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Weyl almost automorphic functions and applications

Les fonctions de Weyl presque automorphes et applications

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ABSTRACT. In this paper, we reconsider the notion of a Weyl p-almost automorphic function introduced by S. Abbas [1] in 2012 and propose several new ways for introduction of the class of Weyl p-almost automorphic functions ($1 \leq p < \infty$). We first analyze the introduced classes of Weyl p-almost automorphic functions of type 1, jointly Weyl p-almost automorphic functions and Weyl p-almost automorphic functions of type 2 in the one-dimensional setting. After that, we introduce and analyze generalizations of these classes in the multi-dimensional setting, working with general Lebesgue spaces with variable exponents. We provide several illustrative examples and applications to the abstract Volterra integro-differential equations.
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1. Introduction and preliminaries

The concept of almost periodicity was introduced by Danish mathematician H. Bohr around 1924-1926 and later generalized by many other authors (cf. [13], [21], [24], [35] and [40] for more details on the subject). Let $I = \mathbb{R}$ or $I = [0, \infty)$, and let $f : I \to X$ be continuous. Given $\epsilon > 0$, we call $\tau > 0$ an ϵ-period for $f(\cdot)$ if and only if $\|f(t+\tau) - f(t)\| \leq \epsilon$, $t \in I$. The set consisting of all ϵ-periods for $f(\cdot)$ is denoted by $\vartheta(f, \epsilon)$. We say that $f(\cdot)$ is almost periodic if and only if for each $\epsilon > 0$ the set $\vartheta(f, \epsilon)$ is relatively dense in I, which means that there exists $l > 0$ such that any subinterval of I of length l meets $\vartheta(f, \epsilon)$.

Let $f : \mathbb{R} \to X$ be continuous. Then we say that $f(\cdot)$ is almost automorphic if and only if for every real sequence (b_n) there exist a subsequence (a_n) of (b_n) and a map $g : \mathbb{R} \to X$ such that

$$\lim_{n\to\infty} f\big(t + a_n\big) = g(t) \ \text{ and } \ \lim_{n\to\infty} g\big(t - a_n\big) = f(t), \tag{1.1}$$

pointwise for $t \in \mathbb{R}$. If this is the case, then $f(\cdot)$ and $g(\cdot)$ are bounded but the limit function $g(\cdot)$ is not necessarily continuous on $\mathbb{R}$. If the convergence of limits appearing in (1.1) is uniform on compact subsets of $\mathbb{R}$, then we say that $f(\cdot)$ is compactly almost automorphic. By Bochner's criterion, any almost periodic function has to be compactly almost automorphic; the converse statement is not true, however ([13]). Let us also recall the well known result of P. Bender (the doctoral dissertation, Iowa State University, 1966) which states that an almost automorphic function $f : \mathbb{R} \to X$ is compactly almost automorphic if and only if $f(\cdot)$ is uniformly continuous.

The notion of Stepanov almost periodicity and the notion of Stepanov almost automorphy have been investigated in many mathematical papers by now; see [24]-[25] and references quoted therein for more

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details about the subject. Assume $1 \leq p < \infty$, $l > 0$ and $f,\ g \in L^p_{loc}(I : X)$, where $I = \mathbb{R}$ or $I = [0, \infty)$. Define the Stepanov metric by

$$D^p_{S_l}\big[f(\cdot), g(\cdot)\big] := \sup_{x \in I} \Bigg[\frac{1}{l} \int_x^{x+l} \big\|f(t) - g(t)\big\|^p \, dt\Bigg]^{1/p}.$$

Then there exists

$$D^p_W\big[f(\cdot), g(\cdot)\big] := \lim_{l \to \infty} D^p_{S_l}\big[f(\cdot), g(\cdot)\big] \tag{1.2}$$

in $[0, \infty]$. The distance appearing in (1.2) is called the Weyl distance of $f(\cdot)$ and $g(\cdot)$. A function $f \in L^p_{loc}(I : X)$ is said to be Stepanov p-bounded if and only if

$$\|f\|_{S^p} := \sup_{t \in I} \Bigg(\int_t^{t+1} \|f(s)\|^p \, ds\Bigg)^{1/p} < \infty.$$

By $L^p_S(I : X)$ we denote the vector space consisting of all Stepanov p-bounded functions; equipped with the above norm, $L^p_S(I : X)$ is a Banach space. We say that a function $f \in L^p_S(I : X)$ is Stepanov p-almost periodic if and only if the function $\hat{f} : I \to L^p([0, 1] : X)$, defined by $\hat{f}(t)(s) := f(t + s)$, $t \in I,\ s \in [0, 1]$ is almost periodic. A function $f \in L^p_{loc}(\mathbb{R} : X)$ is called Stepanov p-almost automorphic if and only if for each real sequence (a_n) there exists a subsequence (a_{n_k}) and a function $g \in L^p_{loc}(\mathbb{R} : X)$ such that

$$\lim_{k \to \infty} \int_t^{t+1} \Big\|f\big(a_{n_k} + s\big) - g(s)\Big\|^p \, ds = 0$$

and

$$\lim_{k \to \infty} \int_t^{t+1} \Big\|g\big(s - a_{n_k}\big) - f(s)\Big\|^p \, ds = 0$$

for each $t \in \mathbb{R}$. Any Stepanov p-almost automorphic function $f(\cdot)$ has to be S^p-bounded ($1 \leq p < \infty$).

In this paper, we reconsider the notion of Weyl p-almost automorphy introduced by S. Abbas [1] in 2012 and propose the following notions of Weyl p-almost automorphy: the Weyl p-almost automorphy of type 1, the Weyl p-almost automorphy of type 2 and the joint Weyl p-almost automorphy ($1 \leq p < \infty$). Furthermore, we introduce and analyze the multi-dimensional analogues of these concepts by using the definitions and results from the theory of Lebesgue spaces with variable exponents (the introduced classes of functions seem to be not considered elsewhere even in the constant coefficient case); in such a way, we continue our recent analysis of multi-dimensional almost periodicity and multi-dimensional almost automorphy carried out in our joint research studies [9]-[11] with A. Chávez, K. Khalil and M. Pinto. Several illustrative examples, open problems and applications to the abstract Volterra integro-differential equations are presented (for the Weyl and Besicovitch generalizations of the almost periodic functions and the almost automorphic functions, we refer the reader to [2, 3, 5, 6, 7, 11, 32, 19, 24, 25, 27] and [28, 29, 33, 34, 35, 37, 38, 39]). It should be also noted that we present some new results about the (equi-)Weyl almost periodic functions here; for example, in [19], we have emphasized that some relations presented in [2, Table 2, p. 56] are stated incorrectly as well as that there is no reasonable information which could tell us whether the class of Weyl-p-almost periodic functions in the sense of extended Kovanko's approach is contained in the class of Besicovitch p-almost periodic functions or not, as well as whether a Weyl-p-almost periodic function $f : \mathbb{R} \rightarrow \mathbb{C}$ has the mean value ($1 \leq p < \infty$). In this paper, we prove by a simple example that for each finite number $p \geq 1$ there exists a Weyl-p-almost periodic function $f : \mathbb{R} \rightarrow [0, \infty)$ satisfying that $f(\cdot)$ is Weyl-p-almost automorphic, neither Weyl-p-almost automorphic of type 1 nor jointly Weyl-p-almost automorphic, as well as that $f(\cdot)$ is not Besicovitch-p-almost periodic (Besicovitch p-bounded) and has no finite mean value (see [24] for the notion; from our recent research study [26] we know that there exists an infinitely differentiable function $f : \mathbb{R} \rightarrow [0, \infty)$ satisfying that for each finite number $p \geq 1$ the function $f(\cdot)$ is Weyl p-almost automorphic and not Besicovitch-p-almost periodic (Besicovitch p-bounded)). See Theorem 2.10 and Example 3.4 for more details (in Theorem 2.12, we analyze the Weyl almost automorphic properties of the Heaviside function; both results, Theorem 2.10 and Theorem 2.12, can be formulated as examples but we have decided to formulate them as theorems because of their indisputable theoretical novelties).

The organization and main ideas of this paper can be briefly summarized as follows. The main definitions and results about the Lebesgue spaces with variable exponent are collected in Subsection 1.1; Subsection 1.2 recalls the fundamental information about multi-dimensional Weyl almost periodic functions we will need later on. The core of this paper is Section 2, in which we introduce three new concepts of Weyl p-almost automorphy for vector-valued functions depending on one real variable. Here we reconsider and give some constructive criticism about the notion introduced by S. Abbas, providing also numerous important examples and relations between the notions of Weyl p-almost automorphy, Weyl p-almost automorphy of type 1, Weyl p-almost automorphy of type 2 and joint Weyl p-almost automorphy (it is worth noting that, in Subsection 2.1, we define the notion of Weyl p-almost automorphy of type 2 without using limit functions, which seems to be competely new and not considered elsewhere in the existing literature; this is motivated by the fact that the spaces of equi-Weyl-p-almost periodic functions are not complete with respect to the Weyl metric). We continue our analysis in Section 3, where we investigate multi-dimensional Weyl almost automorphic functions in Lebesgue spaces with variable exponent; in Subsection 3.1, we specifically analyze Weyl $p(\mathbf{u})$-$(\mathbb{F}, \mathrm{R}, \mathcal{B}, W)$-multi-almost automorphic functions of type 2 and jointly Weyl $p(\mathbf{u})$-$(\mathbb{F}, \mathrm{R}, \mathcal{B}, W)$-multi-almost automorphic functions, which are most important for applications. In Section 4, we apply our theoretical results in the qualitative analysis of solutions for various classes of the abstract Volterra integro-differential equations in Banach spaces; in Section 5, we present some conclusions, remarks and further perspectives for the investigations of Weyl and Besicovitch classes of almost automorphic functions. For better readability, we have repeated some known definitions and results which already exist in other references published so far.

We use the standard notation throughout the paper. We assume henceforth that $(X, \| \cdot \|)$, $(Y, \| \cdot \|_Y)$ and $(Z, \| \cdot \|_Z)$ are complex Banach spaces. By $L(X, Y)$ we denote the Banach algebra of all bounded linear operators from X into Y with $L(X, X)$ being denoted $L(X)$.

For given real number $s \in \mathbb{R}$, we define $\lfloor s \rfloor := \sup\{l \in \mathbb{Z} : l \leq s\}$ and $\lceil s \rceil := \inf\{l \in \mathbb{Z} : s \leq l\}$. The Euler Gamma function is denoted by $\Gamma(\cdot)$; we set $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$, $\zeta > 0$. The Weyl-Liouville fractional derivative $D_{t,+}^{\gamma} u(t)$ of order $\gamma \in (0, 1)$ is defined for those continuous functions $u : \mathbb{R} \rightarrow X$

such that $t \mapsto \int_{-\infty}^{t} g_{1-\gamma}(t-s)u(s)\,ds$, $t \in \mathbb{R}$ is a well-defined continuously differentiable mapping, by

$$D_{t,+}^{\gamma}u(t) := \frac{d}{dt}\int_{-\infty}^{t} g_{1-\gamma}(t-s)u(s)\,ds, \quad t \in \mathbb{R}.$$

For a mapping $a : (0,\infty) \times \mathbb{N} \to \mathbb{C}$, we say that $\lim_{(l,m)\to+\infty} a(l,m) = a \in \mathbb{C}$ if and only if for every $\epsilon > 0$ there exists a positive number $s \in \mathbb{N}$ such that, for every $l \in (s,\infty)$ and $m \in \mathbb{N}$ with $m \geq s$, we have $|a(l,m) - a| < \epsilon$; as usually, we write $a_{l,m}$ for $a(l,m)$.

