Some algebraic identities in 3-prime near-rings

Quelques identités algébriques dans les quasi-anneaux 3-premiers

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ABSTRACT. In this paper, we study the commutativity of 3-prime near-rings satisfying some differential identities on Jordan ideals involving certain additive maps. Some well-known results characterizing the commutativity of 3-prime near-rings by derivations and left multipliers are extended to right multipliers and left generalized derivations. Furthermore, an example is given showing the necessity of the 3-primeness mentioned in the assumptions of our theorems is given.
RÉSUMÉ. Dans ce travail, nous étudions la commutativité des quasi-anneaux 3-premiers satisfaisant certaines identités différentielles sur des idéaux de Jordan impliquant certaines applications additives. Certains résultats bien connus caractérisant la commutativité des quasi-anneaux 3-premiers par l'action des dérivations et des multiplicateurs à gauche ont été étendus aux multiplicateurs à droite et aux dérivations généralisées à gauche. De plus, nous avons enrichi cet article par un exemple qui montre la nécessité de la 3-primalité mentionnée dans les hypothèses de nos théorèmes.
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MOTS-CLÉS. Quasi-anneaux 3-premiers, dérivation à gauche généralisée, commutativité

Introduction

Throughout this paper, $\mathcal{N}$ will represent a right (or left) near-ring and $Z(\mathcal{N})$ will be the multiplicative center of $\mathcal{N}$. We note that for a right near-rings, $-(xy) = (-x)y$ for all $x, y \in \mathcal{N}$. Recal that $\mathcal{N}$ is called 2-torsion free if $(\mathcal{N}, +)$ has no elements of order 2; and $\mathcal{N}$ is called zero-symmetric if $0x = x0 = 0$ for all $x \in \mathcal{N}$ (recall that a right distributivity yields $0.x = 0$). Also, $\mathcal{N}$ is called a 3-prime if for all $x, y \in \mathcal{N}$, $x\mathcal{N}y = \{0\}$ implies $x = 0$ or $y = 0$. A near ring $\mathcal{N}$ is known as 2-torsion free if $2x = 0$ implies $x = 0$ holds for all $x \in \mathcal{N}$. We will write for all $x, y \in \mathcal{N}$, $[x, y] = xy - yx$ and $x \circ y = xy + yx$ for the Lie products and Jordan products, respectively.

In [8] the notion of Jordan ideal defined as the following: An additive subgroup $\mathcal{J}$ of $\mathcal{N}$ is said to be a Jordan ideal of $\mathcal{N}$ if $j \circ n \in \mathcal{J}$ and $n \circ j \in \mathcal{J}$ for all $j \in \mathcal{J}, n \in \mathcal{N}$. A derivation d on a near-ring $\mathcal{N}$ is a group endomorphism on $(\mathcal{N}, +)$ which satisfies $d(xy) = xd(y) + d(x)y$ for all $x, y \in \mathcal{N}$, or equivalently, as noted in [16], that $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{N}$.

Many results in the literature indicate show how the overall structure of a ring $\mathcal{R}$ is often closely related to the behavior of additive maps defined on $\mathcal{R}$. The study of derivations of near-rings was initiated by H. E. Bell and G. Mason in [3]. Motivated by the notion of generalized derivation in rings, see for reference [9, 10, 13, 14, 15], the notion of left generalized derivation in near-ring was defined as follows: An additive mapping $f : \mathcal{N} \longrightarrow \mathcal{N}$ is called a left generalized derivation associated with a derivation d if $f(xy) = xf(y) + d(x)y$ holds for all pairs $x, y \in \mathcal{N}$. Note that the notion of left generalized derivation covers the notion of a right multiplier, i.e., the additive mapping F on $\mathcal{N}$ satisfying $F(xy) = xF(y)$ for all $x, y \in \mathcal{N}$.

Our aim in this paper is to study the commutativity of addition and multiplication of a prime near-ring $\mathcal{N}$ equipped with a right multiplier or left generalized derivation satisfying certain algebraic identities locally on a Jordan ideal $\mathcal{J}$ of $\mathcal{N}$. The obtained results extend some theorems, generalize and unify several results obtained earlier over the entire near-ring.

1 Main results

We facilitate our discussion with the following lemmas, which are needed to develop the proofs of our main theorems.

