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Some remarks on surjections of unit groups

Remarques sur les surjections des groupes unitaires

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ABSTRACT. The main purpose of this paper is to study unital ring homomorphisms of associative rings $\varphi : R \to S$ satisfying one of the following conditions: (a) the unit-preserving property: $\varphi(R^\times) = S^\times$ and (b) the inverse unit-preserving property: $\varphi^{-1}(S^\times) = R^\times$. We establish the relationship between these two conditions. Several characterizations of such conditions are settled. An application to the index of unit groups of rings $R \subset S$ having a nonzero common ideal is given.
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1. introduction

Throughout this paper we assume that all rings are associative with identity and (as usual) that all ring homomorphisms are unital. By an ideal of a ring R, we mean a two-sided ideal. If R is a ring, we let $R^\times$ denote its group of units, $\mathrm{Spec}(R)$ (resp., $\mathrm{Max}(R)$) its set of prime (resp., maximal) ideals. In [2], Chen has studied surjective ring homomorphisms $\varphi : R \to S$ of commutative rings such that $\varphi(R^\times) = S^\times$. So several important results are obtained. We will call a ring homomorphism φ, satisfying the above condition, a homomorphism with the *unit-preserving property*. Motivated by the work of Chen, we will say that a ring homomorphism $\varphi : R \to S$ of arbitrary rings satisfies the *inverse unit-preserving property* if $\varphi^{-1}(S^\times) = R^\times$. In Section 2, we characterize surjective ring homomorphisms satisfying the inverse-unit preserving property (see Corollary 2.4). We show in Proposition 2.7 that any lying over surjective ring homomorphism of commutative rings satisfies the inverse unit-preserving property. Theorem 2.9 states that if R is a von Neumann regular ring, then a ring homomorphism $\varphi : R \to S$ of commutative rings satisfies the inverse unit-preserving property if and only if φ is injective. In Section 3, we extend Chen's definition of ring homomorphisms satisfying the unit-preserving property for arbitrary rings. We establish in Proposition 3.4 a relationship between the unit-preserving and the inverse unit-preserving properties. As a consequence, we recover [2, Corollary 2.1] for arbitrary rings (see Corollary 3.5). We demonstrate in Theorem 3.7 that if $\varphi : R \to S$ is a surjective ring homomorphism with kernel I such that $R/\mathrm{Ann}_R(I)$ is left Artinian, then φ satisfies the unit-preserving property. In Theorem 4.3, we establish that for rings $R \subset S$ sharing an ideal I, if the canonical surjection $\pi : S \to S/I$ satisfies the unit-preserving property, then $[S^\times : R^\times] = [(S/I)^\times : (R/I)^\times]$.

Any unexplained terminology is standard as in [3], [4] and [5].

2. Inverse unit-preserving property

For any ring homomorphism $\varphi : R \to S$, we always have $R^\times \subseteq \varphi^{-1}(S^\times)$. This inclusion relation may be strict, in general, even if φ is surjective or injective. To see this, let us consider, for instance, the canonical surjection $\pi : \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$, where p is a prime number. Clearly, $\pi^{-1}((\mathbb{Z}/p\mathbb{Z})^{\times}) = \mathbb{Z} \setminus p\mathbb{Z} \supsetneq \mathbb{Z}^{\times} = \{\pm 1\}$. We provide another example of an injective ring homomorphism with the above strict inclusion relation. Take, for instance, the canonical injection $\varphi : \mathbb{Z} \to \mathbb{Q}$. Clearly, $\varphi^{-1}(\mathbb{Q}^{\times}) = \mathbb{Z} \setminus \{0\} \supsetneq \mathbb{Z}^{\times}$. These facts encourage us to introduce the following definition.

Definition 2.1. Let R and S be arbitrary rings. We say that a ring homomorphism $\varphi : R \to S$ satisfies the inverse unit-preserving property if $\varphi^{-1}(S^{\times}) = R^{\times}$; or equivalently, if $\varphi^{-1}(S^{\times}) \subseteq R^{\times}$.