1.1. *Lebesgue spaces with variable exponents* $L^{p(x)}$

Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be a nonempty Lebesgue measurable subset and let $M(\Omega : X)$ denote the collection of all measurable functions $f : \Omega \to X$; $M(\Omega) := M(\Omega : \mathbb{R})$. Furthermore, $\mathcal{P}(\Omega)$ denotes the vector space of all Lebesgue measurable functions $p : \Omega \to [1,\infty]$. By $m(\cdot)$ we denote the Lebesgue measure on $\mathbb{R}^n$.

For any $p \in \mathcal{P}(\Omega)$ and $f \in M(\Omega : X)$, we define

$$\varphi_{p(x)}(t) := \begin{cases} t^{p(x)}, & t \geq 0,\ \ 1 \leq p(x) < \infty, \\ 0, & 0 \leq t \leq 1,\ \ p(x) = \infty, \\ \infty, & t > 1,\ \ p(x) = \infty \end{cases}$$

and

$$\rho(f) := \int_{\Omega} \varphi_{p(x)}(\|f(x)\|)\,dx.$$

We define the Lebesgue space $L^{p(x)}(\Omega : X)$ with variable exponent as follows,

$$L^{p(x)}(\Omega : X) := \Big\{ f \in M(\Omega : X) : \lim_{\lambda\to 0+} \rho(\lambda f) = 0 \Big\}.$$

Equivalently,

$$L^{p(x)}(\Omega : X) = \Big\{ f \in M(\Omega : X) : \text{ there exists } \lambda > 0 \text{ such that } \rho(\lambda f) < \infty \Big\};$$

see e.g., [16, p. 73]. For every $u \in L^{p(x)}(\Omega : X)$, we introduce the Luxemburg norm of $u(\cdot)$ by

$$\|u\|_{p(x)} := \|u\|_{L^{p(x)}(\Omega:X)} := \inf\Big\{ \lambda > 0 : \rho(f/\lambda) \leq 1 \Big\}.$$

Equipped with the above norm, $L^{p(x)}(\Omega : X)$ is a Banach space (see e.g. [16, Theorem 3.2.7] for the scalar-valued case), coinciding with the usual Lebesgue space $L^{p}(\Omega : X)$ in the case that $p(x) = p \geq 1$ is a constant function. Further on, for any $p \in M(\Omega)$, we define

$$p^{-} := \text{essinf}_{x\in\Omega} p(x) \quad \text{and} \quad p^{+} := \text{esssup}_{x\in\Omega} p(x).$$

Set

$$D_{+}(\Omega) := \big\{ p \in M(\Omega) : 1 \leq p^{-} \leq p(x) \leq p^{+} < \infty \text{ for a.e. } x \in \Omega \big\}.$$

If $p \in D_+(\Omega)$, then we know that

$$L^{p(x)}(\Omega : X) = \Big\{ f \in M(\Omega : X);\ \text{ for all } \lambda > 0 \text{ we have } \rho(\lambda f) < \infty \Big\}.$$

We will use the following lemma (cf. [16] for the scalar-valued case):

Lemma 1.1. *(i) (The Hölder inequality) Let $p,\ q,\ r \in \mathcal{P}(\Omega)$ such that*

$$\frac{1}{q(x)} = \frac{1}{p(x)} + \frac{1}{r(x)}, \quad x \in \Omega.$$

Then, for every $u \in L^{p(x)}(\Omega : X)$ and $v \in L^{r(x)}(\Omega)$, we have $uv \in L^{q(x)}(\Omega : X)$ and

$$\|uv\|_{q(x)} \leq 2\|u\|_{p(x)}\|v\|_{r(x)}.$$

(ii) Let Ω be of a finite Lebesgue's measure and let $p,\ q \in \mathcal{P}(\Omega)$ such $q \leq p$ a.e. on Ω. Then $L^{p(x)}(\Omega : X)$ is continuously embedded in $L^{q(x)}(\Omega : X)$ with the constant of embedding less or equal to $2(1 + m(\Omega))$.

(iii) Let $f \in L^{p(x)}(\Omega : X)$, $g \in M(\Omega : X)$ and $0 \leq \|g\| \leq \|f\|$ a.e. on Ω. Then $g \in L^{p(x)}(\Omega : X)$ and $\|g\|_{p(x)} \leq \|f\|_{p(x)}$.

We also need the following simple lemma, whose proof can be omitted:

Lemma 1.2. *Suppose that $f \in L^{p(x)}(\Omega : X)$ and $A \in L(X, Y)$. Then $Af \in L^{p(x)}(\Omega : Y)$ and $\|Af\|_{L^{p(x)}(\Omega:Y)} \leq \|A\| \cdot \|f\|_{L^{p(x)}(\Omega:X)}$.*

For further information concerning the Lebesgue spaces with variable exponents $L^{p(x)}$, we refer the reader to the research monograph [16] by L. Diening, P. Harjulehto, P. Hästüso and M. Ruzicka, as well as to [17] and [36]. The basic source of information about generalized almost periodicity and generalized almost automorphy in the Lebesgue spaces with variable exponents can be obtained by consulting [10, 32, 14, 15, 19, 25, 29, 30, 31] and references quoted therein.

1.2. *Weyl almost periodic functions in Lebesgue spaces with variable exponent*

In this subsection, we assume first that the following condition holds:

(A): $I = \mathbb{R}$ or $I = [0, \infty)$, $\phi : [0, \infty) \to [0, \infty)$, $p \in \mathcal{P}(I)$ and $F : (0, \infty) \times I \to (0, \infty)$.

Besides many other notions, the notion of an (equi-)Weyl-(p, ϕ, F)-almost periodic function has been introduced in [29] as follows (see [24] for the case that $p(x) \equiv p \in [1, \infty)$, $\phi(x) \equiv x$ and $F(l, t) \equiv l^{(-1)/p}$, when we obtain the usually considered concept of (equi-)Weyl-p-almost periodicity which will be used henceforth):

Definition 1.3. Suppose that condition (A) holds, $f : I \to X$ and $\phi(\|f(\cdot + \tau) - f(\cdot)\|) \in L^{p(x)}(K)$ for any $\tau \in I$ and any compact subset K of I.

(i) It is said that the function $f(\cdot)$ is equi-Weyl-(p,ϕ,F)-almost periodic, $f \in e-W_{ap}^{(p,\phi,F)}(I:X)$ for short, if and only if for each $\epsilon>0$ we can find two real numbers $l>0$ and $L>0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\sup_{t\in I}\Biggl[F(l,t)\Bigl[\phi\bigl(\bigl\|f(\cdot+\tau)-f(\cdot)\bigr\|\bigr)_{L^{p(\cdot)}[t,t+l]}\Bigr]\Biggr] \leq \epsilon.$$

(ii) It is said that the function $f(\cdot)$ is Weyl-(p,ϕ,F)-almost periodic, $f \in W_{ap}^{(p,\phi,F)}(I:X)$ for short, if and only if for each $\epsilon>0$ we can find a real number $L>0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\limsup_{l\to\infty}\sup_{t\in I}\Biggl[F(l,t)\Bigl[\phi\bigl(\bigl\|f(\cdot+\tau)-f(\cdot)\bigr\|\bigr)_{L^{p(\cdot)}[t,t+l]}\Bigr]\Biggr] \leq \epsilon.$$

In recent joint research article with V. Fedorov [19], we have analyzed the various notions of multidimensional Weyl almost periodicity in Lebesgue spaces with variable exponents $L^{p(x)}$. For our further work, it will be necessary to remind the readers of the following notion: Assume that $\mathcal{B}$ is a non-empty collection of certain subsets of X such that for each $x \in X$ there exists $B \in \mathcal{B}$ with $x \in B$. Suppose that the following condition holds:

(WM1): $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is a Lebesgue measurable set such that $m(\Omega)>0$, $p \in \mathcal{P}(\Lambda)$, $\Lambda'+\Lambda+l\Omega \subseteq \Lambda$, $\Lambda+l\Omega \subseteq \Lambda$ for all $l>0$, $\phi:[0,\infty)\to[0,\infty)$ and $\mathbb{F}:(0,\infty)\times\Lambda \to (0,\infty)$.

We need the following notion:

Definition 1.4. (i) By $e-W_{\Omega,\Lambda',\mathcal{B}}^{(p(\mathbf{u}),\phi,\mathbb{F})}(\Lambda\times X:Y)$ we denote the set consisting of all functions $F:\Lambda\times X\to Y$ such that, for every $\epsilon>0$ and $B\in\mathcal{B}$, there exist two finite real numbers $l>0$ and $L>0$ such that for each $\mathbf{t}_0\in\Lambda'$ there exists $\tau\in B(\mathbf{t}_0,L)\cap\Lambda'$ such that

$$\sup_{x\in B}\sup_{\mathbf{t}\in\Lambda}\mathbb{F}(l,\mathbf{t})\phi\Bigl(\bigl\|F(\tau+\mathbf{u};x)-F(\mathbf{u};x)\bigr\|_Y\Bigr)_{L^{p(\mathbf{u})}(\mathbf{t}+l\Omega)} < \epsilon.$$

(ii) By $W_{\Omega,\Lambda',\mathcal{B}}^{(p(\mathbf{u}),\phi,\mathbb{F})}(\Lambda\times X:Y)$ we denote the set consisting of all functions $F:\Lambda\times X\to Y$ such that, for every $\epsilon>0$ and $B\in\mathcal{B}$, there exists a finite real number $L>0$ such that for each $\mathbf{t}_0\in\Lambda'$ there exists $\tau\in B(\mathbf{t}_0,L)\cap\Lambda'$ such that

$$\limsup_{l\to+\infty}\sup_{x\in B}\sup_{\mathbf{t}\in\Lambda}\mathbb{F}(l,\mathbf{t})\phi\Bigl(\bigl\|F(\tau+\mathbf{u};x)-F(\mathbf{u};x)\bigr\|_Y\Bigr)_{L^{p(\mathbf{u})}(\mathbf{t}+l\Omega)} < \epsilon.$$

If $F:\Lambda\to Y$, then we omit the term "$\mathcal{B}$" from the notation; for example, we write $e-W_{\Omega,\Lambda'}^{(p(\mathbf{u}),\phi,\mathbb{F})}(\Lambda:Y)$ in place of $e-W_{\Omega,\Lambda',\mathcal{B}}^{(p(\mathbf{u}),\phi,\mathbb{F})}(\Lambda\times X:Y)$. Basically, this will be our terminological agreement in many similar situations henceforth. If we write $p\geq 1$ only, then we tacitly assume that the exponent p has a finite value.