Lemma 1.1. [2, Lemma 1.2 (iii)] *Let $\mathcal{N}$ be a 3-prime near-ring. If $z \in Z(\mathcal{N}) \setminus \{0\}$ and $xz \in Z(\mathcal{N})$ then $x \in Z(\mathcal{N})$.*

Lemma 1.2. [4, Lemma 2.2] *Let $\mathcal{N}$ be a 3-prime near-ring. If $\mathcal{N}$ admits a nonzero Jordan ideal $\mathcal{J}$, then $j^2 \neq 0$ for all $j \in \mathcal{J} \setminus \{0\}$.*

Lemma 1.3. [8, Lemma 3] *Let $\mathcal{N}$ be 2-torsion free 3-prime near-ring, and $\mathcal{J}$ a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{J} \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

Lemma 1.4. [7, Theorem 3.1] *Let $\mathcal{N}$ be a 2-torsion free 3-prime left near-ring and J be a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a derivation d such that $\mathrm{d}(\mathcal{J}) \subseteq Z(\mathcal{N})$, then $\mathrm{d} = 0$ or the element of $\mathcal{J}$ commute under the multiplication of $\mathcal{N}$.*

Theorem 1.5. *Let $\mathcal{N}$ be a 3-prime right near-ring and $\mathcal{J}$ be a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a nonzero right multiplier F, then the following assertions are equivalent:*

i) $F(in) + [i, n] \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$,

ii) $F(ni) + [i, n] \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$,

iii) $F(ni) + [n, i] \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$,

iv) $F(in) + [n, i] \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$,

v) $\mathcal{N}$ *is a commutative ring.*

Proof 1.6. *Proving that $i) \Rightarrow v)$. By hypotheses given, we have*

$$F(in) + [i, n] \in Z(\mathcal{N}) \text{ for all } i \in \mathcal{J}, n \in \mathcal{N}. \tag{1}$$

On one hand, in (1) *taking $n = i$, we obtain*

$$iF(i) \in Z(\mathcal{N}) \text{ for all } i \in \mathcal{J}. \tag{2}$$

On the other hand, putting i^2 instead of n in (1), *we find that $iF(i^2) + [i, i^2] \in Z(\mathcal{N})$. Which implies that $iF(i^2) \in Z(\mathcal{N})$ for all $i \in \mathcal{J}$. In view of Lemma 1.1 and* (2), *the latter result shows that either*

$$i \in Z(\mathcal{N}) \text{ or } F(i^2) = 0 \text{ for all } i \in \mathcal{J}. \tag{3}$$

Suppose that there is an element $i_0 \in Z(\mathcal{N}) \cap \mathcal{J}$. According to (1), *we get $F(i_0 n) \in Z(\mathcal{N})$ for all $n \in \mathcal{N}$. In this case, replacing n by nm, where $m \in \mathcal{N}$, and applying Lemma 1.1 together the 3-primeness of $\mathcal{N}$, we arrive at*

$$i_0 = 0 \text{ or } F(m) = 0 \text{ or } n \in \mathcal{N} \text{ for all } n, m \in \mathcal{N}$$

and therefore (2) *shows that* $\mathcal{N}$ *is a commutative ring or* $F(i^2) = 0$ *for all* $i \in \mathcal{J}$*. Let us show that the second condition also affirms the commutativity of* $\mathcal{N}$*. Indeed, substituting* ni^2 *for* n *in* (1)*, we arrive at* $[i, n]i^2 \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$*. Again, taking* $n = ni$ *in the last relation and invoking Lemma 1.1, we obtain*

$$[i, n]i^2 = 0 \text{ or } i \in Z(\mathcal{N}) \text{ for all } i \in \mathcal{J}, n \in \mathcal{N}$$

which reduces to $[i, n]i^2 = 0$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$*. So that* $ini^2 = ni^3$ *for all* $i \in Z(\mathcal{J}), n \in \mathcal{N}$*. Replacing* n *by* nm *in the last expression and applying it again, we get* $[i, n]mi^2 = 0$ *for all* $i \in \mathcal{J}, n, m \in \mathcal{N}$ *and therefore* $[i, n]\mathcal{N}i^2 = \{0\}$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$*. By 3-primeness of* $\mathcal{N}$ *together with Lemma 1.2, we conclude that* $\mathcal{J} \subseteq Z(\mathcal{N})$ *and therefore* $\mathcal{N}$ *is a commutative ring by Lemma 1.3.*

Using the same arguments as in the first part, we can prove that $ii) \Rightarrow v)$*,* $iii) \Rightarrow v)$ *and* $iv) \Rightarrow v)$*. For the opposite direction of all these implications, it is immediate in the case where* $\mathcal{N}$ *is a commutative ring.*

The following corollaries are consequences of the Theorem 1.5.

