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Bifurcation beyond the principal eigenvalues for Neumann problems with indefinite weights

Bifurcation au-delà des valeurs propres principales pour les problèmes de Neumann avec des poids indéfinis

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ABSTRACT. This paper is devoted to the study of the effects of indefinite weights on the following nonlinear Neumann problems

$$(P^{\pm}) \qquad \begin{cases} -\Delta u &= \lambda\, a(x)u \ \pm\ |u|^{p-1}u \qquad \text{in } \Omega \subset \mathbb{R}^N \\ \frac{\partial u}{\partial \nu} &= 0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega \end{cases}.$$

The function $a = a(x)$ is assumed to be continuous and sign-changing. Then the linear part has two sequences of eigenvalues. Our results establish a relation between the position of the parameter λ and the number of nontrivial classical solutions of these problems. The proof combines spectral analysis tools, variational methods and the Clark multiplicity theorem.
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1. Introduction

In this article we study bifurcation results for the equation

$$(P^{\pm}) \qquad \begin{cases} -\Delta u &= \lambda\, a(x)u \ \pm\ |u|^{p-1}u \qquad \text{in } \Omega \subset \mathbb{R}^N \\ \frac{\partial u}{\partial \nu} &= 0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $\dfrac{\partial}{\partial \nu}$ is the normal derivative on $\partial\Omega$, and $a(x)$ is a sign changing continuous function.

For this, let us first discuss the linear eigenvalue problem with Neumann boundary conditions:

$$\begin{cases} -\Delta u &= \lambda\, a(x)u \qquad \text{in } \Omega \subset \mathbb{R}^N \\ \frac{\partial u}{\partial \nu} &= 0 \qquad\qquad \text{on } \partial\Omega. \end{cases} \tag{1}$$

If $a(x) \equiv 1$, or if $a(x)$ is strictly positive, then the description of the spectrum is classical and given by an unbounded sequence of positive eigenvalues. The situation is different when the weight $a(x)$

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changes sign: it is known [18, 11, 13, 14, 15, 16, 21] that in this case there are a positive and a negative (unbounded) sequence of eigenvalues

$$-\infty \leftarrow ... \leq \lambda_k^- \leq \lambda_{k-1}^- \leq \ldots \leq \lambda_2^- < \lambda_1^- \leq 0 \leq \lambda_1^+ < \lambda_2^+ \leq ... \leq \lambda_k^+ \leq ... \rightarrow +\infty \, .$$

A large part of the literature has been dedicated to the study of the principal eigenvalues, that is a value λ having a positive (or negative) eigenfunction (see e.g [3]). For sign changing weights, there are in general two principal eigenvalues: in the specific case of Neumann conditions, one always has that one of the eigenvalues λ_1^- or λ_1^+ is zero, more precisely

- $\lambda_1^+ = 0$ and $\lambda_1^- < 0$, if $\displaystyle\int_\Omega a(x)dx > 0$

and

- $\lambda_1^- = 0$ and $\lambda_1^+ > 0$, if $\displaystyle\int_\Omega a(x)dx < 0.$

This implies in particular that

- $\lambda_1^+ = \lambda_1^- = 0$, if $\displaystyle\int_\Omega a(x)dx = 0.$

The interest in problems with Neumann boundary conditions with sign changing weights is motivated among other by a model of population dynamics. For instance, the following problem has recently been studied by Mazzoleni, Pellacci and Verzini [19]

$$\begin{cases} -\varepsilon\Delta u &= a(x)u - u^2 \qquad \text{in } \Omega \subset \mathbb{R}^N \\ \frac{\partial u}{\partial \nu} &= 0 \qquad\qquad\qquad \text{on } \partial\Omega. \end{cases}$$

Here the positive/negative region of the weight models the spatial arrangement of favorable/unfavorable regions for the survival of a species.

In a recent paper [4], we considered the bifurcation of *positive* solutions from the first eigenvalues λ_1^+ and λ_1^- for the problems

$$(P^\pm) \qquad \begin{cases} -\Delta u &= \lambda\, a(x)u \,\pm\, |u|^{p-1}u \qquad \text{in } \Omega \subset \mathbb{R}^N \\ \frac{\partial u}{\partial \nu} &= 0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega. \end{cases}$$

The results show an interesting complementarity between the non-coercive problem (P^+) and the coercive problem (P^-):

Theorem A. Assume that $a \in C(\overline{\Omega})$ and sign-changing, and $p > 1$. Then problem (P^-) has

- a positive solution, for $\lambda < \lambda_1^-$ and for $\lambda > \lambda_1^+$;
- *no* positive solution, for $\lambda_1^- \leq \lambda \leq \lambda_1^+$.

Theorem B. Assume that $a \in C(\overline{\Omega})$ and sign-changing, and $1 < p < \dfrac{N+2}{N-2}$. Then problem (P^+) has

- *no* positive solution, for $\lambda \leq \lambda_1^-$ and for $\lambda \geq \lambda_1^+$;
- a positive solution, for $\lambda_1^- < \lambda < \lambda_1^+$.

The complementarity is most evident in the *degenerate* case $\displaystyle\int_\Omega a(x)dx = 0$. In this case we have $\lambda_1^- = \lambda_1^+ = 0$, and hence

Corollary. *If* $\displaystyle\int_\Omega a(x)dx = 0$*, then*

- *problem* (P^-) *has a positive solution for every* $\lambda \neq 0$ *(*$p > 1$*);*
- *problem* (P^+) *has no positive solution for any* $\lambda \in \mathbb{R}$ *(*$1 < p < \dfrac{N+2}{N-2}$*).*

Recently, López-Gómez and Rabinowitz [17] studied bifurcation problems associated to indefinite eigenvalue problems, with Dirichlet conditions. In particular, for the model problem

$$(P_D^-) \quad \begin{cases} -\Delta u = \lambda a(x)u - |u|^{p-1}u & \text{in } \Omega \subset \mathbb{R}^N \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $1 < p$ and $a(x)$ continuous and sign-changing, they showed the existence of at least k pairs of solutions for $\lambda > \lambda_k^+$, as well as for $\lambda < \lambda_k^-$, implying that all eigenvalues of the linear equation (1) are also bifurcation points.

In what follows, inspired by the cited work of López-Gómez and Rabinowitz, we extend the result by studying bifurcation from higher eigenvalues of the corresponding nonlinear Neumann problems $(P^\pm)$. We will prove the following results.

