mathcal{P}(Y), \cup, \cap)$, where X is a finite set and Y is an arbitrary nonempty set. For any mapping $f : Y \to X$, we let $f^* : \mathcal{P}(X) \to \mathcal{P}(Y)$ be the mapping defined by $f^*(O) = f^{-1}(O)$ for any $O \in \mathcal{P}(X)$. The principal result of this section is the following.

Theorem 4.2. *For any nonempty set X the following statements are equivalent:*

1. *X is finite.*
2. *The maximal ideals of the semiring $(\mathcal{P}(X), \cup, \cap)$ are exactly the subspace topologies (of the discrete topology $\mathcal{P}(X)$) $m_x = \mathcal{P}(X \setminus \{x\})$, where $x \in X$.*

3. *For each nonempty set Y, the semiring homomorphisms $\mathcal{P}(X) \to \mathcal{P}(Y)$ are exactly the mappings f^*, where f is a arbitrary mapping from Y into X.*

The proof of Theorem 4.2 is based upon four lemmas. The first is the semiring version of a well-known result in ring theory.

Lemma 4.3. *[4, Lemma 7], [13, Proposition 3.2] Let H be a semiring, P a prime ideal of H and $n \geq 2$ an integer. Assume that $I_1, \ldots, I_n$ are ideals of H such that $\bigcap_{k=1}^{n} I_k \subseteq P$. Then $I_\ell \subseteq P$ for some $\ell \in \{1, \cdots, n\}$.*

The next lemma is straightforward.

Lemma 4.4. *If $(X, \mathcal{T})$ is a space, then $\bigcap_{x \in X} m_x = \{\emptyset\}$.*

The following lemma is the semiring version of Krull's Theorem, a well-known result in ring theory.

Lemma 4.5. *[13, Theorem 3.6] Any proper ideal of a commutative semiring H is contained in a maximal ideal of H.*

The proof of the following lemma is easy and then left to the reader.

Lemma 4.6. *Let $\varphi : S_1 \to S_2$ be a semiring homomorphism. If Q is a prime ideal of S_2, then $\varphi^{-1}(Q)$ is a prime ideal of S_1.*

Lemma 4.7. *Each prime ideal of the semiring $(\mathcal{P}(X), \cup, \cap)$ is maximal.*

Proof. Let P be a prime ideal of $\mathcal{P}(X)$ and let J be an ideal properly containing P. Pick an element $A \in J \setminus P$. As $\emptyset = A \cap (X \setminus A) \in P$, $X \setminus A \in P$. Thus $X = A \cup (X \setminus A) \in J$. This implies that $J = \mathcal{P}(X)$ and completes the proof. □

Proof of Theorem 4.2.

(1)⇒(2) First of all, note that each singleton is closed in the discrete topology $\mathcal{P}(X)$. Thus, Lemma 3.1 asserts that, for any $x \in X$, m_x is a maximal ideal of the semiring $(\mathcal{P}(X), \cup, \cap)$. Conversely, let $\mathcal{M}$ be a maximal ideal of the semiring $(\mathcal{P}(X), \cup, \cap)$. We need to show that $\mathcal{M} = m_x$ for some element $x \in X$. To this end, notice first that $\bigcap_{x \in X} m_x = \{\emptyset\}$, by virtue of Lemma 4.4. Thus, $\bigcap_{x \in X} m_x \subseteq \mathcal{M}$. Since X is finite, it follows from Lemma 4.3, that $m_x \subseteq \mathcal{M}$ for some $x \in X$. As m_x is a maximal ideal (see above) and $\mathcal{M}$ is a proper ideal of the semiring $(\mathcal{P}(X), \cup, \cap)$ (since it is maximal), it follows that $m_x = \mathcal{M}$, as desired.