Remark 2.2. (1) Of course, if $\varphi : R \to S$ is a ring isomorphism, then φ satisfies the inverse unit-preserving property.
(2) Let R and S be rings and let $\varphi : R \to S$ be a ring homomorphism satisfying the inverse unit-preserving property. Then $S = \{0\}$ if and only if $R = \{0\}$. Indeed, assume first that $S = \{0\}$. As φ satisfies the inverse unit-preserving property, then $R = R^{\times}$. Thus, $0 \in R^{\times}$ and so $R = \{0\}$. Conversely, suppose $R = \{0\}$. Since $\varphi^{-1}(S^{\times}) = R^{\times} = \{0\}$, then $\varphi(0) = 0 \in S^{\times}$. Hence, $S = \{0\}$. This is why we will assume in the sequel that if $\varphi : R \to S$ is a ring homomorphism satisfying the inverse unit-preserving property, then R and S can be chosen to be nonzero.

In what follows, we will investigate some properties of ring homomorphisms satisfying the inverse unit-preserving property. But, first recall from [5] that the Jacobson radical of a ring R, denoted by $J(R)$, is the intersection of maximal left ideals of R. It follows from [5, Section 4] that $x \in J(R)$ if and only if $1 - yxz \in R^{\times}$ for all $y, z \in R$.

Proposition 2.3. *Let $\varphi : R \to S$ be a ring homomorphism satisfying the inverse unit-preserving property. Then $\mathrm{Ker}(\varphi) \subseteq J(R)$.*

Proof. Let $x \in \mathrm{Ker}(\varphi)$ and $y, z \in R$. According to the above comments, we need to show that $1-yxz \in R^{\times}$. As φ is a ring homomorphism, we have $\varphi(1-yxz) = \varphi(1)-\varphi(y)\varphi(x)\varphi(z) = \varphi(1) = 1$. Thus, $1 - yxz \in \varphi^{-1}(S^{\times})$. But, as φ satisfies the inverse unit-preserving property, then $\varphi^{-1}(S^{\times}) = R^{\times}$. Hence, we infer that $1 - yxz \in R^{\times}$, as desired. This completes the proof. □

As a consequence, we get the following characterization.

Corollary 2.4. *Let $\varphi : R \to S$ be a surjective ring homomorphism. Then the following statements are equivalent:*

1. *φ satisfies the inverse unit-preserving property.*
2. $\mathrm{Ker}(\varphi) \subseteq J(R)$.

Proof. The implication (1)⇒(2) follows from Proposition 2.3 without the “surjective” hypothesis. Let us prove that (2) implies (1). Assume (2). Let $x \in \varphi^{-1}(S^{\times})$. Then $\varphi(x) \in S^{\times}$. So there exists $y \in S$ such that $y\varphi(x) = \varphi(x)y = 1$. But, as φ is surjective, then $y = \varphi(r)$ for some $r \in R$. It follows that $\varphi(r)\varphi(x) = \varphi(x)\varphi(r) = 1$; or equivalently, $\varphi(rx) = \varphi(xr) = 1$. Therefore, $rx - 1, xr - 1 \in \mathrm{Ker}(\varphi)$. Hence $rx, xr \in 1 + \mathrm{Ker}(\varphi) \subseteq 1 + J(R) \subseteq R^{\times}$. So x has both a right inverse, namely $r(xr)^{-1}$, and a left inverse, namely $(rx)^{-1}r$. It follows that $x \in R^{\times}$. The proof is complete. □

Recall from [4] that a ring homomorphism $\varphi : R \to S$ satisfies the *lying-over* property (briefly, LO), if for each $P \in \text{Spec}(R)$, there exists $Q \in \text{Spec}(S)$ such that $\varphi^{-1}(Q) = P$. As an example of such ring homomorphism, one can take for instance, φ to be the inclusion map of an integral ring extension $R \subseteq S$.