Theorem 1. *Assume that* $a \in C(\overline{\Omega})$ *and sign-changing, and* $p > 1$*. Then*

i) for $\lambda < \lambda_k^-$ *and for* $\lambda > \lambda_k^+$*,* (P^-) *possesses at least* k *distinct pairs* $\pm u_1, \ldots, \pm u_k$ *of non trivial solutions;*

ii) for $\lambda \in (\lambda_2^-, \lambda_1^-)$ *as well as for* $\lambda \in (\lambda_1^+, \lambda_2^+)$*,* (P^-) *possesses only solutions with constant sign.*

This results extends Theorem A mentioned above, and can be summarized in the following bifurcation diagram

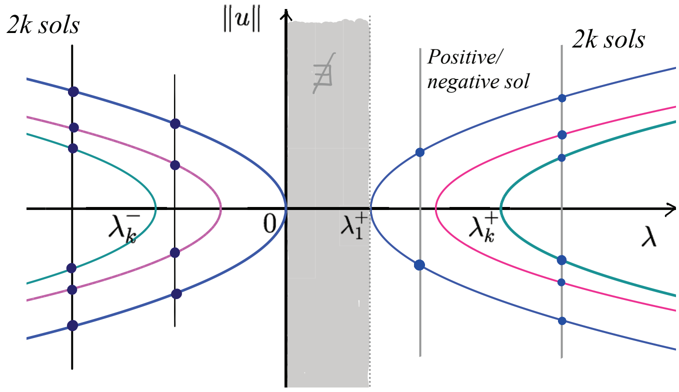

Figure 1. *Bifurcation diagram for equation* (P^-)

We see that

- for λ between λ_1^- and λ_1^+ there exist no solutions;
- for λ between λ_1^+ and λ_2^+, or λ between λ_1^- and λ_2^-, there exist only solutions with constant sign;
- for λ between λ_k^+ and λ_{k+1}^+, or λ between λ_k^- and λ_{k+1}^-, there exist at least k pairs of solutions.

Furthermore, one checks easily in the proofs that for λ approaching λ_k^+ from above, or λ_k^- from below, there is a solution pair $\pm u_k(\lambda)$ with $\|u_k(\lambda)\|$ tending to zero. In Figure 1 we draw the expected bifurcation branches; we caution however that the variational methods do not yield continuous branches of solutions.

For the non coercive problem (P^+) we have a very different, and in some sense complementary, result.

Theorem 2. *Assume that* $a \in C(\overline{\Omega})$ *and sign-changing, and* $1 < p < \dfrac{N+2}{N-2}$*, if* $N \geq 3$*, and* $p > 1$ *otherwise. Then*

for $\lambda \in (\lambda_k^+, \lambda_{k+1}^+)$ *or for* $\lambda \in (\lambda_{k+1}^-, \lambda_k^-)$*,* (P^+) *possesses infinitely many distinct pairs* $(u, -u)$ *of non trivial solutions.*

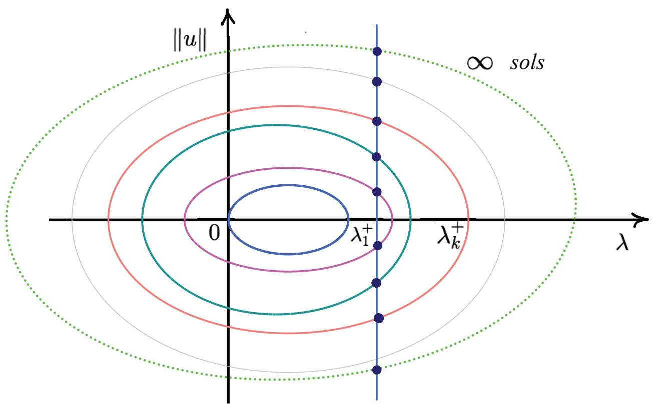

Figure 2. *Bifurcation diagram for equation* (P^+)

In Theorem B above it was shown that equation (P^+) has no positive solutions for $\lambda \notin [\lambda_1^-, \lambda_1^+]$. Furthermore, the positive (and negative) solutions $\pm u_1(\lambda)$, for $\lambda \in (\lambda_1^-, \lambda_1^+)$ and $1 < p < \dfrac{N}{N-2}$, are bounded. Hence the bifurcation branches cannot "escape to infinity", and so it seems plausible that they are connecting $\lambda_1^-(=0)$ and λ_1^+. Whether the other bifurcation branches behave as drawn in *Figure 2* is an interesting open problem.

2. Proof of Theorem 1 (coercive case)

The proof is based in part on the following version of Clark's Theorem (see e.g. [8] and [20], Chapter 9):

Theorem [Clark] *Let $\Phi \in C^1(E, \mathbb{R})$ an even functional satisfying the Palais-Smale (PS) condition and bounded from below. Suppose $\Phi(0) = 0$ and there exists a set $F \subset E$ homeomorphic to the $(k-1)$-dimensional sphere S^{k-1} by an odd map such that $\sup_{u \in F} \Phi(u) < 0$. Then Φ has at least k distinct pairs of critical points.*

For the proof of Theorem 1, $i)$, we proceed by steps. Weak solutions of (P^-) correspond to critical points of the functional

$$\Phi_\lambda : H \to \mathbb{R} \qquad \Phi_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{p+1}\int_\Omega |u|^{p+1} - \frac{\lambda}{2}\int_\Omega a(x)u^2 \tag{2}$$

where $H := H^1(\Omega)$.

Note that the functional is well defined on H only for $p + 1 \leq \dfrac{2N}{N-2}$, but the following Lemma 1 provides an *a priori* estimate, which allows to use a "truncated" functional instead of (2):

$$\widetilde{\Phi}_\lambda : H \to \mathbb{R} \quad , \quad \tilde{\Phi}_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega G(u) - \frac{\lambda}{2}\int_\Omega a(x)u^2,$$

where

$$G(s) = \begin{cases} \frac{|s|^{p+1}}{p+1} & \text{if } 0 \leq |s| \leq C_\lambda \\ \frac{p}{p+1}C_\lambda^{p-1}s^2 - \frac{p-1}{p+1}C_\lambda^p|s| & \text{if } |s| > C_\lambda \end{cases}$$

and $C_\lambda := \left(|\lambda|\, \|a\|_\infty\right)^{\frac{1}{p-1}}$, as suggested by the next lemma.

Lemma 1. *All solutions of (P^-) and of*

$$(\widetilde{P}^-) \quad \begin{cases} -\Delta u &= \lambda\, a(x)u - g(u) \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial\Omega \end{cases}$$

where $g(s) := G'(s)$, satisfy the following a priori estimate

$$0 \leq |u(x)| \leq \left(|\lambda|\, \|a\|_\infty\right)^{\frac{1}{p-1}} := C_\lambda \, , \; x \in \Omega$$

For the proof, see Lemma 1 in [4], which shows the priori estimate for positive solutions. For general solutions the proof follows with minor modifications.

Proof of Theorem 1 (i): The functional $\tilde{\Phi}$ is coercive (see Proposition 2 in [4]) and so it easily follows that

$$c_1 := \inf_{H^1} \tilde{\Phi} > -\infty.$$

We now prove that there exists F homeomorphic to S^{k-1} ($F \sim S^{k-1}$) such that

$$\sup_{u\in F} \Phi_\lambda(u) < 0.$$

For simplicity we prove the result for $k = 2$ (the general case follows in a similar way) and for $\lambda > \lambda_2^+$. It is sufficient to prove that there exists $F \sim S^1$ such that

$$\sup_{u\in F} \Phi_\lambda(u) < 0.$$

We first consider the case $\lambda_1^+ = 0$ (i.e. $\int_\Omega a(x) \geq 0$).