(2)⇒(3) Let Y be a nonempty set. It is readily checked that if f is a map from Y into X then f^* is a semiring homomorphism from $\mathcal{P}(X)$ into $\mathcal{P}(Y)$. Conversely, let $\varphi : \mathcal{P}(X) \to \mathcal{P}(Y)$ be a semiring homomorphism. Our task is to find a mapping $f : Y \to X$ such that $\varphi = f^*$. We will argue as in the proof of [3, Theorem 1]. Let $y \in Y$ and observe that Lemma 4.6 yields that $\varphi^{-1}(m_y)$ is a prime ideal of the semiring $(\mathcal{P}(X), \cup, \cap)$. Thus, Lemma 4.7 guarantees that $\varphi^{-1}(m_y)$ is a maximal ideal of $\mathcal{P}(X)$. By

(2), there exists a unique element $\lambda_y \in X$ such that $\varphi^{-1}(m_y) = m_{\lambda_y}$. Consider the mapping $f : Y \to X$, which maps y to λ_y. We claim that $\varphi = f^*$. Indeed, for any $O \in \mathcal{P}(X)$, we have:

$$\begin{aligned} y \in f^*(O) &\Leftrightarrow f(y) = \lambda_y \in O \\ &\Leftrightarrow X \setminus O \subseteq X \setminus \{\lambda_y\} \\ &\Leftrightarrow X \setminus O \in m_{\lambda_y} \\ &\Leftrightarrow \varphi(X \setminus O) \in m_y \\ &\Leftrightarrow Y \setminus \varphi(O) \subseteq Y \setminus \{y\} \\ &\Leftrightarrow y \in \varphi(O). \end{aligned}$$

Therefore, $\varphi = f^*$, as claimed.

(3)⇒(1) Assume by way of contradiction that X is infinite. Clearly, the set I of all finite subsets of X is an ideal of $\mathcal{P}(X)$. Moreover, $I \neq \mathcal{P}(X)$ since $X \notin I$. Therefore, there exists a maximal ideal $\mathcal{M}$ of $\mathcal{P}(X)$ such that $I \subseteq \mathcal{M}$ accordingly to Lemma 4.5. Consider the mapping $\varphi : \mathcal{P}(X) \to \mathcal{P}(Y)$ given by: $\varphi(O) = \emptyset$ if $O \in \mathcal{M}$ and $\varphi(O) = X$ if $O \notin \mathcal{M}$. Firstly, we claim that φ is a semiring homomorphism. For let, $O, O' \in \mathcal{P}(X)$ with $O \neq O'$. If $\mathcal{M}$ contains exactly one of them, we can suppose without loss of generality, that $O \in \mathcal{M}$ and $O' \notin \mathcal{M}$. Then, $O \cup O' \notin \mathcal{M}$ since otherwise $O' = (O \cup O') \cap O'$ would be in $\mathcal{M}$, which is impossible. Thus $\varphi(O \cup O') = X$. As $\varphi(O) = \emptyset$ and $\varphi(O') = X$, we get $\varphi(O \cup O') = \varphi(O) \cup \varphi(O')$. On the other hand, $O \cap O' \in \mathcal{M}$ since it is contained in O and $O \in \mathcal{M}$. It follows that $\varphi(O \cap O') = \varphi(O) \cap \varphi(O')$. The case where $\mathcal{M}$ contains both of O and O' is trivial, since $O \cap O', O \cup O' \in \mathcal{M}$. It remains to deal with the case where $O \notin \mathcal{M}$ and $O' \notin \mathcal{M}$. In this case $O \cup O' \notin \mathcal{M}$ since otherwise O would be in $\mathcal{M}$, which is absurd. Thus, $\varphi(O \cup O') = \varphi(O) \cup \varphi(O')$. On the other hand, $O \cap O' \notin \mathcal{M}$ since $\mathcal{M}$ is a (maximal) prime ideal of $\mathcal{P}(X)$. Finally, it is clear that $\varphi(X) = X$ since $X \notin \mathcal{M}$ and $\varphi(\emptyset) = \emptyset$ since $\emptyset \in \mathcal{M}$. This proves our claim. Next, we will demonstrate that there is no mapping $f : Y \to X$ such that $\varphi = f^*$. To see this, assume the contrary and let f be a candidate. Pick an element $x \in X$ (such an element exists since X is nonempty). Obviously, $\{f(x)\} \in I \subseteq \mathcal{M}$. Hence, $\varphi(\{f(x)\}) = \emptyset$ and thus, $f^{-1}(\{f(x)\}) = \emptyset$. This contradicts the fact that $x \in f^{-1}(\{f(x)\})$ and completes the proof of Theorem 4.2. □

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