2. Weyl almost automorphic functions of one real variable

The concept of Weyl almost automorphy was introduced by S. Abbas [1, Definition 0.4] in 2012 (see also [24, Definition 3.1.2]):

Definition 2.1. Let $p \geq 1$. Then we say that a function $f \in L^p_{loc}(\mathbb{R} : X)$ is Weyl p-almost automorphic if and only if for every real sequence (s_n), there exist a subsequence (s_{n_k}) and a function $f^* \in L^p_{loc}(\mathbb{R} : X)$ such that

$$\lim_{k\to\infty} \lim_{l\to+\infty} \frac{1}{2l} \int_{-l}^{l} \Big\| f\big(t + s_{n_k} + x\big) - f^*(t + x)\Big\|^p \, dx = 0 \tag{2.1}$$

and

$$\lim_{k\to\infty} \lim_{l\to+\infty} \frac{1}{2l} \int_{-l}^{l} \Big\| f^*\big(t - s_{n_k} + x\big) - f(t + x)\Big\|^p \, dx = 0 \tag{2.2}$$

for each $t \in \mathbb{R}$. The set of all such functions is denoted by $W^pAA(\mathbb{R} : X)$.

Here, by saying that $\lim_{k\to\infty} \lim_{l\to+\infty} a_{k,l} = a$ we mean that for each $k \in \mathbb{N}$ the limit $\lim_{l\to+\infty} a_{k,l}$ exists as well as that for each $\epsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for each $k \in \mathbb{N}_0$ with $k \geq k_0$ we have that the limit $\lim_{l\to+\infty} a_{k,l}$ does not exceed ϵ; the notion of double limit $\lim_{l\to\infty} \lim_{k\to+\infty} a_{k,l} = a$ is understood similarly. See the research article [22] by E. Habil for more details about the subject.

The investigation of Weyl p-almost automorphy, introduced by our friend and colleague S. Abbas, has strongly motivated us to carry out many research studies of this intriguing notion by now. Regrettably, this is our first research article in which we perceive some unclear places in Definition 2.1 and the main result of [1]:

Remark 2.2. 1. In [1, Definition 0.4], it is assumed but not explicitly stated that the integration is taken with respect to the variable x because the limits as $l \to +\infty$ in this definition must tend to zero as $k \to +\infty$, pointwise for every fixed $t \in \mathbb{R}$; hence, t cannot be the variable under which the integration is taken twice. Also, there are two extra left brackets in the integrals mentioned and the considered function has range in X, which are only small typographical errors.

2. The class of Weyl pseudo almost automorphic functions has been introduced and analyzed in [1], as well. But, the above observation becomes crucial in this point because the proof of deduced composition theorem for Weyl pseudo almost automorphic functions is based on the wrong arguments. In this proof, the integration is taken over the variable t and the author has operated with $\sup_{-\infty<x<+\infty} \cdot$ in the expressions appeared on the right hand side of page 3 (see [1, l.-8, -9]), which is completely meaningless because the integration (if we want to take the pointwise limits for $t \in \mathbb{R}$ as $k \to +\infty$) must be taken with the variable x.

3. Suppose that the function $f(\cdot)$ is Stepanov p-bounded as well as that, for a sequence (s_n) given in advance, the limit function $f^*(\cdot)$ from Definition 2.1 is also Stepanov p-bounded. Then for each $k \in \mathbb{N}$ the limits

$$\lim_{l\to+\infty} \frac{1}{2l} \int_{-l}^{l} \Big\| f\big(t + s_{n_k} + x\big) - f^*(t + x)\Big\|^p \, dx$$

and

$$\lim_{l\to+\infty}\frac{1}{2l}\int_{-l}^{l}\Big\|f^{*}\big(t-s_{n_k}+x\big)-f(t+x)\Big\|^{p}\,dx$$

from Definition 2.1 are equal, which cannot be satisfied in any reasonable definition of Weyl p-almost automorphy. Speaking-matter-of-factly, we have

$$\lim_{l\to+\infty}\frac{1}{2l}\int_{-l}^{l}\Big\|f\big(t+s_{n_k}+x\big)-f^{*}(t+x)\Big\|^{p}\,dx$$
$$=\lim_{l\to+\infty}\frac{1}{2l}\int_{t-l+s_{n_k}}^{t+l+s_{n_k}}\Big\|f\big(x\big)-f^{*}\big(x-s_{n_k}\big)\Big\|^{p}\,dx$$
$$=\lim_{l\to+\infty}\frac{1}{2l}\int_{t-l}^{t+l}\Big\|f\big(x\big)-f^{*}\big(x-s_{n_k}\big)\Big\|^{p}\,dx \tag{2.3}$$
$$=\lim_{l\to+\infty}\frac{1}{2l}\int_{-l}^{l}\Big\|f^{*}\big(t-s_{n_k}+x\big)-f(t+x)\Big\|^{p}\,dx,$$

where (2.3) follows from a simple computation involving the Stepanov p-boundedness of functions $f(\cdot)$ and $f^{*}(\cdot)$.

The genesis of this paper is strongly motivated by the above observations and the following question:

Question 2.3. *Let* $1\leq p<\infty$. *If we accept the notion introduced in* Definition 2.1*, is it true that a (compactly, Stepanov* p*-) almost automorphic function is Weyl* p*-almost automorphic?*

In any expected notion of Weyl p-almost automorphy, this must be satisfied. But, unfortunately, there is no reasonable argumentation which could tell us straightforwardly that the answer to Question (2.3) is affirmative. Therefore, we are getting into some unexpected troubles; how to proceed further? Our first idea is to replace the limits in the equations (2.1)-(2.2):

Definition 2.4. Let $p\geq 1$. Then we say that a function $f\in L^{p}_{loc}(\mathbb{R}:X)$ is Weyl p-almost automorphic of type 1 if and only if for every real sequence (s_n), there exist a subsequence (s_{n_k}) and a function $f^{*}\in L^{p}_{loc}(\mathbb{R}:X)$ such that

$$\lim_{l\to+\infty}\lim_{k\to\infty}\frac{1}{2l}\int_{-l}^{l}\Big\|f\big(t+s_{n_k}+x\big)-f^{*}(t+x)\Big\|^{p}\,dx=0$$

and

$$\lim_{l\to+\infty}\lim_{k\to\infty}\frac{1}{2l}\int_{-l}^{l}\Big\|f^{*}\big(t-s_{n_k}+x\big)-f(t+x)\Big\|^{p}\,dx=0$$

for each $t\in\mathbb{R}$. The set of all such functions is denoted by $W^{p}AA_{1}(\mathbb{R}:X)$.

Accepting this definition, it is very simple to show that a Stepanov p-almost automorphic function is Weyl p-almost automorphic of type 1 because for every fixed real numbers t and l we have

$$\lim_{k\to\infty} \frac{1}{2l} \int_{-l}^{l} \Big\| f\big(t+s_{n_k}+x\big) - f^*(t+x)\Big\|^p \, dx = 0$$

and

$$\lim_{k\to\infty} \frac{1}{2l} \int_{-l}^{l} \Big\| f^*\big(t-s_{n_k}+x\big) - f(t+x)\Big\|^p \, dx = 0. \tag{2.4}$$

In actual fact, we have

$$\frac{1}{2l} \int_{-l}^{l} \Big\| f\big(t+s_{n_k}+x\big) - f^*(t+x)\Big\|^p \, dx \le \frac{1}{2l} \sum_{j=0}^{\lfloor 2l \rfloor} \int_{t-l+j}^{t-l+j+1} \Big\| f\big(s_{n_k}+x\big) - f^*(x)\Big\|^p \, dx,$$

which simply implies by definition of Stepanov p-almost automorphy that for any real number $\epsilon > 0$ we can always find a positive integer $k_0 \in \mathbb{N}$ such that

$$\frac{1}{2l} \int_{-l}^{l} \Big\| f\big(t+s_{n_k}+x\big) - f^*(t+x)\Big\|^p \, dx \le \frac{1+\lfloor 2l \rfloor}{2l} \frac{\epsilon}{2} \le \epsilon, \quad k \ge k_0;$$

we can similarly prove the limit equation (2.4). On the other hand, it can be easily shown by using the Bochner criterion [24] that any Stepanov p-almost periodic function $f : \mathbb{R} \to X$ is Weyl p-almost automorphic, Weyl p-almost automorphic of type 1, as well as jointly Weyl p-almost automorphic in the sense of the following definition (with the limit function $f^* \equiv f$):

Definition 2.5. Let $p \ge 1$. Then we say that a function $f \in L^p_{loc}(\mathbb{R} : X)$ is jointly Weyl p-almost automorphic if and only if for every real sequence (s_n), there exist a subsequence (s_{n_k}) and a function $f^* \in L^p_{loc}(\mathbb{R} : X)$ such that

$$\lim_{(l,k)\to\infty} \frac{1}{2l} \int_{-l}^{l} \Big\| f\big(t+s_{n_k}+x\big) - f^*(t+x)\Big\|^p \, dx = 0 \tag{2.5}$$

and

$$\lim_{(l,k)\to\infty} \frac{1}{2l} \int_{-l}^{l} \Big\| f^*\big(t-s_{n_k}+x\big) - f(t+x)\Big\|^p \, dx = 0 \tag{2.6}$$

for each $t \in \mathbb{R}$. The set of all such functions is denoted by $W^pAA_j(\mathbb{R} : X)$.

Remark 2.6. We feel it is our duty to say that the limits in the previous two definitions are nice but a little bit unnatural. It will be difficult to extract a subsequence that verifies the above properties and it would be very interesting to find other more practical criteria characterizing these concepts.

It can be simply verified that the joint Weyl p-almost automorphy of function $f \in L^p_{loc}(\mathbb{R} : X)$ implies its Weyl p-almost automorphy provided that for each $k \in \mathbb{N}$ the both limits in the equations (2.1)-(2.2) exist as $l \to +\infty$; a similar comment holds for the notion of Weyl p-almost automorphy of type 1 (see also [22, Theorem 2.13]). Before proceeding any further, we would like to note that the joint Weyl p-almost automorphy of a function $f \in L^p_{loc}(\mathbb{R} : X)$ does not imply its Stepanov p-almost automorphy:

Example 2.7. Let $p \geq 1$. Then it is well known that the function $f(t) := \chi_{[0,1/2]}(t)$, $t \in \mathbb{R}$ is not Stepanov p-almost automorphic as well as that this function is equi-Weyl-p-almost periodic ([24]). It is also jointly Weyl p-almost automorphic with the limit function $f^* \equiv 0$, as easily shown (futhermore, this function is jointly Weyl p-almost automorphic in the sense of Definition 3.1(iii) below for any function $\mathbb{F}(l)$ satisfying $\lim_{l\to+\infty} \mathbb{F}(l) = 0$, with the meaning clear). The use of zero limit function shows that $L^p(\mathbb{R} : X) \subseteq W^pAA_j(\mathbb{R} : X)$ and $L^p(\mathbb{R} : X) \subseteq AAW^{\mathbb{F},p,j}_{\mathrm{R}}(\mathbb{R} : X)$, provided that $\lim_{l\to+\infty} \mathbb{F}(l) = 0$ and R denotes the collection of all real sequences; see Definition 3.1(iii) for the notion. The above conclusions remain valid for the Weyl p-almost automorphy and the Weyl p-almost automorphy of type 1, with the same choice of the limit function.