Let $\Omega^+ = \{x \in \Omega : \quad a(x) > 0\}$. Since $a = a(x)$ is continuous, there exists a ball $B \subset\subset \Omega^+$. Let $\phi_1 := 1+\delta\eta$ where $\delta > 0$, $\eta \in C_0^\infty(\Omega)$ is a positive smooth function with compact support in B and such that $\int_\Omega |\nabla\eta|^2 = 1$, and let $\phi_2 = \phi_2^+$ be an eigenfunction associated to the second (positive) eigenvalue λ_2^+, with $\int_\Omega |\nabla\phi_2|^2 = 1$. Define the set

$$F = F_\varepsilon = \{u \in H^1 : \ u = \alpha_1\phi_1 + \alpha_2\phi_2, \quad \alpha_1^2 + \alpha_2^2 = \varepsilon^2\},$$

where ε will be chosen later. F is homeomorphic to S^1 by an odd map for any $\varepsilon > 0$. Since F is compact, it is sufficient to prove that there exists $\varepsilon > 0$ such that for all $u \in F_\varepsilon$, $\Phi_\lambda(u) < 0$.

Indeed, for $u = \alpha_1\phi_1 + \alpha_2\phi_2$, using that $\int_\Omega a(x) \geq 0$, and $\int_\Omega a(x)\phi_2 = 0$ (see Remark 1, a) in the Appendix, using that $1 = \phi_1^+$),

$$\begin{aligned}
\Phi_\lambda(u) &= \tfrac{1}{2}\int_\Omega |\nabla(\alpha_1\delta\eta + \alpha_2\phi_2)|^2 - \tfrac{\lambda}{2}\int_\Omega a(x)[\alpha_1(1+\delta\eta) + \alpha_2\phi_2]^2 \\
&\quad + \tfrac{1}{p+1}\int_\Omega |\alpha_1\phi_1 + \alpha_2\phi_2|^{p+1} \\
&= \tfrac{1}{2}\alpha_1^2\delta^2 + \tfrac{1}{2}\alpha_2^2 + \alpha_1\alpha_2\delta\int_\Omega \nabla\eta\nabla\phi_2 - \tfrac{\lambda}{2}\int_\Omega a(x)\left(\alpha_1 + \alpha_1\delta\eta + \alpha_2\phi_2\right)^2 + O(\varepsilon^{p+1}) \\
&\leq \tfrac{1}{2}\alpha_1^2\delta^2 + \tfrac{1}{2}\alpha_2^2 + \alpha_1\alpha_2\delta\int_\Omega[\nabla\eta\nabla\phi_2 - \lambda a(x)\eta\phi_2] \\
&\quad - \tfrac{\lambda}{2}\int_\Omega a(x)\left[2\alpha_1^2\delta\eta + \alpha_1^2\delta^2\eta^2 + \alpha_2^2(\phi_2)^2\right] + O(\varepsilon^{p+1}) \\
&= \tfrac{1}{2}\alpha_2^2\left(1 - \tfrac{\lambda}{\lambda_2^+}\right) + \alpha_1\alpha_2\delta(\lambda_2^+ - \lambda)\int_\Omega a(x)\eta\phi_2 \\
&\quad + \tfrac{\alpha_1^2\delta}{2}\left[-2\lambda\int_\Omega a(x)\eta + \delta(1 - \lambda\int_\Omega a(x)\eta^2)\right] + O(\varepsilon^{p+1}) \\
&= \mathcal{A}\alpha_2^2 + \mathcal{B}\alpha_1\alpha_2\delta + \mathcal{C}\alpha_1^2\delta + O(\varepsilon^{p+1})
\end{aligned}$$

where

$$\mathcal{A} := \tfrac{1}{2}\left(1 - \tfrac{\lambda}{\lambda_2^+}\right) < 0$$
$$\mathcal{B} := (\lambda_2^+ - \lambda)\int_\Omega a(x)\eta\phi_2$$
$$\mathcal{C} := \left[-\lambda\int_\Omega a(x)\eta + \tfrac{\delta}{2}(1 - \lambda\int_\Omega a(x)\eta^2)\right] < 0, \quad \text{for } 0 < \delta < \tfrac{2\lambda\int_\Omega a(x)\eta}{|1-\lambda\int_\Omega a(x)\eta^2|}.$$

Now, recall that α_1, α_2 are $O(\varepsilon)$. Choose $\delta = \varepsilon^{\frac{p-1}{2}}$: if $\alpha_1 = O(\alpha_2)$ (i.e. $\alpha_1 = o(\alpha_2)$ or $\alpha_1 \asymp \alpha_2$) we have $\alpha_2^2 \asymp \varepsilon^2$, so that

$$\Phi_\lambda(u) = \mathcal{A}\,\alpha_2^2 \; + \; o(\varepsilon^2) < 0, \;\; \varepsilon \text{ small.}$$

On the other hand, if $\alpha_2 = o(\alpha_1)$, $\alpha_1^2 \asymp \varepsilon^2$, we have

$$\Phi_\lambda(u) = \mathcal{A}\,\alpha_2^2 \; + \mathcal{C}\alpha_1^2\delta + o(\alpha_1^2\delta) + O(|\alpha_1|^{p+1}) \leq \mathcal{C}\alpha_1^2\delta + o(\alpha_1^2\delta) + O(\varepsilon^{p+1}) < 0, \; \varepsilon \text{ small.}$$

For the case $\int_\Omega a(x)dx < 0$ the proof is easier: it is sufficient to take

$$F = F_\varepsilon = \{u \in H^1 : \; u = \alpha_1\phi_1 + \alpha_2\phi_2, \quad \alpha_1^2 + \alpha_2^2 = \varepsilon^2\},$$

where $\phi_1 = \phi_1^+$ and $\phi_2 = \phi_2^+$ are the (normalized) eigenfunctions associated respectively to λ_1^+ and λ_2^+.

For the general case $k \geq 2$ it is sufficient to take

$$F = F_\varepsilon = \{u \in H^1 : \; u = \sum_{i=1}^k \alpha_i\phi_i, \quad \sum_{i=1}^k \alpha_i^2 = \varepsilon^2\},$$

with the same choices for ϕ_1 (according to the sign of $\int_\Omega a(x)$), and $\phi_i = \phi_i^+$, $i = 2, .., k$, normalized eigenfunctions associated to λ_i^+ $i = 2, ..., k$.

To conclude the proof of (i) in Theorem 1, it is sufficient to use *Clark's Theorem*.

$ii)$ Now we prove that for $\lambda \in (\lambda_1^+, \lambda_2^+)$ the solutions have a definite sign.