Proposition 2.5. *Let $\varphi : R \to S$ be an homomorphism of commutative rings satisfying LO. Then φ satisfies the inverse unit-preserving property.*

Proof. By using Remark 2.2 (2), we can assume that $R \neq \{0\}$ and so $R \setminus R^\times \neq \emptyset$. Now, suppose that the assertion fails. Then there exists $r \in R \setminus R^\times$ such that $y := \varphi(r) \in S^\times$. An application of Krull's theorem ensures the existence of a maximal ideal M of R such that $r \in M$. As φ satisfies LO, there exists $Q \in \text{Spec}(S)$ such that $\varphi^{-1}(Q) = M$. Hence $y = \varphi(r) \in \varphi(M) = \varphi(\varphi^{-1}(Q)) \subseteq Q$. Since $y \in S^\times$, this gives the desired contradiction completing the proof. □

Proposition 2.6. *Let $\varphi : R \to S$ be an homomorphism of commutative rings such that the induced map $\varphi_M : R_M \to S_M$ satisfies the inverse unit-preserving property for all $M \in \text{Max}(R)$. Then φ satisfies the inverse unit-preserving property.*

Proof. Without loss of generality, we can assume that $R \neq \{0\}$ accordingly to Remark 2.2 (2). We need to show that if $y \in S^\times$ and $y = \varphi(x)$ for some $x \in R$, then $x \in R^\times$. Deny. Then we can choose $M \in \text{Max}(R)$ such that $x \in M$. As $\varphi_M(x/1) = \varphi(x)/1 = y/1 \in S_M^\times$ and $\varphi_M : R_M \to S_M$ satisfies the inverse unit-preserving property, we get $x/1 \in R_M^\times$, which is a contradiction since $x/1 \in MR_M$. □

Recall that a ring R is called *von Neumann regular* if for every element $a \in R$ there exists an element x in R with $a = axa$. It is well known that for every von Neumann regular ring R, we have $J(R) = 0$.

Theorem 2.7. *Let $\varphi : R \to S$ be an homomorphism of commutative rings. Assume that R is a von Neumann regular ring, then the following statements are equivalent:*

1. *φ satisfies the inverse unit-preserving property.*
2. *φ is injective.*

Proof. As φ satisfies the inverse unit-preserving property, then Proposition 2.3 guarantees that $\text{Ker}(\varphi) \subseteq J(R)$. Since R is a von Neumann regular ring, then $J(R) = 0$. It follows that $\text{Ker}(\varphi) = 0$. Hence, φ is injective. This proves that (1) implies (2). Conversely, assume (2). According to Remark 2.2 (2), we can assume that $R \neq \{0\}$. By Proposition 2.6, it is enough to prove that the induced map $\varphi_M : R_M \to S_M$ satisfies the inverse unit-preserving property for all $M \in \text{Max}(R)$. Fix any such M. Since R_M is a flat R-module, the monomorphism φ induces a monomorphism $\varphi_M : R_M \to S \otimes_R R_M \cong S_M$. As R is von Neumann regular, $K := R_M$ is a field and $B := S_M$ is a nonzero K-algebra. Since $B \neq \{0\}$, we get $0 \notin B^\times$ and so, since K is a field, it follows that $K \to B$ satisfies the inverse unit-preserving property. This proves assertion (1). The proof is complete. □

3. Some remarks on Unit-preserving property

In the nice paper [2], Chen has introduced the following definition.

Definition 3.1. (1) Let R and S be commutative rings. We say that a surjective ring homomorphism $\varphi : R \to S$ has $(*)$ if the induced map $\varphi^\times : R^\times \to S^\times$ is surjective.
(2) We say that the ring R has $(*)$ if every surjective ring homomorphism $\varphi : R \to S$ (for any ring S) has $(*)$.
(3) An ideal I of a commutative ring R is said to have $(*)$ if the canonical surjection $R \to R/I$ has $(*)$.

Next, we extend Chen's definition for arbitrary rings.

Definition 3.2. (1) Let R and S be rings (not necessarily commutative). We say that a surjective ring homomorphism $\varphi : R \to S$ satisfies the unit-preserving property if $\varphi^\times(R^\times) = S^\times$; or equivalently, if $S^\times \subseteq \varphi(R^\times)$.
(2) We say that the ring R satisfies the unit-preserving property if every surjective ring homomorphism $\varphi : R \to S$ (for any ring S) satisfies the unit-preserving property.
(3) An ideal I of an arbitrary ring R is said to satisfy the unit-preserving property if the canonical surjection $R \to R/I$ satisfies the unit-preserving property.