Corollary 1.7. *Let* $\mathcal{N}$ *be a 3-prime near-ring and* $\mathcal{J}$ *be a nonzero Jordan ideal of* $\mathcal{N}$*. Then the following assertions are equivalent:*

i) $2in - ni \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$*,*

ii) $ni + [i, n] \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$*,*

iii) $2ni - in \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$*,*

iv) $in + [n, i] \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$*,*

v) $\mathcal{N}$ *is a commutative ring.*

Corollary 1.8. *Let* $\mathcal{N}$ *be a 3-prime near-ring and* $\mathcal{J}$ *be a nonzero Jordan ideal of* $\mathcal{N}$*. If* $\mathcal{N}$ *admits a nonzero right multiplier* F*, then the following assertions are equivalent:*

i) $F(xy) + [x, y] \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{N}$*,*

ii) $F(yx) + [x, y] \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{N}$*,*

iii) $F(yx) + [y, x] \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{N}$*,*

iv) $F(xy) + [y, x] \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{N}$*,*

v) $\mathcal{N}$ *is a commutative ring.*

Corollary 1.9. *Let* $\mathcal{N}$ *be a 3-prime near-ring and* $\mathcal{J}$ *be a nonzero Jordan ideal of* $\mathcal{N}$*. Then the following assertions are equivalent:*

i) $xy + [x, y] \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{N}$*,*

ii) $yx + [x, y] \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{N}$*,*

iii) $yx + [y, x] \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{N}$*,* *iv)* $xy + [y, x] \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{N}$*,*

v) $\mathcal{N}$ *is a commutative ring.*

Theorem 1.10. *Let* $\mathcal{N}$ *be a 2-torsion free 3-prime left near-ring. If* $\mathcal{N}$ *admits a nonzero right multiplier* F*, then the following proprieties are equivalent:*

(i) $F(i \circ n) + [i, n] \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$*,*

(ii) $F(i \circ n) + [n, i] \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$*,*

(iii) $F(n \circ i) + [i, n] \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$*,*

(iv) $F(n \circ i) + [n, i] \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$*,*

(v) $\mathcal{N}$ *is a commutative ring.*

Proof 1.11. *For* $(i) \Rightarrow (v)$*. By hypothesis, we have*

$$F(i \circ n) + [i, n] \in Z(\mathcal{N}) \text{ for all } i \in \mathcal{J}, n \in \mathcal{N}. \quad (4)$$

Taking i *instead of* n *in* (4)*, we get* $F(2i^2) \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}$*. For a second time, putting* i^2 *instead of* n *in* (4)*, we get* $iF(2i^2) \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}$*. In view of Lemma 1.1 and 2-torsion freeness of* $\mathcal{N}$*, we find that*

$$F(i^2) = 0 \text{ or } i \in Z(\mathcal{N}) \text{ for all } i \in \mathcal{J}. \quad (5)$$

Let $i_0 \in Z(\mathcal{N}) \cap \mathcal{J}$*. In this case, taking* $i = i_0$ *in* (4) *we infer that* $i_0F(2n) \in Z(\mathcal{N})$ *for all* $n \in \mathcal{N}$*. Replacing* n *by* mn *in the last result, we obtain* $mi_0F(2n) \in Z(\mathcal{N})$ *for all* $n, m \in \mathcal{N}$ *which reduces, because of Lemma 1.1 together 2-torsion freeness of* $\mathcal{N}$*, to*

$$m \in Z(\mathcal{N}) \text{ or } i_0 = 0 \text{ or } F(n) = 0 \text{ for all } n, m \in \mathcal{N}. \quad (6)$$