Suppose that u is a (non trivial) solution of (P^-). We may read u as an eigenfunction (associated to the eigenvalue λ) of the following problem

$$(P_b) \quad \begin{cases} -\Delta\phi_b = \lambda b(x)\phi_b, & \text{in } \Omega \\ \frac{\partial\phi_b}{\partial\nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $b(x) = a(x) - \dfrac{|u|^{p-1}}{\lambda} \leq a(x)$ in Ω, with $b(x) \not\equiv a(x)$.

If $b(x) \leq 0$, we are done: $\lambda > 0$ cannot be an eigenvalue.

If $b(x) > 0$ somewhere, it is a sign changing weight. But since $b(x) \leq a(x)$, by monotonicity of the eigenvalues (see *f)* in appendix) we have

$$\lambda_2^+(b) \geq \lambda_2^+(a) > \lambda > 0.$$

Therefore, $\lambda = \lambda_1^+(b)$, and $u = \phi_b$ is an eigenfunction associated to the first eigenvalue, and has a fixed sign. □

3. Proof of Theorem 2 (non-coercive case)

It is well known that weak solutions of (P^+) correspond to critical points of the functional

$$\Psi_\lambda : H^1(\Omega) \to \mathbb{R} \qquad \Psi_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1} - \frac{\lambda}{2}\int_\Omega a(x)u^2.$$

We can apply the *Generalized Mountain Pass theorem* (Rabinowitz), see [20]. We need to prove some geometric estimates. Theorem 2 then follows in a standard way, since we have compactness due to the subcritical growth.

We prove that the functional Ψ_λ has a *linking geometry*:

We first consider the case $\bar{a} := \int_\Omega a(x)dx < 0$ and $\lambda \in [\lambda_k^+, \lambda_{k+1}^+)$.

For $j \in \mathbb{N}$, denote by $\phi_j = \phi_j^+$ (resp. $\phi_{-j} = \phi_j^-$) the normalized eigenfunction associated to the j^{th} positive eigenvalue (resp. j^{th} negative eigenvalue). In this case the normalization is meant as $\int_\Omega a(x)\phi^2 = 1$.

Let $E_k = span\{\phi_1, \dots, \phi_k\}$, and $F(\ni \phi_{k+1})$ so that $H^1 = E_k \oplus F$.

Since, for every k fixed, E_k is a finite dimensional space, we have that the following expressions

$$\|v\|_a := \Big(\int_\Omega a(x)v^2dx\Big)^{1/2}, \quad \Big(\int_\Omega |v|^2dx\Big)^{1/2}, \quad \|v\|_* =: \Big(\int_\Omega |\nabla v|^2 + \big(\int_\Omega v\big)^2\Big)^{1/2},$$

define equivalent norms.

Now, choose $e = \phi_{k+1}(\in F)$. For $0 < \rho < R_1$ and $R_2 > 0$, define

$$S^\perp = \{u \in F : \ \|u\| = \rho\} \quad \text{and} \quad Q = \{se + v : \ 0 \le s \le R_1, \ v \in E_k, \|v\|_a \le R_2\},$$

where $\|v\|_a := \big(\int_\Omega a(x)v^2dx\big)^{1/2}$ is actually a norm. It is known (see [20]) that the sets $S^\perp$ and ∂Q are *topologically linked sets*. To apply the linking theorem, we need the following estimates.

Lemma 2. *There are* $\rho > 0$, $R_1 > 0$, $R_2 > 0$ *and* $\alpha > 0$ *such that*

$$\sup_{u \in \partial Q} \Psi_\lambda(u) \le 0 < \alpha \le \inf_{u \in S^\perp} \Psi_\lambda(u). \tag{3}$$

Proof. We first consider the case $\bar{a} := \int_\Omega a(x)dx < 0$. We can assume, without loss of generality, that $\bar{a} = -1$.

Step 1 (*Estimates on* $S^\perp$): Let $J_k = \mathbb{Z} \setminus \{0, 1, ..., k\}$. Let K be the set of functions in H^1 vanishing outside $\Omega_0 = \{x \in \Omega : \ a(x) = 0\}$;

Every $u \in F$ can be represented by

$$u = \sum_{j \in J} \alpha_j \phi_j + w,$$

where w belongs to K. If $K = \emptyset$ the proof is easier, so let us consider the case $K \neq \emptyset$.

It is sufficient to prove that there exist $\alpha > 0$ and $\rho > 0$ such that

$$\Psi_\lambda(u) \geq \alpha > 0\ , \ \text{ for } \ u = \sum_{j \in J_k} \alpha_j \phi_j + w \ \text{ with } \ \|u\| = \rho$$

For $u = \sum_{j \in J_k} \alpha_j \phi_j + w$, since the eigenfunctions ϕ_j and w are a-orthogonal (see appendix), that is

$$\int_\Omega a(x)\phi_j\phi_i = 0,\ \int_\Omega a(x)\phi_j w = 0,\ \int_\Omega \nabla\phi_j\nabla\phi_i = 0,\ \int_\Omega \nabla\phi_j\nabla w = 0,$$

we have that

$$\begin{aligned}
\Psi_\lambda(u) &= \tfrac{1}{2}\sum_{j\in J_k}\alpha_j^2\Big(\int_\Omega|\nabla\phi_j|^2 - \lambda\int_\Omega a(x)\phi_j^2\Big) + \tfrac{1}{2}\int_\Omega|\nabla w|^2 - \tfrac{1}{p+1}\int_\Omega|u|^{p+1}\\
&= \tfrac{1}{2}\sum_{j\geq k+1}\alpha_j^2\left(\lambda_j^+ - \lambda\right) + \tfrac{1}{2}\sum_{j\leq -1}\alpha_j^2\left(-\lambda_j^- + \lambda\right) + \tfrac{1}{2}\int_\Omega|\nabla w|^2 - \tfrac{1}{p+1}\int_\Omega|u|^{p+1}\\
&\text{(here we used the normalization } \textstyle\int_\Omega a(x)\phi_j^2 = \mathrm{sign}(j))\\
&\geq \tfrac{1}{2}\left(\lambda_{k+1}^+ - \lambda\right)\sum_{j\geq k+1}\alpha_j^2 + \tfrac{1}{2}\sum_{j\leq -1}\lambda\alpha_j^2 + \tfrac{1}{2}\int_\Omega|\nabla w|^2 - \tfrac{1}{p+1}\int_\Omega|u|^{p+1}\\
&\geq \tfrac{1}{2}\Big\{\min\{\lambda_{k+1}-\lambda,\lambda\}\sum_{j\in J_{nm}}\alpha_j^2 + \int_\Omega|\nabla w|^2\Big\} - \tfrac{1}{p+1}\int_\Omega|u|^{p+1}\\
&\geq C\|u\|^2 - \tfrac{1}{p+1}\int_\Omega|u|^{p+1} = C\rho^2 + O(\rho^{p+1})
\end{aligned}$$

since $\Big\{\sum_j \alpha_j^2 + \int_\Omega |\nabla w|^2\Big\}^{\frac{1}{2}}$ is an equivalent norm on H^1.