From application point of view, the main drawback of the notion of Weyl p-almost automorphy (Weyl p-almost automorphy of type 1) presents the fact that we cannot so simply state satisfactory results about the invariance of Weyl p-almost automorphy (Weyl p-almost automorphy of type 1) under the actions of convolution products; cf. [24] and [25] for more details about the subject. Concerning the joint Weyl p-almost automorphy, the best we can do in the present situation is to state the following result with $p = 1$; the proof is very similar to that of [27, Proposition 7] and therefore omitted:

Proposition 2.8. *Suppose that $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family satisfying that*

$$\int_0^\infty (1+t)\|R(t)\|_{L(X,Y)}\,dt < \infty.$$

Let $f \in W^1AA_j(\mathbb{R} : X)$, and let $f(\cdot)$ be essentially bounded. Then the function $F(\cdot)$, given by

$$F(t) := \int_{-\infty}^{t} R(t-s)f(s)\,ds, \quad t \in \mathbb{R}, \tag{2.7}$$

is bounded and belongs to the class $W^1AA_j(\mathbb{R} : Y)$.

The situation in which the exponent p is strictly greater that one is a bit complicated. Concerning this problematic, we would like to raise the following issue:

Question 2.9. *Can we deduce an analogue of [18, Theorem 1] for joint Weyl p-almost automorphy?*

Further on, it should be noted that the Weyl p-almost automorphy does not imply Weyl p-almost automorphy of type 1 or joint Weyl p-almost automorphy:

Theorem 2.10. *Suppose that $\sigma \in (0,1)$, $p \in [1,\infty)$ and $(1-\sigma)p < 1$. Define $f(x) := |x|^\sigma$, $x \in \mathbb{R}$. Then the function $f(\cdot)$ is Weyl p-almost automorphic, not Weyl p-almost automorphic of type 1 nor joint Weyl p-almost automorphic; furthermore, the function $f(\cdot)$ is Weyl p-almost periodic, Besicovitch p-unbounded and has no mean value.*

Proof. Suppose first that $a > 1-(1-\sigma)p$. It is clear that, for every real numbers ω and t, we have

$$\lim_{l\to+\infty} l^{-a}\int_{-l}^{l}\Bigl|\bigl|t+x+\omega\bigr|^{\sigma}-|t+x|^{\sigma}\Bigr|^{p}\,dx=0,$$

which implies that the function $f(\cdot)$ is Weyl p-almost automorphic with the limit function $f^*\equiv f$ (moreover, $f\in AAW_{\mathrm{R}}^{\mathbb{F},p}(\mathbb{R}:\mathbb{C})$ in the sense of Definition 3.1(i) with $\mathbb{F}(l)\equiv l^{-a}$ and R being the collection of all real sequences). In order to see that, we can apply the Lagrange mean value theorem and the following computation ($l>\max(|t|,|t\pm\omega|)$):

$$\begin{aligned}
&l^{-a}\int_{t-l}^{t+l}\Bigl|\bigl|x+\omega\bigr|^{\sigma}-|x|^{\sigma}\Bigr|^{p}\,dx\\
&\quad\le l^{-a}\sigma^{p}|\omega|^{p}\int_{t-l}^{t+l}\max_{v\in[|x|,|x+\omega|]\cup[|x+\omega|,|x|]}v^{(\sigma-1)p}\,dx\\
&\quad\le l^{-a}\sigma^{p}|\omega|^{p}\int_{t-l}^{t+l}\Bigl[|x|^{(\sigma-1)p}+|x+\omega|^{(\sigma-1)p}\Bigr]\,dx\\
&\quad= l^{-a}\sigma^{p}|\omega|^{p}\Bigl[(t+l)^{1-(1-\sigma)p}+(l-t)^{1-(1-\sigma)p}+(l-t)^{1-(1-\sigma)p}\\
&\quad+(t+l+\omega)^{1-(1-\sigma)p}+(l-t-\omega)^{1-(1-\sigma)p}\Bigr]\\
&\quad\le l^{-a}\sigma^{p}|\omega|^{p}\Bigl[4t^{1-(1-\sigma)p}+4l^{1-(1-\sigma)p}+2\omega^{1-(1-\sigma)p}\Bigr]\to 0,\quad l\to+\infty.
\end{aligned}$$

To see that $f(\cdot)$ is not Weyl p-almost automorphic of type 1, it suffices to show that, for every $l>0$, for every $f^*\in L^p_{loc}(\mathbb{R}:\mathbb{C})$ and for every strictly increasing real sequence (s_k) tending to plus infinity, we have

$$\lim_{k\to+\infty}\int_{-l}^{l}\Bigl|\bigl|x+s_k\bigr|^{\sigma}-f^*(x)\Bigr|^{p}\,dx=+\infty.$$

This follows from the next computation:

$$\begin{aligned}
&\int_{-l}^{l}\Bigl|\bigl|x+s_k\bigr|^{\sigma}-f^*(x)\Bigr|^{p}\,dx\ge\int_{0}^{l}\Bigl|\bigl(x+s_k\bigr)^{\sigma}-f^*(x)\Bigr|^{p}\,dx\\
&\ge\int_{0}^{l}2^{1-p}\bigl(x+s_k\bigr)^{\sigma p}\,dx-\int_{0}^{l}\bigl|f^*(x)\bigr|^{p}\,dx\\
&\ge 2^{1-p}ls_k^{\sigma p}-\int_{0}^{l}\bigl|f^*(x)\bigr|^{p}\,dx\to+\infty,\quad k\to+\infty.
\end{aligned}$$

We can similarly prove that $f(\cdot)$ is not jointly Weyl p-almost automorphic.

Concerning the Weyl almost periodic properties of the function $f(\cdot)$, we first observe that this function is not equi-Weyl-(p,x,F)-almost periodic for any function $F(l,t)$ which does not depend on t because, for every real numbers $l > 0$ and $t \in \mathbb{R}$, we have

$$\lim_{\tau \to +\infty} \int_t^{t+l} \Big| |x+\tau|^{\sigma} - |x|^{\sigma} \Big| \, dx = +\infty.$$

On the other hand, if $a > (1-(1-\sigma)p)/p$, then the function $f(\cdot)$ is Weyl-(p, x, F)-almost periodic with $F(l,t) \equiv l^{-a}$. Towards this end, we will prove the following estimate:

$$\int_t^{t+l} \Big| |x+\tau|^{\sigma} - |x|^{\sigma} \Big|^p \, dx$$
$$\leq \sigma^p |\tau|^p \frac{l^{1-(1-\sigma)p}}{1-(1-\sigma)p} \Big[1 + 2^{1-(\sigma-1)p} \Big] + |\tau|^{\sigma p+1} \cdot \big(2^{\sigma} + 1 \big)^p, \tag{2.8}$$

provided $t,\ \tau \in \mathbb{R}$ and $l > |\tau|$. This estimate is clearly satisfied for $\tau = 0$ and, since the right hand side of estimate does not depend on $t \in \mathbb{R}$, it suffices to verify its validity for $\tau > 0$ (we can apply the substitution $x \mapsto x + \tau$). Let it be the case; then we recognize the following subcases:

1. $t \leq -\tau$ and $t + l \leq 0$. Then we have two possibilities:

1.1. $-\tau \leq t + l$. Then we have $0 \leq -(t+\tau) \leq l$ and

$$\int_t^{t+l} \Big| |x+\tau|^{\sigma} - |x|^{\sigma} \Big|^p \, dx \leq \int_t^{0} \Big| |x+\tau|^{\sigma} - |x|^{\sigma} \Big|^p \, dx$$
$$= \int_t^{-\tau} \Big| |x+\tau|^{\sigma} - |x|^{\sigma} \Big|^p \, dx + \int_{-\tau}^{0} \Big| |x+\tau|^{\sigma} - |x|^{\sigma} \Big|^p \, dx$$
$$\leq \int_t^{-\tau} \Big| |x+\tau|^{\sigma} - |x|^{\sigma} \Big|^p \, dx + |\tau|^{\sigma p+1} \cdot \big(2^{\sigma} + 1 \big)^p.$$

Applying the Lagrange mean value theorem, we can continue the computation as follows:

$$\leq \sigma^p |\tau|^p \int_t^{-\tau} \max_{v \in (-x-\tau, -x)} v^{p(\sigma-1)} \, dx + |\tau|^{\sigma p+1} \cdot \big(2^{\sigma} + 1 \big)^p$$
$$= \sigma^p |\tau|^p \int_t^{-\tau} (-x-\tau)^{p(\sigma-1)} \, dx + |\tau|^{\sigma p+1} \cdot \big(2^{\sigma} + 1 \big)^p$$
$$= \sigma^p |\tau|^p \frac{[-(t+\tau)]^{1-(1-\sigma)p}}{1-(1-\sigma)p} + |\tau|^{\sigma p+1} \cdot \big(2^{\sigma} + 1 \big)^p$$
$$\leq \sigma^p |\tau|^p \frac{l^{1-(1-\sigma)p}}{1-(1-\sigma)p} + |\tau|^{\sigma p+1} \cdot \big(2^{\sigma} + 1 \big)^p.$$

1.2. $t + l \leq -\tau$. Then we have $0 \leq -(t+\tau)$, $0 \leq -(t+\tau+l)$ and we can apply the Lagrange mean value theorem in order to see that:

$$\int_t^{t+l} \Big| |x+\tau|^{\sigma} - |x|^{\sigma} \Big|^p \, dx \leq \sigma^p |\tau|^p \int_t^{t+l} \max_{v \in (-x-\tau, -x)} v^{p(\sigma-1)} \, dx$$

$$= \sigma^p |\tau|^p \int\limits_t^{t+l} (-x-\tau)^{(\sigma-1)p}\, dx = \sigma^p |\tau|^p \frac{(-t-\tau)^{1-(1-\sigma)p} - (-t-\tau-l)^{1-(1-\sigma)p}}{1-(1-\sigma)p}$$

$$\le \sigma^p |\tau|^p \frac{l^{1-(1-\sigma)p}}{1-(1-\sigma)p}.$$

2. $t \le -\tau$ and $t+l > 0$. Then we have $l \ge |t| \ge |\tau|$ and arguing as in case 1.1, we get:

$$\int\limits_t^{t+l} \Big| \big|x+\tau\big|^\sigma - |x|^\sigma \Big|^p \, dx$$

$$\le \int\limits_t^{-\tau} \Big| \big|x+\tau\big|^\sigma - |x|^\sigma \Big|^p \, dx + \int\limits_{-\tau}^{0} \Big| \big|x+\tau\big|^\sigma - |x|^\sigma \Big|^p \, dx + \int\limits_0^{t+l} \Big| \big|x+\tau\big|^\sigma - |x|^\sigma \Big|^p \, dx$$

$$\le \sigma^p |\tau|^p \frac{l^{1-(1-\sigma)p}}{1-(1-\sigma)p} + |\tau|^{\sigma p+1} \cdot \big(2^\sigma + 1\big)^p + \int\limits_0^{t+l} \Big[\big(x+\tau\big)^\sigma - x^\sigma \Big]^p \, dx$$

$$\le \sigma^p |\tau|^p \frac{l^{1-(1-\sigma)p}}{1-(1-\sigma)p} + |\tau|^{\sigma p+1} \cdot \big(2^\sigma + 1\big)^p + \sigma^p |\tau|^p \int\limits_0^{t+l} x^{(\sigma-1)p}\, dx$$

$$\le \sigma^p |\tau|^p \frac{l^{1-(1-\sigma)p}}{1-(1-\sigma)p} + |\tau|^{\sigma p+1} \cdot \big(2^\sigma + 1\big)^p + \sigma^p |\tau|^p \frac{(t+l)^{1-(1-\sigma)p}}{1-(1-\sigma)p}$$

$$\le \sigma^p |\tau|^p \frac{l^{1-(1-\sigma)p}}{1-(1-\sigma)p} + |\tau|^{\sigma p+1} \cdot \big(2^\sigma + 1\big)^p + \sigma^p |\tau|^p \frac{(2l)^{1-(1-\sigma)p}}{1-(1-\sigma)p}.$$