Consequently, from (5) *and* (6) *we conclude that* $\mathcal{N}$ *is a commutative ring or* $F(i^2) = 0$ *for all* $i \in \mathcal{J}$*. Suppose that the second case holds; in particular taking* $F(i)n$ *instead of* n *in* (4)*, we obtain* $F\big(i \circ F(i)n\big) + [i, F(i)n] \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$*. Developing this expression and taking account that* $F(i^2) = 0$*, we arrive at* $F(i)n\big(F(i) - i\big) \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$*. Now, replacing* n *by* $F(i)n$ *in the last expression and invoking Lemma 1.1, we find that*

$$F(i) \in Z(\mathcal{N}) \text{ or } F(i)n\big(F(i) - i\big) = 0 \text{ for all } i \in \mathcal{J}, n \in \mathcal{N}.$$

Using the 3-primeness of $\mathcal{N}$*, the latter result reduces to*

$$F(i) \in Z(\mathcal{N}) \text{ or } F(i) = i \text{ for all } i \in \mathcal{J}.$$

If there exists a nonzero element $i_0 \in \mathcal{J}$ *satisfies* $F(i_0) = i_0$*, then* $F(i_0^2) = i_0^2 = 0$ *which, in view of Lemma 1.2, is a contradiction; and therefore* $F(i) \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}$*. Consequently,* $0 = F(i^2) = iF(i) = inF(i)$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$*. So that,* $i\mathcal{N}F(i) = \{0\}$ *for all* $i \in \mathcal{J}$*. In the light of the 3-primeness of* $\mathcal{N}$*, the latter result gives* $F(i) = 0$ *for all* $i \in \mathcal{J}$*. In particular, for* $i = i \circ n$ *we get* $F(in + ni) = 0$ *which implies that* $iF(n) = 0$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$*. Substituting* nm *for* n *and using 3-primeness of* $\mathcal{N}$ *we infer that* $\mathcal{J} = \{0\}$ *or* $F = 0$*. But in the both cases,* (4) *shows that* $[i, n] \in Z(\mathcal{N})$

for all $i \in \mathcal{J}, n \in \mathcal{N}$. *Putting* in *instead of* n *and using the fact that* $[i, in] = i[i, n]$ *and applying Lemma 1.1, we conclude that*

$$i \in Z(\mathcal{N}) \text{ or } [i, n] = 0 \text{ for all } i \in \mathcal{J}, n \in \mathcal{N}.$$

It follows that $\mathcal{J} \subseteq Z(\mathcal{N})$ *which forces that* $\mathcal{N}$ *is a commutative ring by Lemma 1.3.*

For $(v) \Rightarrow (i)$ *this implication is clair.*

For the equivalences $(ii) \Leftrightarrow v)$, $(iii) \Leftrightarrow (v)$ *and* $(iv) \Leftrightarrow (v)$*, to avoid multiple repetitions, just use the same techniques as above with minor modifications.*

As an application of the previous theorem, we get the following corollaries.

Corollary 1.12. *Let* $\mathcal{N}$ *be a 2-torsion free 3-prime near-ring. Then the following proprieties are equivalent:*

(i) $i \circ n + [i, n] \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$,

(ii) $i \circ n + [n, i] \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$,

(iii) $n \circ i + [i, n] \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$,

(iv) $n \circ i + [n, i] \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$,

(v) $\mathcal{N}$ *is a commutative ring.*

Corollary 1.13. *Let* $\mathcal{N}$ *be a 2-torsion free 3-prime near-ring. If* $\mathcal{N}$ *admits a nonzero right multiplier* F*, then the following proprieties are equivalent:*

(i) $F(x \circ y) + [x, y] \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{N}$,

(ii) $F(x \circ y) + [y, x] \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{N}$,

(iii) $F(y \circ x) + [x, y] \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{N}$,

(iv) $F(y \circ x) + [y, x] \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{N}$,

(v) $\mathcal{N}$ *is a commutative ring.*

Corollary 1.14. *Let* $\mathcal{N}$ *be a 2-torsion free 3-prime near-ring. Then the following proprieties are equivalent:*

(i) $x \circ y + [x, y] \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{N}$,

(ii) $x \circ y + [y, x] \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{N}$,

(iii) $y \circ x + [x, y] \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{N}$,

(iv) $y \circ x + [y, x] \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{N}$,

(v) $\mathcal{N}$ *is a commutative ring.*

The following example shows the necessity of the 3-primeness used in the Theorem 1.5 and in the Theorem 1.10.