Therefore, for ρ small, there exists $\alpha > 0$ such that $= C\rho^2 + O(\rho^{p+1}) > \alpha > 0$.

Step 2 *(Estimates on ∂Q):*

We observe that

$$\partial Q = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2,$$

where $\Gamma_0 = \{v \in E_k :\ \|v\| \leq R_2\}$, $\Gamma_1 = \{R_1 e + v :\ v \in E_k,\ \|v\| \leq R_2\}$ and $\Gamma_2 = \{se + v :\ \|v\| = R_2,\ 0 \leq s \leq R_1\}$.

(Γ_0): For $v = \sum_{j=1}^{k} \alpha_j \phi_j \in \Gamma_0$

$$\begin{aligned}
\Psi_\lambda(v) &= \tfrac{1}{2}\sum_{j=1}^k\alpha_j^2\Big(\int_\Omega|\nabla\phi_j|^2 - \lambda\int_\Omega a(x)\phi_j^2\Big) - \tfrac{1}{p+1}\int_\Omega|v|^{p+1}\\
&\leq \tfrac{1}{2}\sum_{j=1}^k\alpha_j^2\left(\lambda_j^+ - \lambda\right) \leq \tfrac{\left(\lambda_k^+ - \lambda\right)}{2}\sum_{j=1}^k\alpha_j^2 \leq 0
\end{aligned}$$

(Γ_1) For $u = R_1 e + v = \sum_{j=1}^{k+1} \alpha_j \phi_j \in \Gamma_1$ $(\alpha_{k+1} = R_1)$, using equivalence of norms on E_{k+1} and almost orthogonality, we have

$$\begin{aligned}
\Psi_\lambda(u) &= \tfrac{1}{2}\sum_{j=1}^{k+1} \alpha_j^2 \Big(\int_\Omega |\nabla \phi_j|^2 - \lambda \int_\Omega a(x)\phi_j^2 \Big) - \tfrac{1}{p+1}\int_\Omega |v + R_1 e|^{p+1} \\
&\leq \tfrac{(\lambda_{k+1}^+ - \lambda)}{2} R_1^2 - \tfrac{1}{p+1}\int_\Omega |v + R_1 e|^{p+1} \\
&\leq \tfrac{\lambda}{2} R_1^2 - C_1 \Big(\int_\Omega |v + R_1 e|^2 \Big)^{\frac{p+1}{2}} \\
&\leq \tfrac{\lambda_{k+1}^+ - \lambda}{2} R_1^2 - C_2 \Big(\int_\Omega a(x)(v + R_1 e)^2 \Big)^{\frac{p+1}{2}} \\
&\leq \tfrac{\lambda}{2} R_1^2 - C_2 R_1^{p+1} \leq 0, \text{ for large } R_1.
\end{aligned}$$

(Γ_2) For $u = se + v = se + \sum_{j=1}^{k} \alpha_j \phi_j \in \Gamma_2$

$$\begin{aligned}
\Psi_\lambda(u) &= \tfrac{1}{2}\sum_{j=1}^{k} \alpha_j^2 \Big(\int_\Omega |\nabla \phi_j|^2 - \lambda \int_\Omega a(x)\phi_j^2 \Big) + \tfrac{\lambda_{k+1} - \lambda}{2} s^2 - \tfrac{1}{p+1}\int_\Omega |v + se|^{p+1} \\
&\leq \tfrac{\left(\lambda_k^+ - \lambda\right)}{2} R_2^2 + \tfrac{\lambda_{k+1}^+ - \lambda}{2} s^2 \\
&\leq \tfrac{\left(\lambda_k^+ - \lambda\right)}{2}\, R_2^2 + \tfrac{\lambda_{k+1}^+ - \lambda}{2}\, R_1^2 \leq 0 \text{ , for } R_2 \geq \Big(\tfrac{\lambda_{k+1}^+ - \lambda}{\lambda - \lambda_k^+} \Big)^{\frac{1}{2}} R_1;
\end{aligned}$$

and if we want to include the case $\lambda = \lambda_k^+$:

- for $u = se + v = se + \sum_{j=1}^{k} \alpha_j \phi_j \in \Gamma_2$, with the argument used above

$$\begin{aligned}
\Psi_\lambda(u) &= \tfrac{1}{2}\sum_{j=1}^{k} \alpha_j^2 \Big(\int_\Omega |\nabla \phi_j|^2 - \lambda \int_\Omega a(x)\phi_j^2 \Big) + \tfrac{\lambda_{k+1} - \lambda}{2} s^2 - \tfrac{1}{p+1}\int_\Omega |v + se|^{p+1} \\
&\leq \tfrac{\left(\lambda_k^+ - \lambda\right)}{2} R_2^2 + \tfrac{\lambda_{k+1}^+ - \lambda}{2} R_1^2 - C_2 R_2^{p+1} \leq 0 \text{ , for } R_2 \text{ sufficiently large.}
\end{aligned}$$

Now we consider the degenerate case $\int_\Omega a(x) = 0$.

The estimates on $S^\perp$ and in Γ_0 are still valid. In this case $\|u\|_a = \Big(\int_\Omega a(x)u^2 \Big)^{1/2}$ is no longer a norm on E_k, (resp. on E_{k+1}), since $\|u\|_a = 0$ on constant functions.

- For $u = R_1 e + v = \sum_{j=1}^{k+1} \alpha_j \phi_j \in \Gamma_1$ $(\alpha_{k+1} = R_1)$, using almost orthogonality, Hölder inequality, and the equivalence of norm $\| \cdot \|_*$ on E_{k+1}, we have

$$
\begin{aligned}
\Psi_\lambda(u) &= \frac{1}{2}\sum_{j=1}^{k+1}\alpha_j^2\Big(\int_\Omega|\nabla\phi_j|^2-\lambda\int_\Omega a(x)\phi_j^2\Big)-\frac{1}{p+1}\int_\Omega|v+R_1e|^{p+1}\\
&\le \frac{(\lambda_{k+1}^+-\lambda)}{2}R_1^2-\frac{1}{p+1}\int_\Omega|v+R_1\phi_{k+1}|^{p+1}\\
&\le \frac{\lambda}{2}R_1^2-C_1R_1^{p+1}\Big(\int_\Omega|\frac{v}{R_1}+\phi_{k+!}|^2\Big)^{\frac{p+1}{2}}\\
&\le \frac{\lambda_{k+1}^+-\lambda}{2}R_1^2-C_2R_1^{p+1}\left(\|\phi_{k+1}+\frac{v}{R_1}\|_*\right)^{p+1}\\
&\le \frac{\lambda}{2}R_1^2-C_2R_1^{p+1}\left(\int_\Omega|\nabla v/R_1\|^2+|\nabla\phi_{k+!}|^2\right)^{\frac{p+1}{2}}\\
&\le \frac{\lambda}{2}R_1^2-C_2R_1^{p+1}\le 0,\ \text{for large}\ R_1.
\end{aligned}
$$