3. $t > -\tau$ and $t+l > 0$ (case $t > -\tau$ and $t+l \le 0$ cannot happen because it implies $-\tau < t < t+l \le 0$, which contradicts our assumption $l > |\tau|$). We consider the following two subcases of this case:

3.1. $t \ge 0$. Then the situation is clear since

$$\int\limits_t^{t+l} \Big| \big|x+\tau\big|^\sigma - |x|^\sigma \Big|^p \, dx = \int\limits_t^{t+l} \Big[\big(x+\tau\big)^\sigma - x^\sigma \Big]^p \, dx$$

$$\le \sigma^p |\tau|^p \int\limits_t^{t+l} x^{(\sigma-1)p}\, dx = \sigma^p |\tau|^p \frac{(t+l)^{1-(1-\sigma)p} - t^{1-(1-\sigma)p}}{1-(1-\sigma)p} \le \sigma^p |\tau|^p \frac{l^{1-(1-\sigma)p}}{1-(1-\sigma)p}.$$

3.2. $-\tau < t < 0$. Then $l > |t|$ and we have

$$\int\limits_t^{t+l} \Big| \big|x+\tau\big|^\sigma - |x|^\sigma \Big|^p \, dx$$

$$\le \int\limits_{-\tau}^{0} \Big| \big|x+\tau\big|^\sigma - |x|^\sigma \Big|^p \, dx + \int\limits_0^{2l} \Big[\big(x+\tau\big)^\sigma - x^\sigma \Big]^p \, dx$$

$$\le |\tau|^{\sigma p+1} \cdot \big(2^\sigma + 1\big)^p + \sigma^p |\tau|^p \frac{(2l)^{1-(1-\sigma)p}}{1-(1-\sigma)p}.$$

Therefore, the estimate (2.8) is proved. Fix now $\tau \in \mathbb{R}$ and $l > |\tau|$. The estimate (2.8) implies

$$\sup_{t\in\mathbb{R}} l^{-a}\left[\int_t^{t+l}\Big||x+\tau|^{\sigma}-|x|^{\sigma}\Big|^p\,dx\right]^{1/p}$$
$$\leq l^{-a}\left[\sigma|\tau|\frac{l^{(1-(1-\sigma)p)/p}}{(1-(1-\sigma)p)^{1/p}}\Big[1+2^{1-(\sigma-1)p}\Big]^{1/p}+|\tau|^{(\sigma p+1)/p}\cdot\big(2^{\sigma}+1\big)\right]. \tag{2.9}$$

It is clear that (2.9) implies the required conclusion, because for any $\epsilon > 0$ in the corresponding definition of Weyl-(p, x, F)-almost periodicity we can take $L = 1$ and after that, for any $\tau \in I'$ we can take

$$l \geq \max\left(\left(\epsilon|\tau|^{-\frac{\sigma p+1}{p}}\right)^{(-1)/a}, \left(\epsilon|\tau|^{-1}\right)^{\frac{p}{1-(1-\sigma)p-ap}}\right).$$

In particular, $f(\cdot)$ is Weyl-p-almost periodic; further on, this function is not Besicovitch p-bounded since

$$\lim_{l\to+\infty}\frac{1}{l}\int_0^l x^{\sigma p}\,dx = \lim_{l\to+\infty}\frac{1}{l}\frac{l^{\sigma p+1}}{\sigma p+1} = +\infty.$$

Therefore, the function $f(\cdot)$ is not Besicovitch-p-almost periodic (Besicovitch-Doss-p-almost periodic, equivalently, see [24, Definition 2.13.1]) and the function $f(\cdot)$ has no finite mean value since

$$\lim_{l\to+\infty}\frac{1}{l}\int_0^l x^{\sigma}\,dx = \lim_{l\to+\infty}\frac{1}{l}\frac{l^{\sigma+1}}{\sigma+1} = +\infty.$$

□

Before going any further, we would like to note that we can prove that the function $f(\cdot)$ is not Weyl p-almost automorphic of type 1 by applying the following general result:

Proposition 2.11. *Suppose $p \geq 1$ and $f : \mathbb{R} \to Y$ is not Stepanov p-bounded. Then $f(\cdot)$ is not Weyl p-almost automorphic of type 1 and not jointly Weyl p-almost automorphic.*

Proof. We will only prove that $f(\cdot)$ is not Weyl p-almost automorphic of type 1. Suppose the contrary. Then there exist a real number $l \geq 1$ and a p-locally integrable function $f^* : \mathbb{R} \to Y$ such that

$$\lim_{k\to+\infty}\int_{-l}^{l}\big\|f\big(x+s_{n_k}\big)-f^*(x)\big\|^p\,dx \leq 1. \tag{2.10}$$

Since $f(\cdot)$ is not Stepanov p-bounded, we may assume without loss of generality that there exists a strictly increasing sequence (l_n) tending to plus infinity such that $\lim_{n\to+\infty}\int_{-l+l_n}^{l+l_n}\|f(x)\|^p\,dx = +\infty$. Define $s_n := l + l_n$, $n \in \mathbb{N}$. Then for each subsequence (s_{n_k}) of (s_n) we have

$$\int_{-l+s_{n_k}}^{l+s_{n_k}}\big\|f(x)\big\|^p\,dx = \int_{l_{n_k}}^{l_{n_k}+2l}\big\|f(x)\big\|^p\,dx \to +\infty,$$

as $k \to +\infty$. This contradicts (2.10) since

$$\frac{1}{2l}\int_{-l}^{l}\left\|f(x+s_{n_k})-f^*(x)\right\|^p dx \geq \frac{1}{2l}\left[2^{1-p}\int_{-l}^{l}\left\|f(x+s_{n_k})\right\|^p dx - \int_{-l}^{l}\left\|f^*(x)\right\|^p dx\right] \\ \to +\infty, \quad k \to +\infty.$$

□

Let $p \geq 1$. Then we know that the Heaviside function $f(t) := \chi_{[0,\infty)}(t)$, $t \in \mathbb{R}$ is not equi-Weyl-p-almost periodic as well as that this function is Weyl-p-almost periodic ([24] and [29, Example 2.12, Example 2.13]). It is very interesting what is going on with the Weyl p-almost automorphic properties of the Heaviside function. Concerning this question, we have the following:

Theorem 2.12. *The Heaviside function $f(\cdot)$ is not jointly Weyl p-almost automorphic but $f(\cdot)$ is both Weyl p-almost automorphic and Weyl p-almost automorphic of type 1.*

Proof. Suppose that $f(\cdot)$ is jointly Weyl p-almost automorphic; let (s_n) be any strictly increasing sequence of real numbers tending to plus infinity and let $\epsilon \in (0, 2^{-p}/3)$ be given. By definition, we know that there exist a subsequence (s_{n_k}) and a function $f^* \in L^p_{loc}(\mathbb{R} : X)$ such that (2.5) and (2.6) hold true. Hence, there exists a finite real number $m > 0$ such that, for every $l \geq m$ and for every $k \in \mathbb{N}$ with $k \geq m$, we have

$$\frac{1}{2l}\left[\int_{-l}^{l}\left|f\left(t+s_{n_k}+x\right)-f^*(t+x)\right|^p dx + \int_{-l}^{l}\left|f^*\left(t-s_{n_k}+x\right)-f(t+x)\right|^p dx\right] < \epsilon.$$

This implies

$$\frac{1}{2l}\left[\int_{[-l,l]\cap(-\infty,-t-s_{n_k}]}\left|f^*(t+x)\right|^p dx + \int_{[-l,l]\cap[-t-s_{n_k},+\infty)}\left|1-f^*(t+x)\right|^p dx\right.$$
$$\left. + \int_{[-l,l]\cap(-\infty,-t]}\left|f^*\left(t+x-s_{n_k}\right)\right|^p dx + \int_{[-l,l]\cap[-t,+\infty)}\left|1-f^*\left(t+x-s_{n_k}\right)\right|^p dx\right]$$
$$= \frac{1}{2l}\left[\int_{[-l+t,l+t]\cap(-\infty,-s_{n_k}]}\left|f^*(x)\right|^p dx + \int_{[-l+t,l+t]\cap[-s_{n_k},+\infty)}\left|1-f^*(x)\right|^p dx\right.$$
$$+ \int_{[-l+t-s_{n_k},l+t-s_{n_k}]\cap(-\infty,-s_{n_k}]}\left|f^*\left(x\right)\right|^p dx$$
$$\left. + \int_{[-l+t-s_{n_k},l+t-s_{n_k}]\cap[-s_{n_k},\infty)}\left|1-f^*\left(x\right)\right|^p dx\right] < \epsilon.$$

This particularly holds with $t = 0$, so that letting $k \to +\infty$ in the last estimate (cf. the second addend) yields

$$\frac{1}{2l}\int_{-l}^{l}\left|1-f^*(x)\right|^p dx < \epsilon, \quad l \geq m.$$

With fixed $k=\lceil m\rceil$, the last estimate in the previous calculation also yields (cf. the first addend) that

$$\frac{1}{2l}\int\limits_{-l}^{-s_{n_{\lceil m\rceil}}}\bigl|f^{*}(x)\bigr|^{p}\,dx<\epsilon,\quad l\geq m$$

so that there exists a finite real number $m_1>m$ such that

$$\frac{1}{2l}\int\limits_{-l}^{l}\bigl|1-f^{*}(x)\bigr|^{p}\,dx+\frac{1}{2l}\int\limits_{-l}^{0}\bigl|f^{*}(x)\bigr|^{p}\,dx<3\epsilon,\quad l\geq m_1.$$

As a consequence, we have

$$\frac{1}{2l}\int\limits_{-l}^{0}\bigl|1-f^{*}(x)\bigr|^{p}\,dx+\frac{1}{2l}\int\limits_{-l}^{0}\bigl|f^{*}(x)\bigr|^{p}\,dx<3\epsilon,\quad l\geq m_1.$$