Example 1.15. *Define $\mathcal{N}$, $\mathcal{J}$ and F by:*

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, z \in \mathbb{Z} \right\}, \quad \mathcal{J} = \left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid x \in \mathbb{Z} \right\},$$

$$F \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $\mathcal{N}$ is a near-ring which is not 3-prime, $\mathcal{J}$ is a nonzero Jordan ideal of $\mathcal{N}$ and F is a nonzero right multiplier of $\mathcal{N}$. Then, we can see that

i) $F(in) + [i, n] \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$,

ii) $F(ni) + [i, n] \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$,

iii) $F(ni) + [n, i] \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$,

iv) $F(in) + [n, i] \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$,

v) $F(i \circ n) + [i, n] \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$,

vi) $F(i \circ n) + [n, i] \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$,

vii) $F(n \circ i) + [i, n] \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$,

viii) $F(n \circ i) + [n, i] \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$.

However, $\mathcal{N}$ is not a commutative ring.

The following example illustrates that the hypothesis "2-torsion freeness of $\mathcal{N}$" is essential in the Theorem 1.10.

Example 1.16. *Define $\mathcal{N}$ and $\mathcal{J}$ as follows:*

$$\mathcal{N} = M_2(\mathbb{Z}_2), \quad \mathcal{J} = \left\{ \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \mid k, 0 \in \mathbb{Z}_2 \right\}.$$

Then $\mathcal{N}$ is a prime ring which is not 2-torsion free, and $\mathcal{J}$ is a nonzero Jordan ideal of $\mathcal{N}$. And let $F = id_{\mathcal{N}}$, then F is a right multiplier on $\mathcal{N}$ which satisfies the following conditions:

(i) $F(i \circ n) + [i, n] \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$,

(ii) $F(i \circ n) + [n, i] \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$,

(iii) $F(n \circ i) + [i, n] \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$,

(iv) $F(n \circ i) + [n, i] \in Z(\mathcal{N})$ *for all* $i \in \mathcal{J}, n \in \mathcal{N}$.

But, $\mathcal{N}$ is not a commutative ring.

2 Some results on jordan ideals with left generalized derivation in left near-ring

Motivated by the results in [5, Theorem 3.3], [6, Theorem 3.3] and [7, Theorem2], our objective in this section is to generalize these results of Jordan ideals on 3-prime near-rings admitting left generalized derivations.

Theorem 2.1. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring, $\mathcal{J}$ a Jordan ideal of $\mathcal{N}$, and F be a left generalized derivation of $\mathcal{N}$ associated with a derivation d. If $\mathcal{J}$ has one of the following properties:*

(i) $F(n \circ j) = [n, j]$ *for all* $j \in \mathcal{J},\ n \in \mathcal{N}$,

(ii) $F([n, j] = n \circ j$ *for all* $j \in \mathcal{J},\ n \in \mathcal{N}$,

then $\mathcal{J}$ is commutative.

Proof 2.2. (i) *Assume that*

$$F(n \circ j) = [n, j] \text{ for all } j \in \mathcal{J},\ n \in \mathcal{N}. \tag{1}$$

Replacing n by jn in (1) *and using the fact that $[jn, j] = j[n, j]$ together with $jn \circ j = j(n \circ j)$, we obtain*

$$F(jn \circ j) = [jn, j] \text{ for all } j \in \mathcal{J},\ n \in \mathcal{N}.$$

Which implies that

$$F(j(n \circ j)) = j[n, j] \text{ for all } j \in \mathcal{J}, n \in \mathcal{N}.$$

Using the definition of F, we obtain

$$d(j)(n \circ j) + jF(n \circ j) = j[n, j] \text{ for all } j \in \mathcal{J}, n \in \mathcal{N}.$$

This expression gives us $d(j)(n \circ j) = 0$ for all $j \in \mathcal{J}, n \in \mathcal{N}$, it follows that

$$d(j)nj = -d(j)jn \text{ for all } j \in \mathcal{J}, n \in \mathcal{N}. \tag{2}$$

Substituting nm in place of n in (2)*, we get*

$$\begin{aligned} d(j)nmj &= -d(j)jnm \\ &= d(j)jn(-m) \\ &= -d(j)nj(-m) \\ &= d(j)njm \quad \text{for all } j \in \mathcal{J},\ n \in \mathcal{N}. \end{aligned}$$