Remark 3.3. It is worth noticing that we can limit ourselves in Definition 3.2 to only nonzero rings S. In fact, let R be a ring and let $\varphi : R \to \{0\}$ be the only possible surjective ring homomorphism from R to $\{0\}$. Then $\varphi(R^\times) = \{0\}$. So, φ satisfies the the unit-preserving property.

We start our investigation with the following straightforward result.

Proposition 3.4. *Let $\varphi : R \to S$ be a surjective ring homomorphism satisfying the inverse unit-preserving property, then φ satisfies the unit-preserving property.*

It is worth mentioning that the converse to Proposition 3.4 does not hold in general. To see this, it is enough to consider the mapping $\varphi : R \to \{0\}$ for some nonzero ring R. Then φ satisfies the unit-preserving property by virtue of Remark 3.3, however φ does not satisfy the inverse unit-preserving property according to Remark 2.2 (2).

As a consequence of Corollary 2.4 and Proposition 3.4, we recover [2, Corollary 2.1] for arbitrary rings.

Corollary 3.5. *Let R be a ring and let I be an ideal of R. If $I \subseteq J(R)$, then I satisfies the unit-preserving property.*

Proposition 3.6. *Let R and S be commutative rings and let $\varphi : R \to S$ be a surjective ring homomorphism satisfying LO. Then φ satisfies the unit-preserving property.*

Proof. Combine Propositions 2.7 and 3.4. □

Recall that if R is a ring and A is a subset of R, then the (*left*) *annihilator* of A, denoted $\mathrm{Ann}_R(A)$, is the set of all elements r in R such that, for all a is A, $ra = 0$. In set notation, $\mathrm{Ann}_R(A) := \{r \in R \mid \forall a \in A, ra = 0\}$. Clearly, $\mathrm{Ann}_R(A)$ is a left ideal of R. If moreover, A is a left ideal of R, then $\mathrm{Ann}_R(A)$ is an ideal of R.

Next, we establish the titular result of this section.

Theorem 3.7. *Let R and S be rings and let $\varphi : R \to S$ be a surjective ring homomorphism with kernel I. If $R/\mathrm{Ann}_R(I)$ is left Artinian, then φ satisfies the unit-preserving property.*

Proof. Let $\mathrm{End}(I)$ be the endomorphism ring of the additive group $(I, +)$ and let $\rho : R \to \mathrm{End}(I)$ be the ring homomorphism defined by $\rho(r) = L_r$, where L_r is the left multiplication by r. Set $J := \mathrm{Ker}(\rho)$. Clearly, $J = \mathrm{Ann}_R(I)$ is an ideal of R. Set $E := \mathrm{Im}(\rho)$. By the first ring isomorphism theorem, we have $R/J \cong E$. It follows from [1, Lemm 3.5] that the combined map $R \to S \times E$ induces a ring isomorphism $\phi : R/(I \cap J) \to S \times_{R/(I+J)} E$. We claim that the map $(R/(I \cap J))^\times \to S^\times$ is surjective. Indeed, let $u \in S^\times$. Write v for the image of u in $(R/(I + J))^\times$. Since E is left Artinian, we can choose by virtue of [1, Lemma 3.4] an element $w \in E^\times$ mapping to $v \in (R/(I + J))^\times$. Thus, $(u, w) \in S^\times \times_{(R/(I+J))^\times} E^\times = (S \times_{R/(I+J)} E)^\times$. Hence, $\phi^{-1}(u, w)$ is a unit of $R/(I \cap J)$ that maps to $u \in S^\times$. This proves our claim. Since $(I \cap J)(I \cap J) \subseteq JI = 0$, we infer that for any $x \in I \cap J$ the element $1 + x$ has inverse $1 - x$ and therefore belongs to $R^\times$. This yields that $I \cap J \subseteq J(R)$, so by Corollary 3.5 the map $R^\times \to (R/(I \cap J))^\times$ is surjective. It follows that the composed map $R^\times \to S^\times$ is also surjective. □

4. The case of rings sharing a nonzero ideal

We start with the following straightforward result. We include a proof for the sake of completeness.