This contradicts the following estimate

$$\frac{1}{2l}\int\limits_{-l}^{0}\bigl|1-f^{*}(x)\bigr|^{p}\,dx+\frac{1}{2l}\int\limits_{-l}^{0}\bigl|f^{*}(x)\bigr|^{p}\,dx\geq\frac{1}{2l}\int\limits_{-l}^{0}2\cdot 2^{-p}\,dx=2^{-p},\quad l\geq m_1,$$

so that $f(\cdot)$ is not jointly Weyl p-almost automorphic. In order to see that $f(\cdot)$ is Weyl p-almost automorphic, we can take $f^{*}\equiv f$ in the corresponding definition and here it is only worth noting that, for every fixed real numbers t and a, we have that the mapping

$$l\mapsto\int\limits_{-l}^{l}\bigl|f(t+x+a)-f(t+x)\bigr|^{p}\,dx,\quad l\in\mathbb{R}$$

is bounded, which follows from a simple analysis concerning the support of the function $x\mapsto f(t+x+a)-f(t+x)$, $x\in\mathbb{R}$; let us also stress that the above also shows that $f\in AAW^{\mathbb{F},p}_{\mathrm{R}}(\mathbb{R}:\mathbb{C})$, provided that $\lim_{l\rightarrow+\infty}\mathbb{F}(l)=0$ and R denotes the collection of all real sequences (see Definition 3.1(i) for the notion). It remains to be proved that $f(\cdot)$ is Weyl p-almost automorphic of type 1. More generally, let $\mathbb{F}:(0,\infty)\rightarrow(0,\infty)$ be such that $\lim_{l\rightarrow+\infty}\mathbb{F}(l)=0$ and let R denote the collection of all real sequences; we will prove that $f\in AAW^{\mathbb{F},p,1}_{\mathrm{R}}(\mathbb{R}:\mathbb{C})$. If the sequence (s_n) is bounded, then the situation is very simple and we can take $f^{*}(\cdot)$ to be a certain translation of $f(\cdot)$ after applying the Bolzano-Weierstrass theorem. If the sequence (s_n) is unbounded, then it has a strictly monotone subsequence (s_n') tending to plus infinity or minus infinity. The consideration in both cases is similar and we may assume without loss of generality that $\lim_{n\rightarrow+\infty}s_n'=+\infty$ and $s_n'>0$ for all $n\in\mathbb{N}$. Choose any strictly increasing sequence (a_n) of positive real numbers such that $\lim_{n\rightarrow+\infty}a_n=+\infty$. After that, we construct inductively a subsequence (s_{n_k}) of (s_n') so that $s_{n_1}=s_1'$ and the following conditions are satisfied:

$$s_{n_{k+1}}>s_{n_k}+2a_{n_k},\quad k\in\mathbb{N};\tag{2.11}$$

$$|\mathbb{F}(v)|<\frac{1}{(a_{n_1}+a_{n_2}+\cdots+a_{n_k})^2},\ \text{provided } v\geq s_{n_k} \text{ and } k\in\mathbb{N}\setminus\{1\}.\tag{2.12}$$

Define now $f^*(t) := 1$ for $t \geq -s_{n_1}$ or $t \in [-s_{n_{k+1}}, -a_{n_k} - s_{n_k}]$ for some $k \geq 2$, and $f^*(t) := 0$ if there exists $k \in \mathbb{N}$ such that $t \in (-a_{n_k} - s_{n_k}, -s_{n_k})$. By the corresponding definition, the dominated convergence theorem and a simple argumentation, it suffices to show that

$$\lim_{l \to +\infty} \mathbb{F}(l) \int_{-l}^{l} \bigl|1 - f^*(t+x)\bigr|^p \, dx = 0 \tag{2.13}$$

and

$$\forall l > 0 \, : \lim_{k \to +\infty} \int_{-l}^{l} \bigl|f^*\bigl(t + x - s_{n_k}\bigr) - f(t+x)\bigr|^p \, dx = 0. \tag{2.14}$$

In order to prove (2.14), it suffices to repeat verbatim the argumentation used in the proof of joint Weyl p-almost automorphy of $f(\cdot)$. We actually need to prove that there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ we have

$$\int_{[-l+t-s_{n_k}, l+t-s_{n_k}] \cap (-\infty, -s_{n_k}]} \bigl|f^*\bigl(x\bigr)\bigr|^p \, dx$$
$$+ \int_{[-l+t-s_{n_k}, l+t-s_{n_k}] \cap [-s_{n_k}, \infty)} \bigl|1 - f^*\bigl(x\bigr)\bigr|^p \, dx < \epsilon.$$

But, for sufficiently large values of parameter k we have that $a_{n_k} > l + |t| + |l - t|$ so that the above sum is equal to zero due to our construction of function f^* and condition (2.11). To prove (2.13), it suffices to consider the case $t = 0$ due to the boundedness of function $f^*(\cdot)$. We need to prove that

$$\lim_{l \to +\infty} \mathbb{F}(l) \int_{-l}^{-s_{n_1}} \bigl|1 - f^*(x)\bigr|^p \, dx = 0.$$

Let $l > s_{n_3}$ and let $l \in [s_{n_{k+1}}, s_{n_{k+2}})$ for some $k \in \mathbb{N} \setminus \{1\}$. Taking into account (2.12), we have

$$\mathbb{F}(l) \int_{-l}^{-s_{n_1}} \bigl|1 - f^*(x)\bigr|^p \, dx \leq \left[\max_{v \geq s_{n_k}} \mathbb{F}(v)\right] \cdot \sum_{j=1}^{n_k} a_{n_j}$$
$$\leq \frac{1}{(a_{n_1} + a_{n_2} + \cdots + a_{n_k})^2} \cdot \sum_{j=1}^{n_k} a_{n_j} \leq \frac{1}{(a_{n_1} + a_{n_2} + \cdots + a_{n_k})}.$$

This simply implies the required. □

2.1. *Concept without limit functions*

The set $W^pAA(\mathbb{R} : X)$, equipped with the usual operations of pointwise addition of functions and multiplication of functions with scalars, has a linear vector structure. As we have observed in [24], it is not clear how one can prove (see also [1, p. 5, l. 2-3]) that an equi-Weyl-p-almost periodic function $f : \mathbb{R} \to X$, in the sense of [24, Definition 2.3.1(i)], is Weyl-p-almost automorphic. The main problem lies in the fact that it is not clear how one can prove that, for a given real sequence (s_n), there exist a subsequence (s_{n_k}) of (s_n) and a p-locally integrable function $f^* : \mathbb{R} \to X$ such that the sequence of translations $(f(\cdot + s_{n_k}))$ converges to $f^*(\cdot)$ in the Weyl metric, i.e., that

$$\lim_{k\to+\infty} \lim_{l\to+\infty} \sup_{t\in\mathbb{R}} \frac{1}{2l} \int_{-l}^{l} \big\| f\big(t + x + s_{n_k}\big) - f^*(t+x) \big\|^p \, dx = 0.$$

We can only prove that the family of translations $\{f(\cdot + h) : h \in \mathbb{R}\}$ is totally bounded with respect to the Weyl metric (see e.g, [3, Theorem 2] for case $p = 1$) as well as that, for a given real sequence (s_n), there exists a subsequence (s_{n_k}) of (s_n) such that the sequence of translations $(f(\cdot + s_{n_k}))$ is a Cauchy sequence in the Weyl metric (see e.g., [19, Theorem 3.8]), i.e., that for each $\epsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that, for every $k,\ m \in \mathbb{N}$, the assumptions $k \geq k_0$ and $m \geq k_0$ imply that

$$\lim_{l\to+\infty} \sup_{t\in\mathbb{R}} \left[\frac{1}{2l} \int_{-l}^{l} \big\| f\big(t + s_{n_k} + x\big) - f\big(t + s_{n_m} + x\big) \big\|^p \, dx \right] < \epsilon. \tag{2.15}$$

Before proceeding any further, we would like to note that the existence of a subsequence (s_{n_k}) of (s_n) such that the sequence of translations $(f(\cdot + s_{n_k}))$ is a Cauchy sequence in the Weyl metric does not imply that the function $f(\cdot)$ is equi-Weyl-p-almost periodic; for example, this is not true for the Heaviside function $f(\cdot)$:

Example 2.13. Let (s_n) be a real sequence, let (s_{n_k}) be the same as (s_n), and let $\epsilon > 0$. We choose $k_0 = 1$ in the above definition. Then, for every $k,\ m \in \mathbb{N}$, we have

$$\lim_{l\to+\infty} \sup_{t\in\mathbb{R}} \left[\frac{1}{2l} \int_{-l}^{l} \big\| f\big(t + s_{n_k} + x\big) - f\big(t + s_{n_m} + x\big) \big\|^p \, dx \right] = 0, \tag{2.16}$$

and therefore, (2.15) is satisfied. In order to see that (2.16) holds, observe that for each $t \in \mathbb{R}$ the integal

$$\int_{-l}^{l} \big\| f\big(t + s_{n_k} + x\big) - f\big(t + s_{n_m} + x\big) \big\|^p \, dx \tag{2.17}$$

is taken with respect to the variable x which belongs to the interval $[-l, l]$ but the integrand is not equal to zero only for those values of x for which the numbers $t + s_{n_k} + x$ and $t + s_{n_m} + x$ have different signs; hence, the measure of set of integration in (2.17) is less or equal than $2|s_{n_k} - s_{n_m}|$. Since the essential bound of the integrand is less or equal than 1 for each $t \in \mathbb{R}$, we get

$$\sup_{t\in\mathbb{R}} \left[\int_{-l}^{l} \big\| f\big(t + s_{n_k} + x\big) - f\big(t + s_{n_m} + x\big) \big\|^p \, dx \right] \leq 2\big|s_{n_k} - s_{n_m}\big|,$$

which simply implies the required.

In our new concept, which generalizes the concept of equi-Weyl-p-almost periodicity, we do not use the limit functions with the respect to the Weyl metric but only Cauchy sequences with respect to the Weyl metric (our idea is, in fact, to remove the operation $\sup_{t\in\mathbb{R}} \cdot$ from the expression (2.15)):

Definition 2.14. Let $p \geq 1$ and $f \in L^p_{loc}(\mathbb{R} : X)$. Then we say that $f(\cdot)$ is Weyl p-almost automorphic of type 2 if and only if for each real sequence (s_n) there exists a subsequence (s_{n_k}) of (s_n) such that for each $\epsilon > 0$ and $t \in \mathbb{R}$ there exists $k_0 \in \mathbb{N}$ such that, for every $k,\ m \in \mathbb{N}$ with $k \geq k_0$ and $m \geq k_0$, there exists $l_0 > 0$ such that

$$\frac{1}{2l}\int_{-l}^{l}\big\|f\big(t+s_{n_k}+x\big)-f\big(t+s_{n_m}+x\big)\big\|^p\,dx<\epsilon,\quad l\geq l_0. \tag{2.18}$$

It is clear that the Weyl p-almost automorphic functions of type 2 form a vector space under the usual operations as well as that [24, Lemma 2.2.13] implies that any Weyl p'-almost automorphic function of type 2 is Weyl p-almost automorphic of type 2, provided that $1 \leq p \leq p' < +\infty$ (the same holds for all other classes of Weyl almost automorphic functions considered so far).