- For $u = se+v = se+\sum_{j=1}^{k}\alpha_j\phi_j\in\Gamma_2$

$$
\begin{aligned}
\Psi_\lambda(u) &= \frac{1}{2}\sum_{j=1}^{k}\alpha_j^2\left(\int_\Omega|\nabla\phi_j|^2-\lambda\int_\Omega a(x)\phi_j^2\right)+\frac{\lambda_{k+1}-\lambda}{2}s^2-\frac{1}{p+1}\int_\Omega|v+se|^{p+1}\\
&\le \frac{(\lambda_k^+-\lambda)}{2}R_2^2+\frac{\lambda_{k+1}^+-\lambda}{2}s^2\\
&\le \frac{(\lambda_k^+-\lambda)}{2}R_2^2+\frac{\lambda_{k+1}^+-\lambda}{2}R_1^2\le 0\ ,\quad \text{for}\quad R_2\ge\left(\frac{\lambda_{k+1}^+-\lambda}{\lambda-\lambda_k^+}\right)^{1/2};
\end{aligned}
$$

and if we want to include the case $\lambda=\lambda_k^+$:

- For $u = se+v = se+\sum_{j=1}^{k}\alpha_j\phi_j\in\Gamma_2$, with the argument used above

$$
\begin{aligned}
\Psi_\lambda(u) &= \frac{1}{2}\sum_{j=1}^{k}\alpha_j^2\Big(\int_\Omega|\nabla\phi_j|^2-\lambda\int_\Omega a(x)\phi_j^2\Big)+\frac{\lambda_{k+1}-\lambda}{2}s^2-\frac{1}{p+1}\int_\Omega|v+se|^{p+1}\\
&\le \frac{(\lambda_k^+-\lambda)}{2}R_2^2+\frac{\lambda_{k+1}^+-\lambda}{2}R_1^2-C_2R_2^{p+1}\le 0\ ,\quad \text{for } R_2 \text{ sufficiently large}
\end{aligned}
$$

The case $\bar a = \displaystyle\int_\Omega a(x)dx > 0$, let's say $\bar a = 1$, follows with minor modifications. Note that for $\lambda\in(\lambda_{k+1}^-,\lambda_k^-)$ it is sufficient to exchange $a=a(x)$ with $-a(x)$. □

4. Appendix

4.1. *The eigenvalue problem with indefinite weights*

For the convenience of the reader, in this section we recall some features for the following eigenvalue problem

$$
\begin{cases}
-\Delta\phi &= \lambda a(x)\phi \qquad \text{in } \Omega\\
\frac{\partial\phi}{\partial\nu} &= 0 \qquad\qquad\ \ \text{on } \partial\Omega,
\end{cases}
\tag{4}
$$

where $\Omega\subset\mathbb{R}^N$ is an open bounded domain, with $\partial\Omega$ of class C^1, and $a=a(x)$ is a non trivial continuous function. If $a=a(x)$ changes sign then there exist two sequences

i) $\{\lambda_j^+\}$ of positive eigenvalues, with associated eigenfunctions $\{\phi_j^+\}$,

ii) $\{\lambda_j^-\}$ of negative eigenvalues, with associated eigenfunctions $\{\phi_j^-\}$.

Indefinite eigenvalue problems are often generated by the linearization of nonlinear maps that needs to be controlled in every point of the domain space. For interesting generalizations of these methods, see [5, 6, 7, 22].

In what follows, we outline some properties of the eigen-pairs $\left(\lambda_j^{\pm}, \phi_j^{\pm}\right)$, and rephrase the variational characterization for the eigenvalues given by Manes-Micheletti ([18], see also [3]).

We define the following bilinear form

$$S(u,v) := \int_\Omega a(x)uv\,dx$$

Let $\mathcal{B}_+ = \{u:\ S(u,u)=1\}, \quad \mathcal{B}_- = \{u:\ S(u,u)=-1\}$.

Remark 1. *Since $a=a(x)$ changes sign, both $\mathcal{B}_+$ and $\mathcal{B}_-$ are nonempty.*

a) *(a-orthogonality)* if λ_* and λ^* are two different eigenvalues of (4), and resp. ϕ_*, ϕ^* two associated eigenvectors, then ϕ_*, ϕ^* are orthogonal:

$$\int_\Omega \nabla\phi_*\nabla\phi^*\,dx = 0 \quad \text{and} \quad \int_\Omega a(x)\,\phi_*\phi^*\,dx = 0.$$

b) *(the first eigenvalues)*

$$\lambda_1^+ = \inf_{u\in\mathcal{B}_+}\int_\Omega |\nabla u|^2 dx \geq 0 \quad \text{and} \quad \lambda_1^- = -\inf_{u\in\mathcal{B}_-}\int_\Omega |\nabla u|^2 dx \leq 0$$

are simple, with associated positive eigenfunctions ϕ_1^+ and ϕ_1^-. Moreover,

i) If $\int_\Omega a(x)\,dx < 0$ $\big($resp. $>0\big)$, then

$$\lambda_1^- = 0 \quad \text{and} \quad \lambda_1^+ > 0 \quad \big(resp. \quad \lambda_1^- < 0 \quad \text{and} \quad \lambda_1^+ = 0\big).$$

ii) If $\int_\Omega a(x)\,dx = 0$, then

$$\lambda_1^- = 0 = \lambda_1^+.$$

c) *(higher eigenvalues)* For $k \geq 2$

$$\lambda_k^+ = \inf_{\dim F = k}\sup_{u\in\mathcal{B}_+\cap F}\int_\Omega |\nabla u|^2 dx > 0 \quad \text{and} \quad \lambda_k^- = -\inf_{\dim F = k}\sup_{u\in\mathcal{B}_-\cap F}\int_\Omega |\nabla u|^2 dx < 0$$

or equivalently, using the characterization of Manes-Micheletti [18]

$$\frac{1}{\lambda_k^+} = \sup_{\dim F = k}\min_{u\in F, u\neq 0}\frac{\int_\Omega a(x)u^2}{\int_\Omega |\nabla u|^2} \quad \text{and} \quad \frac{1}{\lambda_k^-} = -\sup_{\dim F = k}\min_{u\in F, u\neq 0} -\frac{\int_\Omega a(x)u^2}{\int_\Omega |\nabla u|^2};$$

d) $\lambda_k^+ \to +\infty \quad \text{and} \quad \lambda_k^- \to -\infty \quad \text{as} \quad k\to+\infty.$

e) *(positivity of first eigenfunctions)* The eigenfunctions corresponding to the first eigenvalues have constant sign. Moreover, the eigenvalues $\lambda \neq \lambda_1^{\pm}$ *do not posses a positive eigenfunction.*

f) *(monotonicity)* if $a = a(x)$ and $b = b(x)$ are two continuous sign changing functions such that $a(x) \geq b(x)$, and $a(x) \not\equiv b(x)$, then

$$\lambda_k^+(a) \leq \lambda_k^+(b) \quad \text{and} \quad \lambda_k^-(a) \geq \lambda_k^-(b).$$

proof of (f). Let $\mathcal{G}^a =: \{u : \int_\Omega a(x)u^2 > 0\}$ and $\mathcal{G}^b := \{u : \int_\Omega b(x)u^2 > 0\}$. Clearly, $\mathcal{G}^a \supset \mathcal{G}^b$. Therefore,

$$\lambda_k^+(a) = \inf_{\dim F = k} \sup_{u \in F \cap \mathcal{G}^a} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega a(x)u^2} \leq \inf_{\dim F = k} \sup_{u \in F \cap \mathcal{G}^b} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega a(x)u^2}.$$