Lemma 4.1. *Let $R \subset S$ be an extension of rings having I as a common ideal. Then $I \subseteq J(R)$ if and only if $I \subseteq J(S)$.*

Proof. For the "Only if" part, assume that $I \subseteq J(R)$. Our task is to show that $I \subseteq J(S)$. For, let $x \in I$ and $y, z \in S$. As $yxz \in I \subseteq J(R)$, then $1 - yxz = 1 - 1(yxz)1 \in R^\times$. In particular, $1 - yxz \in S^\times$. This proves that $x \in J(S)$. Thus $I \subseteq J(S)$, as desired.

For the "If" part, suppose that $I \subseteq J(S)$ and let $x \in I$ and $y, z \in R$. As $yxz \in I \subseteq J(S)$, then $1 - yxz \in S^\times$. Thus, $(1 - yxz)\alpha = 1$ for some $\alpha \in S$. This implies $\alpha = (yxz)\alpha \in IS \subseteq I \subseteq R$. Therefore, $1 - yxz \in R^\times$. This proves that $x \in J(R)$. Thus $I \subseteq J(R)$. This completes the proof. □

Corollary 4.2. *Let $R \subset S$ be an extension of rings having I as a common ideal. Then the canonical surjection $\pi : S \to S/I$ satisfies the inverse unit-preserving property if and only if so does the restriction $\pi_{|R} : R \to R/I$.*

Proof. Combine Corollary 2.4 and Lemma 4.1. □

Theorem 4.3. *Let $R \subset S$ be an extension of rings having I as a common ideal. Then the following hold true:*

1. $[S^\times : R^\times] \leq [(S/I)^\times : (R/I)^\times]$.
2. *If moreover the canonical surjection $\pi : S \to S/I$ satisfies the unit-preserving property, then* $[S^\times : R^\times] = [(S/I)^\times : (R/I)^\times]$

Proof.

1. The canonical surjection $\pi : S \to S/I$ induces a group homomorphism $\pi^\times : S^\times \to (S/I)^\times$. We claim that ${\pi^\times}^{-1}((R/I)^\times) = R^\times$. Indeed, the inclusion relation $R^\times \subseteq {\pi^\times}^{-1}((R/I)^\times)$ is clear. Conversely, let $x \in {\pi^\times}^{-1}((R/I)^\times)$. Then $\pi(x) = x + I \in (R/I)^\times$. Hence, there exists $r \in R$ such that $(x+I)(r+I) = (r+I)(x+I) = 1 + I$. This implies that $xr = 1 + a$ for some element $a \in I$. But $x \in S^\times$, so $sx = xs = 1$ for some $s \in S$. Therefore, $sxr = s + sa$. Hence, $s = r - sa \in R + aS \subseteq R + I = R$. This shows that x is invertible in R with inverse s. This completes the proof of our claim. Now, since ${\pi^\times}^{-1}((R/I)^\times) = R^\times$, the homomorphism $\pi^\times$ reduces to an embedding of sets $\overline{\pi^\times} : S^\times/R^\times \to (S/I)^\times/(R/I)^\times$. It follows that $[S^\times : R^\times] \leq [(S/I)^\times : (R/I)^\times]$.

2. As the canonical surjection $\pi : S \to S/I$ satisfies the unit-preserving property, then the group homomorphism $\pi^\times : S^\times \to (S/I)^\times$ is surjective. Thus, $\overline{\pi^\times}$ is also surjective. This yields the equality of unit indexes. □

We close the paper with the following corollary.

Corollary 4.4. *Let $R \subset S$ be an extension of rings having I as a common ideal. If $S/\mathrm{Ann}_S(I)$ is left Artinian, then $[S^\times : R^\times] = [(S/I)^\times : (R/I)^\times]$.*

Proof. This follows readily from Theorems 3.7 and 4.3 since the canonical surjection $\pi : S \to S/I$ satisfies the unit-preserving property in case $S/\mathrm{Ann}_S(I)$ is left Artinian. □

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