We have the following result:

Proposition 2.15. *Suppose that $p \geq 1$ and $f \in L^p_{loc}(\mathbb{R} : X)$ is Weyl p-almost automorphic or jointly Weyl p-almost automorphic. Then $f(\cdot)$ is Weyl p-almost automorphic of type 2.*

Proof. We will consider the class of Weyl p-almost automorphic functions, only. Let a real sequence (s_n) be given. Then there exist a subsequence (s_{n_k}) and a function $f^* \in L^p_{loc}(\mathbb{R} : X)$ such that (2.1) holds. Let the numbers $\epsilon > 0$ and $t \in \mathbb{R}$ be given. Then we have the existence of a positive integer $k_0 \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ with $k \geq k_0$ there exists $l_k > 0$ such that for every $l \geq l_k$ we have $|a_{k,l}| < \epsilon/(2(2^p - 1))$, where

$$a_{k,l} := \frac{1}{2l}\int_{-l}^{l}\big\|f\big(t+s_{n_k}+x\big)-f^*(t+x)\big\|^p\,dx.$$

Let $k',\ k'' \geq k_0$. Then there exists a finite real number $l_0 := \max(l_{k'}, l_{k''}) > 0$ such that for every $l \geq l_0$ we have $|a_{k',l}| < \epsilon/(2(2^p-1))$ and $|a_{k'',l}| < \epsilon/(2(2^p-1))$. Using the inequality $(a+b)^p \leq 2^{p-1}(a^p+b^p)$, $a,\ b \geq 0$, the above simply implies (2.18) with the numbers k and m replaced therein with the numbers k' and k'', which completes the proof. ☐

The proof of Proposition 2.15 does not work for Weyl p-almost automorphic functions of type 1 and we would like to ask:

Question 2.16. *Suppose that $p \geq 1$ and $f : \mathbb{R} \rightarrow Y$ is (Stepanov p-almost automorphic) Weyl p-almost automorphic functions of type 1. Is it true that $f(\cdot)$ is Weyl p-almost automorphic of type 2?*

The following question is also meaningful:

Question 2.17. *Suppose $p \geq 1$. Construct, if possible, a jointly Weyl p-almost automorphic function which is not (equi-)Weyl-p-almost periodic?*

Before switching to the next section, we shall revisit the following examples from our recent paper [26]:

Example 2.18. Suppose that $p \geq 1$. Let us recall that A. Haraux and P. Souplet have considered, in [23, Theorem 1.1], the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(t) := \sum_{n=1}^{\infty} \frac{1}{n} \sin^2\Big(\frac{t}{2^n}\Big), \quad t \in \mathbb{R}.$$

This function is bounded, uniformly continuous, uniformly recurrent, Besicovitch 1-unbounded and Weyl p-almost automorphic for any finite exponent (cf. [26] for the notion and more details). By Proposition 2.11, we immediately get that $f(\cdot)$ is not Weyl p-almost automorphic of type 1 nor jointly Weyl p-almost automorphic. On the other hand, an application of Proposition 2.15 shows that $f(\cdot)$ is Weyl p-almost automorphic of type 2. In the present situation, we do not know to tell whether, for a given real sequence (s_n), there exists a subsequence (s_{n_k}) of (s_n) such that the sequence of translations $(f(\cdot + s_{n_k}))$ is a Cauchy sequence in the Weyl metric (cf. also the remarkable example by H. Bohr [8, pp. 113–115, part I] and [26, Theorem 1.4, Example 3], which will not be reexamined here).

Example 2.19. (J. de Vries, [12, point 6., p. 208; Figure 3.7.3, p. 208], M. Kostić [26, Theorem 1.5]) Let $(p_i)_{i\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers such that $p_i | p_{i+1},\ i \in \mathbb{N}$ and $\lim_{i\to\infty} p_i/p_{i+1} = 0$. Define the function $f_i : [-p_i, p_i] \to [0, 1]$ by $f_i(t) := |t|/p_i$, $t \in [-p_i, p_i]$ and extend the function $f_i(\cdot)$ periodically to the whole real axis; the obtained function, denoted by the same symbol $f_i(\cdot)$, is of period $2p_i$ ($i \in \mathbb{N}$). Set

$$f(t) := \sup\big\{f_i(t) : i \in \mathbb{N}\big\}, \quad t \in \mathbb{R}. \tag{2.19}$$

Then we know that the function $f : \mathbb{R} \to \mathbb{R}$, given by (2.19), is bounded, uniformly continuous, uniformly recurrent and not asymptotically (Stepanov) almost automorphic ([26]). In this paper, we will only prove the following new property of function $f(\cdot)$:

$$\limsup_{l\to+\infty} \frac{1}{2l} \int_{-l}^{l} |1 - f(x)|\, dx = \limsup_{l\to+\infty} \frac{1}{l} \int_{0}^{l} |1 - f(x)|\, dx \geq \frac{1}{4}. \tag{2.20}$$

In order to do that, fix a positive integer $i \in \mathbb{N}$ and consider the straight line $y = x/p_{i+1}$ and the straight line $y = [(-1)/p_i](x - 2p_i)$ connecting the points $(p_i, 1)$ and $(2p_i, 0)$. The intersection of these lines is the point $(2p_ip_{i+1}/(p_i + p_{i+1}), 2p_i/(p_i + p_{i+1}))$. Set $l_i := 2p_ip_{i+1}/(p_i + p_{i+1})$; then $\lim_{i\to+\infty} l_i = +\infty$ and $f(x) \geq f_i(x)$, $x \in [p_i, l_i]$. This implies (2.20), because

$$\begin{aligned}
&\frac{1}{l_i} \int_{0}^{l_i} |1 - f(x)|\, dx \geq \frac{1}{l_i} \int_{p_i}^{l_i} |1 - f(x)|\, dx \\
&= \frac{1}{2} \frac{p_i + p_{i+1}}{2p_ip_{i+1}} \left[\frac{2p_ip_{i+1}}{p_i + p_{i+1}} - p_i\right] \cdot \left[1 - \frac{2p_i}{p_i + p_{i+1}}\right] \\
&= \frac{1}{2} \frac{p_ip_{i+1} - p_i^2}{2p_ip_{i+1}} \cdot \left[1 - \frac{2p_i}{p_i + p_{i+1}}\right] \to \frac{1}{4}, \quad i \to +\infty.
\end{aligned}$$

Finally, we would like to ask whether the function $f(\cdot)$ is (equi-)Weyl-p-almost periodic [(jointly) Weyl p-almost automorphic (of type $1, 2$)] for some (each) finite exponent $p \geq 1$?

3. Multi-dimensional Weyl almost automorphy in Lebesgue spaces with variable exponent

The main aim of this section is to introduce and analyze multi-dimensional Weyl almost automorphy in Lebesgue spaces with variable exponent. Here, we basically follow the approach used in our investigations of (equi-)Weyl-$(p, \phi, \mathbb{F})$-almost periodic functions (see [19] and [29] for more details) and our considerations from Section 2. Unless stated otherwise, we will always assume henceforth that $\Omega := [-1, 1]^n \subseteq \mathbb{R}^n$, $p \in \mathcal{P}(\mathbb{R}^n)$ and $\mathbb{F} : (0, \infty) \times \mathbb{R}^n \to (0, \infty)$; in contrast with the above-mentioned researches, we will always assume here $\phi(x) \equiv x$ for simplicity.

We start by introducing the following notion:

Definition 3.1. Suppose that $F : \mathbb{R}^n \times X \to Y$ satisfies that for each $x \in X$, $l > 0$ and $\mathbf{t} \in \mathbb{R}^n$ we have $F(\mathbf{t} + \mathbf{u}; x) \in L^{p(\mathbf{u})}(l\Omega : Y)$. Let for every $B \in \mathcal{B}$ and $(\mathbf{b}_k = (b_k^1, b_k^2, \cdot \cdot \cdot, b_k^n)) \in \mathrm{R}$ there exist a subsequence $(\mathbf{b}_{k_m} = (b_{k_m}^1, b_{k_m}^2, \cdot \cdot \cdot, b_{k_m}^n))$ of $(\mathbf{b}_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that for each $x \in B$, $l > 0$ and $\mathbf{t} \in \mathbb{R}^n$ we have $F^*(\mathbf{t} + \mathbf{u}; x) \in L^{p(\mathbf{u})}(l\Omega : Y)$, as well as:

(i)

$$\lim_{m \to +\infty} \lim_{l \to +\infty} \mathbb{F}(l, \mathbf{t}) \Big\| F(\mathbf{t} + \mathbf{u} + (b_{k_m}^1, \cdot \cdot \cdot, b_{k_m}^n); x) - F^*(\mathbf{t} + \mathbf{u}; x) \Big\|_{L^{p(\mathbf{u})}(l\Omega:Y)} = 0 \qquad (3.1)$$

and

$$\lim_{m \to +\infty} \lim_{l \to +\infty} \mathbb{F}(l, \mathbf{t}) \Big\| F^*(\mathbf{t} + \mathbf{u} - (b_{k_m}^1, \cdot \cdot \cdot, b_{k_m}^n); x) - F(\mathbf{t} + \mathbf{u}; x) \Big\|_{L^{p(\mathbf{u})}(l\Omega:Y)} = 0, \qquad (3.2)$$

pointwise for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$, or

(ii)

$$\lim_{l \to +\infty} \lim_{m \to +\infty} \mathbb{F}(l, \mathbf{t}) \Big\| F(\mathbf{t} + \mathbf{u} + (b_{k_m}^1, \cdot \cdot \cdot, b_{k_m}^n); x) - F^*(\mathbf{t} + \mathbf{u}; x) \Big\|_{L^{p(\mathbf{u})}(l\Omega:Y)} = 0$$

and

$$\lim_{l \to +\infty} \lim_{m \to +\infty} \mathbb{F}(l, \mathbf{t}) \Big\| F^*(\mathbf{t} + \mathbf{u} - (b_{k_m}^1, \cdot \cdot \cdot, b_{k_m}^n); x) - F(\mathbf{t} + \mathbf{u}; x) \Big\|_{L^{p(\mathbf{u})}(l\Omega:Y)} = 0,$$

pointwise for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$, or

(iii)

$$\lim_{(l,m) \to +\infty} \mathbb{F}(l, \mathbf{t}) \Big\| F(\mathbf{t} + \mathbf{u} + (b_{k_m}^1, \cdot \cdot \cdot, b_{k_m}^n); x) - F^*(\mathbf{t} + \mathbf{u}; x) \Big\|_{L^{p(\mathbf{u})}(l\Omega:Y)} = 0$$

and

$$\lim_{(l,m) \to +\infty} \mathbb{F}(l, \mathbf{t}) \Big\| F^*(\mathbf{t} + \mathbf{u} - (b_{k_m}^1, \cdot \cdot \cdot, b_{k_m}^n); x) - F(\mathbf{t} + \mathbf{u}; x) \Big\|_{L^{p(\mathbf{u})}(l\Omega:Y)} = 0,$$

pointwise for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$.