For every F, there exists $u_F \in G^b \cap F$ such that

$$\begin{aligned} \inf{}_{\dim F = k} \ \sup{}_{u \in F \cap \mathcal{G}^b} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega a u^2} &= \inf{}_{\dim F = k} \frac{\int_\Omega |\nabla u_F|^2 dx}{\int_\Omega a u_F^2} \\ &\leq \inf{}_{\dim F = k} \frac{\int_\Omega |\nabla u_F|^2 dx}{\int_\Omega b(x) u_F^2} \leq \inf{}_{\dim F = k} \sup{}_{u \in F \cap \mathcal{G}^b} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega b(x) u^2} = \lambda_k^+(b) \ . \end{aligned}$$

□

4.2. *An explicit example*

In this section we solve explicitly the eigenvalue problem

$$\begin{cases} -\phi'' = \lambda a(x) \phi, & \text{in } \Omega \\ \phi'(0) = \phi'(2) = 0 \end{cases} \tag{5}$$

in the following cases

(a_0) $\Omega = (0, 2)$, and $a(x) = \text{sign}(1 - x)$, i.e. $\int_\Omega a(x)dx = 0$;

(a_-) $\Omega = (0, 2)$, and $a(x) = \text{sign}(1/2 - x)$, i.e. $\int_\Omega a(x)dx < 0$.

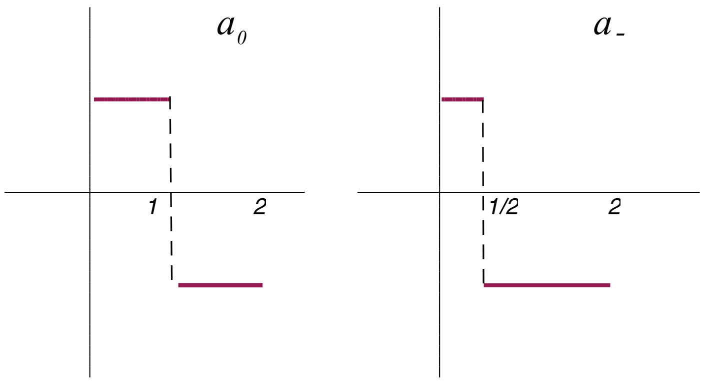

Figure 3. *The weights a_0 and a_-*

(a_0) **The degenerate case.** First, we look for strictly positive eigenvalues. By integrating the equation separately on $(0,1)$ and on $(1,2)$ resp., we obtain the following families of functions, taking into account the boundary conditions

$$u_1(x) = A\ \cos(\sqrt{\lambda}x), \qquad u_2(x) = B\ \mathrm{Ch}\left(\sqrt{\lambda}(2-x)\right).$$

Now, by gluing the solutions in $x=1$, we obtain the following conditions

$$A\cos(\sqrt{\lambda}) = B\ \mathrm{Ch}(\sqrt{\lambda}) \qquad A\sin(\sqrt{\lambda}) = B\ \mathrm{Sh}(\sqrt{\lambda})$$

which lead to the following equation in $\sqrt{\lambda}$, with $\lambda \geq 0$

$$\tan(\sqrt{\lambda}) = \mathrm{Th}(\sqrt{\lambda})$$

whose solutions form an increasing sequence $0 = \lambda_1^+ < \lambda_2^+ < \dots < \lambda_k^+ < \dots$ with $(\lambda_k^+)^{\frac{1}{2}} \in \big((k-1)\pi, \frac{\pi}{2} + (k-1)\pi\big)$ (see fig. 4).

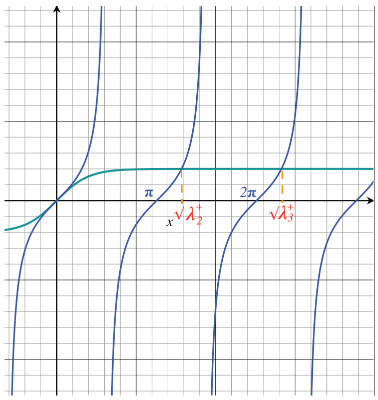

Figure 4. *in blue* $y = \tan(t)$, *in green* $y = \mathrm{Th}(t)$, $\quad t = \sqrt{\lambda}$

In this case the eigenfunctions are

$$\phi_k^+(x) = \begin{cases} A\cos\left((\lambda_k^+)^{\frac{1}{2}}x\right) & 0 \leq x \leq 1 \\ A\dfrac{\cos\left((\lambda_k^+)^{\frac{1}{2}}\right)}{\mathrm{Ch}\left((\lambda_k^+)^{\frac{1}{2}}\right)}\mathrm{Ch}\left((\lambda_k^+)^{\frac{1}{2}}(2-x)\right) & 1 < x \leq 2 \end{cases}.$$

The first eigenfunction (corresponding to $\lambda_1^+ = 0$) is a constant, while the eigenfunction ϕ_2^+ associated to the second (the first strictly positive) eigenvalue changes sign and behaves as in fig. 5.

Figure 5. *The eigenfunctions* ϕ_1^+ *and* ϕ_2^+

By using the transformation $x \mapsto 2-x$ one can find the sequence of negative eigenvalues $\mu_k = -\lambda_k$.

(a_-) **The non degenerate case.**

In this case, by a similar argument, the positive eigenvalues are the solutions of

$$\tan \frac{\sqrt{\lambda}}{2} = \mathrm{Th} \frac{3}{2}\sqrt{\lambda},$$

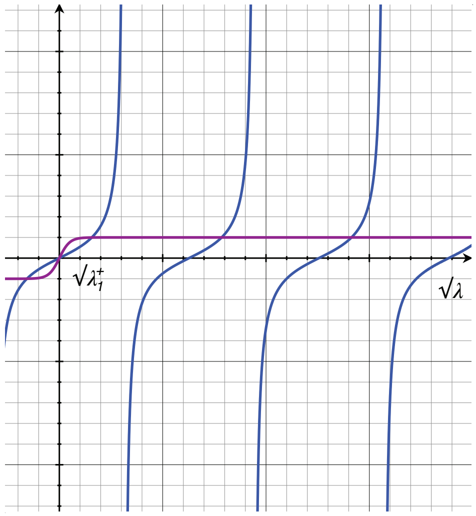

Figure 6. *in blue* $y = \tan \frac{t}{2}$, *in purple* $y = \mathrm{Th} \frac{3t}{2}$

The eigenfunction associated to the first (strictly) positive eigenvalue does not change sign. On the other hand, the negative eigenvalues are the (negative) solutions of

$$\tan \frac{3}{2}\sqrt{|\lambda|} = \mathrm{Th} \frac{\sqrt{|\lambda|}}{2}$$

and the eigenfunction associated to the first negative eigenvalue (ϕ_2^-) changes sign: roughly speaking, the first "negative" eigenvalue in this case is the trivial one.