Which can be rewritten as $d(j)\mathcal{N}[m, j] = \{0\}$ for all $j \in \mathcal{J},\ m \in \mathcal{N}$. By the 3-primeness of $\mathcal{N}$, we have

$$d(j) = 0 \text{ or } j \in Z(\mathcal{N}) \text{ for all } j \in \mathcal{J}. \tag{3}$$

Suppose there exists $j_0 \in \mathcal{J}$ such that $j_0 \in Z(\mathcal{N})$. Then, (1) *becomes $2F(nj_0) = 0$ for all $n \in \mathcal{N}$, using the 2-torsion freeness of $\mathcal{N}$, we get*

$$F(j_0 n) = 0 \text{ for all } n \in \mathcal{N}. \tag{4}$$

By the definition of F, (4) *gives*

$$d(j_0)n + j_0F(n) = 0 \text{ for all } n \in \mathcal{N}. \tag{5}$$

Replacing n by nj_0 in (5) *and using* (4) *we obtain $d(j_0)nj_0 = 0$ for all $n \in \mathcal{N}$, which implies that $d(j_0)nj_0 = \{0\}$. In view of 3-primeness of $\mathcal{N}$, the latter results assures that $d(j_0) = 0$. In this case,* (3) *becomes $d(\mathcal{J}) = \{0\}$ which forces that $\mathcal{J}$ is commutative or $d = 0$ by Lemma 1.4.* (ii) *Suppose that*

$$F([n, j]) = n \circ j \text{ for all } j \in \mathcal{J},\ n \in \mathcal{N}. \tag{6}$$

Replacing n by jn in (6)*, we obtain*

$$F([jn, j]) = jn \circ j \text{ for all } j \in \mathcal{J},\ n \in \mathcal{N},$$

it follows that

$$F(j[n, j]) = j(n \circ j) \text{ for all } j \in \mathcal{J},\ n \in \mathcal{N}.$$

That is,

$$d(j)([n, j]) + jF([n, j]) = j(n \circ j) \text{ for all } j \in \mathcal{J},\ n \in \mathcal{N}.$$

By (6)*, we can see that $d(j)([n, j]) = 0$ for all $j \in \mathcal{J},\ n \in \mathcal{N}$, which leads to*

$$d(j)nj = d(j)jn \text{ for all } j \in \mathcal{J},\ n \in \mathcal{N}. \tag{7}$$

Substituting nm in place of n in (7)*, we get*

$$\begin{aligned} d(j)nmj &= d(j)jnm \\ &= d(j)njm \quad \text{for all } j \in \mathcal{J},\ n, m \in \mathcal{N}. \end{aligned}$$

Then the above equation leads to $d(j)\mathcal{N}[m, j] = \{0\}$, for all $j \in \mathcal{J}, m \in \mathcal{N}$ and by the 3-primeness of $\mathcal{N}$, we have

$$d(j) = 0 \quad \text{or } j \in Z(\mathcal{N}) \text{ for all } j \in \mathcal{J}. \tag{8}$$

Suppose there exists $j_0 \in \mathcal{J}$ such that $j_0 \in Z(\mathcal{N})$. Then (6) *becomes $2nj_0 = 0$ for all $n \in \mathcal{N}$ and using the 2-torsion freeness of $\mathcal{N}$, we get $nj_0 = 0$ for all $n \in \mathcal{N}$, which implies that $j_0 = 0$. In this case,* (8) *implies that $d(\mathcal{J}) = \{0\}$, which forces that $\mathcal{J}$ is commutative or $d = 0$ by Lemma 1.4.*

Corollary 2.3. [5, Theorem 3.3] *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring, $\mathcal{J}$ a Jordan ideal of $\mathcal{N}$, and d be a nonzero derivation. If $\mathcal{J}$ has one of the following properties:*

(i) $d(n \circ j) = [n, j]$ *for all* $j \in \mathcal{J}, n \in \mathcal{N}$,

(ii) $d([n, j]) = n \circ j$ *for all* $j \in \mathcal{J}, n \in \mathcal{N}$,

then $\mathcal{J}$ is commutative.