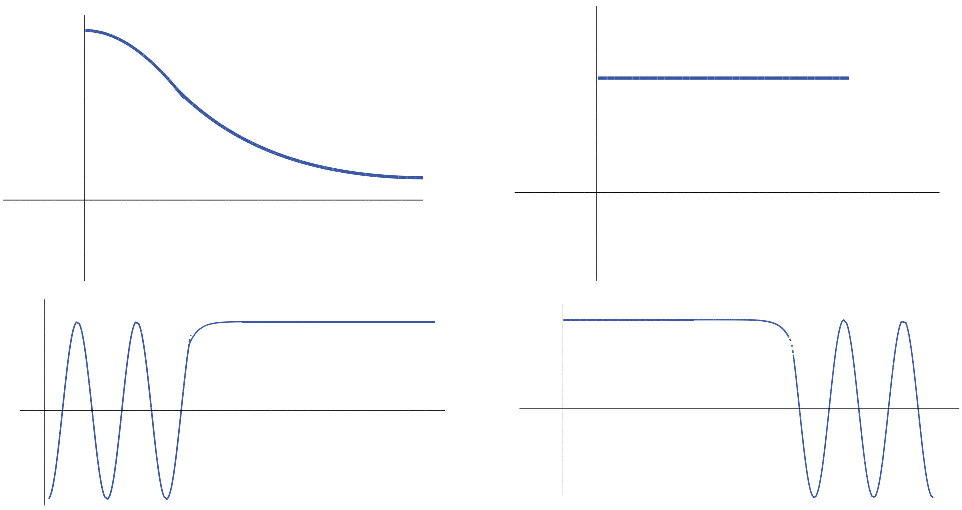

Figure 7. *Eigenfunctions for* $\int_\Omega a(x) < 0$*: above* ϕ_1^+ *and* ϕ_1^-*; below* ϕ_5^+ *and* ϕ_5^-

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Bibliography

[1] A. Ambrosetti , G. Prodi, *On the inversion of some differentiable mappings with singularities between Banach spaces*, Ann. Mat. Pura Appl. 93 (1972) 231-246.

[2] M. Bôcher, *Boundary problems in one dimension*, Proc. Fifth Internat. Congress Math. (Cambridge, 1912), Vol. I, Cambridge Univ. Press, New York, 1913, pp. 163-195

[3] K.J. Brown, S.S. Lin *On the existence of positive eigenfunctions for an eigenvalue problem with indefinite weight function*, J Math Anal Appl 75(1) (1980), pp 112–120.

[4] M. Calanchi, B. Ruf, *Eigenvalues and bifurcation for Neumann problems with indefinite weights*, Electronic Journal of Differential Equations, Special Issue 01 (2021), pp. 255–268.

[5] M. Calanchi, C. Tomei, A. Zaccur, *Fibers and global geometry of functions*, PNLDE 86, Springer, Cham (2015) 55-75.

[6] M. Calanchi, C. Tomei, A. Zaccur, *Cusps and a converse to the Ambrosetti-Prodi theorem*, Ann. Sc. Norm. Sup. Pisa (5) XVIII (2018) 483–507.

[7] M. Calanchi, C. Tomei, *Positive eigenvectors and simple nonlinear maps*, Journal of Functional Analysis Vol 280, Issue 7 (2021).

[8] D. C. Clark, *A variant of the Ljusternik–Shnirelmann theory*, Indiana Univ. Math. J., 22 (1972), 65–74.

[9] D. G. de Figueiredo, *Positive solutions of semilinear elliptic problems*, Lecture Notes in Math., vol. 957, Springer-Verlag, Berlin, 1982, pp. 34-87

[10] D.G. de Figueiredo, P.L. Lions, R. Nussbaum, *A priori estimates and existence of positive solutions of semilinear elliptic equations*. J. Math. Pures Appl. (9) 61 (1982), no. 1, 41–63.

[11] J. Fleckinger, M. L. Lapidus, *Eigenvalues of Elliptic Boundary Value Problems With an Indefinite Weight Function* Trans. AMS, May, 1986, Vol. 295, No. 1 (May, 1986), pp. 305-324

[12] B. Gidas, J. Spruck, *A priori bounds for positive solutions of nonlinear elliptic equations*, Comm. Partial Differential Equations 6 (1981), no. 8, 883?901.

[13] P. Hess, T. Kato, *On some linear and nonlinear eigenvalue problems with an indefinite weight function*, Comm. Partial Differential Equations 5 (1980), 999-1030

[14] D. Hilbert, Grundzuge einer allgemeinen Theorie der linearen Integralgleichungen, Teubner, Leipzig, 1912; Chelsea, New York, 1953.

[15] E. Holmgren, *Über Randwertaufgaben bei einer linearen Differentialgleichung zweiter Ordnung*, Ark. Mat., Astro och Fysik 1 (1904), 401-417.

[16] M. L. Lapidus, Valeurs propres du laplacien avec un poids qui change de signe, C. R. Acad. Sci. Paris Ser. I Math. 298 (1984), 265-268.

[17] J. López-Gómez, P. H. Rabinowitz, *The effects of spatial heterogeneities on some multiplicity results* Discrete and Continuous Dyn. Systems, Volume 36, Number 2, (2016).

[18] A. Manes, A.M. Micheletti, *Un'estensione della teoria variazionale classica degli autovalori per operatori ellittici del secondo ordine*, Boll. Unione Mat. Ital. 7 (1973) 285-301.

[19] D. Mazzoleni, B. Pellacci, G. Verzini, *Asymptotic spherical shapes in some spectral optimization problems*, Journal de Mathématiques Pures et Appliquées 135, (2020) 256–283.

[20] P.H. Rabinowitz, *Minimax Methods in Critical Point Theory with Application to Differential Equations*, Conference board of the mathematical sciences. Regional conference series in mathematics 65, Amer. Math. Soc., Providence, RI, 1986.

[21] R. G. D. Richardson, *Contributions to the study of oscillation properties of the solutions of linear differential equations of the second order*, Amer. J. Math. 40 (1918), 283-316.

[22] B. Ruf, *Singularity theory and bifurcation phenomena in differential equations*, Topological Nonlinear Analysis, II, Progr. Nonlinear Differential Equations Appl., 27, Birkhäuser, Boston, MA, (1997) 315-395.