Corollary 2.4. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If $\mathcal{N}$ admits a left generalized derivation F associated with a derivation d satisfying any one of the following properties:*

(i) $F(x \circ y) = [x, y]$ *for all* $x, y \in \mathcal{N}$,

(ii) $F([x, y]) = x \circ y$ *for all* $x, y \in \mathcal{N}$,

then $\mathcal{N}$ *is a commutative ring.*

Corollary 2.5. *Let* $\mathcal{N}$ *be a 2-torsion free 3-prime near-ring. If* $\mathcal{N}$ *admits a derivation* d *satisfying any one of the following properties:*

(i) $d(x \circ y) = [x, y]$ *for all* $x, y \in \mathcal{N}$,

(ii) $d([x, y] = x \circ y$ *for all* $x, y \in \mathcal{N}$,

then $\mathcal{N}$ *is a commutative ring.*

Theorem 2.6. *Let* $\mathcal{N}$ *be a 2-torsion free 3-prime near-ring,* $\mathcal{J}$ *a nonzero Jordan ideal of* $\mathcal{N}$*, and* F *be a left generalized derivation associated with a derivation* d*. If* $F(n \circ j) = 0$ *for all* $n \in \mathcal{N},\ j \in \mathcal{J}$*, then* $\mathcal{J}$ *is commutative or* $d = 0$.

Proof 2.7. *We assuming that*

$$F(n \circ j) = 0 \text{ for all } j \in \mathcal{J}, n \in \mathcal{N}. \tag{9}$$

Replacing n *by* jn *in* (9) *and using the fact that* $jn \circ j = j(n \circ j)$*, we obtain*

$$d(j)(n \circ j) = 0 \text{ for all } j \in \mathcal{J}, n \in \mathcal{N}. \tag{10}$$

It follows that

$$d(j)nj = -d(j)jn \text{ for all } j \in \mathcal{J}, n \in \mathcal{N}. \tag{11}$$

Substituting nm *for* n *in* (11) *and using it again, we obtain*

$$\begin{aligned} d(j)nmj &= -d(j)jnm \\ &= d(j)jn(-m) \\ &= -d(j)nj(-m) \\ &= d(j)njm \text{ for all } j \in \mathcal{J}, n \in \mathcal{N}. \end{aligned}$$

Then, the last equation gives

$$d(j)\mathcal{N}[n, j] = \{0\} \text{ for all } j \in \mathcal{J}, n \in \mathcal{N}. \tag{12}$$

By the 3-primeness of $\mathcal{N}$*, we conclude that*

$$d(j) = 0 \text{ or } j \in Z(\mathcal{N}) \text{ for all } j \in \mathcal{J}, n \in \mathcal{N} \tag{13}$$

If there exists $j_0 \in \mathcal{J}$ *such that* $j_0 \in Z(\mathcal{N})$*, then* (9) *yields* $F(n \circ j_0) = 0$ *for all* $n \in \mathcal{N}$*, which implies that*

$$2F(j_0 n) = 0 \text{ for all } n \in \mathcal{N}.$$

By the 2-torsion freeness of $\mathcal{N}$, we get

$$F(j_0 n) = 0 \text{ for all } n \in \mathcal{N}$$

and therefore

$$d(j_0)n + j_0 F(n) = 0 \text{ for all } n \in \mathcal{N}.$$

Replacing n by $n \circ j_0$ in the above equation and using (9), we get

$$d(j_0)(n \circ j_0) = 0 \text{ for all } n \in \mathcal{N}.$$

It follows that

$$2d(j_0)nj_0 = 0 \text{ for all } n \in \mathcal{N}.$$

By the 2-torsion freeness of $\mathcal{N}$, we obtain

$$d(j_0)nj_0 = 0 \text{ for all } n \in \mathcal{N}.$$

Which means that $d(j_0)\mathcal{N}j_0 = \{0\}$. Again by the 3-primeness of $\mathcal{N}$, we conclude that $d(j_0) = 0$. Hence, (13) *implies that $d(\mathcal{J}) = \{0\}$ it follows that $\mathcal{J}$ is commutative or $d = 0$ by Lemma 1.4.*

Corollary 2.8. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring, $\mathcal{J}$ a nonzero Jordan ideal of $\mathcal{N}$, and d be a nonzero derivation of $\mathcal{N}$. If $d(n \circ j) = 0$ for all $n \in \mathcal{N}$, $j \in \mathcal{J}$, then $\mathcal{J}$ is a commutative.*

Corollary 2.9. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If $\mathcal{N}$ admits a left generalized derivation associated with a derivation d satisfying the following identities $F(x \circ y) = 0$ for all $x, y \in \mathcal{N}$, then $F = d = 0$.*

Corollary 2.10. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If $\mathcal{N}$ admits a derivation d such that $d(x \circ y) = 0$ for all $x, y \in \mathcal{N}$, then $d = 0$.*

The following example proves that the 3-primeness of $\mathcal{N}$ in Theorem 2.1 and Theorem 2.6 cannot be omitted.

Example 2.11. *Let $\mathcal{S}$ be a 2-torsion free left near-ring which is not abelian. Define $\mathcal{N}$, $\mathcal{J}$, d and F by:*

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} \mid x, y, z, 0 \in \mathcal{S} \right\}, \quad \mathcal{J} = \left\{ \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid x, 0 \in \mathcal{S} \right\},$$

$$F \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad d = F.$$

Then, $\mathcal{N}$ is a left near-ring which is not 3-prime, $\mathcal{J}$ is a nonzero Jordan ideal of $\mathcal{N}$ and F is a left generalized derivation associated with the derivation d. We can see that

(i) $F(n \circ j) = [n, j]$ *for all* $j \in \mathcal{J}$, $n \in \mathcal{N}$,

(ii) $F([n, j] = n \circ j$ *for all* $j \in \mathcal{J}$, $n \in \mathcal{N}$,

(iii) $F(n \circ j) = 0$ *for all* $j \in \mathcal{J}$, $n \in \mathcal{N}$.

But neither $d = 0$ nor $\mathcal{J}$ is commutative.

References

[1] BELL H. E., BOUA A., OUKHTITE L., *Semigroup ideals and commutativity in 3-prime near-rings.* Comm. Alg., 43 (5) (2015), 1757-1770.

[2] BELL H. E., *On derivations in near-rings II. near-rings, near-fields and K-loops.* (Hamburg, 1995) Math. Appl, Dordrecht: Kluwer Acad. Publ., 426 (1997), 191-197.

[3] BELL H. E., MASON G., *On derivations in near-rings.* North-Holland Math. Stud. 137 (1987), 31-35.

[4] BOUA A., BELL H. E., *Jordan Ideals and Derivations Satisfying Algebraic Identities.* Iranian Mathematical Society, 44 (2018), 1543-1554.

[5] BOUA A., ALI A., HUQUE Ul., *Several Algebriac Identities in 3-Prime Near-ring.* Kragujevac Journal of Mathematics, 42(2) (2018), 249-258.

[6] BOUA A., *Commutativity of Near-rings With Certain Constrains on Jordan Ideals.* Bol. Soc. Paran Mat., 36(4) (2018), 159-170.

[7] BOUA A., *Commutivity of Jordan Ideals in 3-Prime Near-Rings with Derivations.* Commun. Korean Math., 33 (1)(2018), 37-44.

[8] BOUA A., OUKHTITE L., RAJI A., *Jordan ideals and derivations in prime near-rings.* Comment. Math. Univ. Carolin., 55 (2) (2014), 131-139.

[9] BRESAR M., *On The distance of Composition of Two Derivations To The Generalized Derivations.* Glasgow Math. J., 33, (1991), 89-93.

[10] BRESAR M., VUKMAN J., *On some additive mappings in rings with involution.* Aequat. Math. 38 (1989), 178-185.

[11] EN-GUADY A., BOUA A., FARHAN E., *Some functional identities on near-rings involving Jordan ideals and left generalized derivations.* Iraqi Journal of science, 62 (6) (2021), 1961-1967.

[12] EN-GUADY A., BOUA A., *On Jordan ideals with left derivations in 3-prime near-rings.* Extracta Math., 38 (1) (2023), 51-66.

[13] HVALA B., *Generalized derivations in rings.* Comm. Algebra, 26(4), 1988, 1147-1166.

[14] OUKHTITE L., *Left multipliers and Jordan ideals in rings with involution.* Afr. Diaspora J. Math., 11 (2011), no. 1, 24–28.

[15] OUKHTITE L., *On Jordan ideals and derivations in rings with involution.* Comment. Math. Univ. Carolin., 51 (2010), no. 3, 389–395.

[16] WANG X.K., *Derivations in prime near-rings.* Proc. Amer. Math. Soc, 121 (2)(1994), 361-366.