In the case that (i), resp. [(ii); (iii)], holds, then we say that the function $F(\cdot; \cdot)$ is Weyl-$(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphic, resp. [Weyl-$(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphic of type 1; jointly Weyl-$(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphic]. By $AAW_{(\mathrm{R},\mathcal{B})}^{\mathbb{F},p(\mathbf{u})}(\mathbb{R}^n \times X : Y)$, resp. $[AAW_{(\mathrm{R},\mathcal{B})}^{\mathbb{F},p(\mathbf{u}),1}(\mathbb{R}^n \times X : Y)$;

$AAW^{\mathbb{F},p(\mathbf{u}),j}_{(\mathrm{R},\mathcal{B})}(\mathbb{R}^n \times X : Y)$] we denote the collection of all Weyl-$(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphic [Weyl-$(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphic of type 1; jointly Weyl-$(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphic] functions $F : \mathbb{R}^n \times X \to Y$.

We omit the term "$\mathcal{B}$" from the notation for the functions of the form

From our analysis of multi-dimensional Weyl almost pe$F : \mathbb{R}^n \to Y$.riodicity carried out in [19], it follows that the case $p(\mathbf{u}) \equiv p \in [1, \infty)$ and $\mathbb{F}(l, \mathbf{t}) \equiv l^{-n/p}$ is the most important, when we say that the function $F : \mathbb{R}^n \times X \to Y$ is (jointly) Weyl p-$(\mathrm{R}, \mathcal{B})$-multi-almost automorphic (of type 1).

In the next definition, we continue our analysis from Subsection 2.1 by introducing the following non-trivial class of functions:

Definition 3.2. Suppose that $\emptyset \neq W \subseteq \mathbb{R}^n$, $\mathbb{F} : (0, \infty) \times \mathbb{R}^n \to (0, \infty)$ and $F : \mathbb{R}^n \times X \to Y$ satisfies that for each $x \in X$, $l > 0$ and $\mathbf{t} \in \mathbb{R}^n$ we have $F(\mathbf{t} + \mathbf{u}; x) \in L^{p(\mathbf{u})}(l\Omega : Y)$. If for every $B \in \mathcal{B}$ and $(\mathbf{b}_k = (b_k^1, b_k^2, \cdot \cdot \cdot, b_k^n)) \in \mathrm{R}$ there exists a subsequence $(\mathbf{b}_{k_m} = (b_{k_m}^1, b_{k_m}^2, \cdot \cdot \cdot, b_{k_m}^n))$ of $(\mathbf{b}_k)$ such that for each $\epsilon > 0$, $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$ there exists $m_0 \in \mathbb{N}$ such that, for every $m,\ m' \in \mathbb{N}$ with $m \geq m_0$ and $m' \geq m_0$, there exists $l_0 > 0$ such that, for every $l \geq l_0$ and $w \in lW$, we have

$$\Big\| F\Big(\mathbf{t} + \mathbf{u} + (b_{k_m}^1, \cdot \cdot \cdot, b_{k_m}^n) - w; x\Big) - F\Big(\mathbf{t} + \mathbf{u} + (b_{k_{m'}}^1, \cdot \cdot \cdot, b_{k_{m'}}^n) - w; x\Big)\Big\|_{L^{p(\mathbf{u})}(l\Omega:Y)} < \epsilon / \mathbb{F}(l, \mathbf{t} - w), \quad (3.3)$$

then we say that $F(\cdot;\cdot)$ is Weyl $p(\mathbf{u})$-$(\mathbb{F}, \mathrm{R}, \mathcal{B}, W)$-multi-almost automorphic of type 2.

We can also introduce the concepts in which the parameters $x \in B$ and $\mathbf{t} \in \mathbb{R}$ are separated with respect to the use of quantifiers, as well, but we will not go into further details concerning this notion. If $F : \mathbb{R}^n \to Y$, then we omit the term $\mathcal{B}$ from the notation, as accepted before.

Remark 3.3. (i) Since the introduced classes of almost automorphic functions are translation invariant, we do not need to follow the second approach, obeyed for the introduction of various classes of (equi-)Weyl-$[p, \phi, \mathbb{F}]$-almost periodic functions. In this concept, we assume that $\Omega := [-1, 1]^n \subseteq \mathbb{R}^n$, $p \in \mathcal{P}(\Omega)$ and $\mathbb{F} : (0, \infty) \times \mathbb{R}^n \to (0, \infty)$. We can consider the following notion: Suppose that $F : \mathbb{R}^n \times X \to Y$ satisfies that for each $x \in X$, $l > 0$ and $\mathbf{t} \in \mathbb{R}^n$ we have $F(\mathbf{t} + l\mathbf{u}; x) \in L^{p(\mathbf{u})}(\Omega : Y)$. We say that the function $F(\cdot;\cdot)$ is Weyl-$[\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B}]$-multi-almost automorphic if and only if for every $B \in \mathcal{B}$ and $(\mathbf{b}_k = (b_k^1, b_k^2, \cdot \cdot \cdot, b_k^n)) \in \mathrm{R}$ there exist a subsequence $(\mathbf{b}_{k_m} = (b_{k_m}^1, b_{k_m}^2, \cdot \cdot \cdot, b_{k_m}^n))$ of $(\mathbf{b}_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that for each $x \in B$, $l > 0$ and $\mathbf{t} \in \mathbb{R}^n$ we have $F^*(\mathbf{t} + l\mathbf{u}; x) \in L^{p(\mathbf{u})}(\Omega : Y)$, as well as

$$\lim_{m \to +\infty} \lim_{l \to +\infty} \mathbb{F}(l, \mathbf{t}) \Big\| F(\mathbf{t} + l\mathbf{u} + (b_{k_m}^1, \cdot \cdot \cdot, b_{k_m}^n); x) - F^*(\mathbf{t} + l\mathbf{u}; x) \Big\|_{L^{p(\mathbf{u})}(\Omega:Y)} = 0$$

and

$$\lim_{m \to +\infty} \lim_{l \to +\infty} \mathbb{F}(l, \mathbf{t}) \Big\| F^*(\mathbf{t} + l\mathbf{u} - (b_{k_m}^1, \cdot \cdot \cdot, b_{k_m}^n); x) - F(\mathbf{t} + l\mathbf{u}; x) \Big\|_{L^{p(\mathbf{u})}(\Omega:Y)} = 0,$$

pointwise for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$. For the sake of brevity, we will skip all related details concerning this class of functions and related classes of functions defined in a similar way, with the replaced limits or just one joint limit.

(ii) Suppose that $F : \mathbb{R}^n \times X \rightarrow Y$ satisfies that for each $x \in X$, $l > 0$ and $\mathbf{t} \in \mathbb{R}^n$ we have $F(\mathbf{t}+\mathbf{u};x) \in L^{p(\mathbf{u})}(l\Omega : Y)$, as well as that, for every $B \in \mathcal{B}$ and $(\mathbf{b}_k = (b_k^1, b_k^2, \cdot\cdot\cdot, b_k^n)) \in \mathrm{R}$, there exist a subsequence $(\mathbf{b}_{k_m} = (b_{k_m}^1, b_{k_m}^2, \cdot\cdot\cdot, b_{k_m}^n))$ of $(\mathbf{b}_k)$ and a function $F^* : \mathbb{R}^n \times X \rightarrow Y$ such that for each $x \in B$, $l > 0$ and $\mathbf{t} \in \mathbb{R}^n$ we have $F^*(\mathbf{t}+\mathbf{u};x) \in L^{p(\mathbf{u})}(l\Omega : Y)$, and

$$\lim_{m\rightarrow+\infty} \lim_{l\rightarrow+\infty} \sup_{\mathbf{t}\in\mathbb{R}^n, x\in B} \mathbb{F}(l) \Big\| F(\mathbf{t}+\mathbf{u}+(b_{k_m}^1, \cdot\cdot\cdot, b_{k_m}^n);x) - F^*(\mathbf{t}+\mathbf{u};x) \Big\|_{L^{p(\mathbf{u})}(l\Omega:Y)} = 0.$$

Using the substitution $\mathbf{t} \mapsto \mathbf{t} + (b_{k_m}^1, \cdot\cdot\cdot, b_{k_m}^n)$, $\mathbf{t} \in \mathbb{R}^n$ we get that

$$\lim_{m\rightarrow+\infty} \lim_{l\rightarrow+\infty} \sup_{\mathbf{t}\in\mathbb{R}^n, x\in B} \mathbb{F}(l) \Big\| F^*(\mathbf{t}+\mathbf{u}-(b_{k_m}^1, \cdot\cdot\cdot, b_{k_m}^n);x) - F(\mathbf{t}+\mathbf{u};x) \Big\|_{L^{p(\mathbf{u})}(l\Omega:Y)} = 0.$$

Hence, $F(\cdot,\cdot)$ is Weyl-$(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphic. It is also worth noting that, in the case that $p(\mathbf{u}) \equiv p \in [1,\infty)$ and $\mathbb{F}(l) \equiv l^{-n/p}$, the assumptions used in this remark imply the Weyl-$(\mathrm{R}, \mathcal{B}, p)$-normality of function $F(\cdot,\cdot)$ in the sense of [19, Definition 3.5].

We continue by providing the following illustrative examples:

Example 3.4. (J. Stryja [39, pp. 42–47]; see also [2, Example 4.28]) Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) := 0$ for $x \leq 0$, $f(x) := \sqrt{n/2}$ if $x \in (n-2, n-1]$ for some $n \in 2\mathbb{N}$ and $f(x) := -\sqrt{n/2}$ if $x \in (n-1, n]$ for some $n \in 2\mathbb{N}$. It is clear that the function $f(\cdot)$ is not Stepanov 1-bounded, which immediately implies that the function $f(\cdot)$ is not asymptotically Stepanov 1-almost automorphic, not Weyl 1-almost automorphic of type 1 and not jointly Weyl 1-almost automorphic; furthermore, arguing in the same way as in the proof of Proposition 2.11, we get that the function $f(\cdot)$ is not Weyl-$(\mathbb{F}, 1, \mathrm{R})$-multi-almost automorphic if R is any collection of real sequences containing a strictly increasing sequence (s_n) tending to plus infinity. We already know that the function $f(\cdot)$ is not equi-Weyl-1-almost periodic as well as that the function $f(\cdot)$ is Weyl-1-almost periodic and not Weyl 1-normal, as well as that for each $n \in 2\mathbb{N}$ we have

$$\lim_{l\rightarrow+\infty} \frac{1}{2l} \sup_{t\in\mathbb{R}} \int_{-l}^{l} \big| f(t+n+x) - f(t+x) \big| \, dx = 0, \tag{3.4}$$

so that the function $f(\cdot)$ is Weyl-$(\mathbb{F}, 1, \mathrm{R})$-multi-almost automorphic with $\mathbb{F}(l,t) \equiv 1/l$ and R being the collection of all real sequences (a_m) satisfying that $a_m \in 2\mathbb{N}$ for all $m \in \mathbb{N}$. Now we will prove that the function $f(\cdot)$ is not Weyl-$(\mathbb{F}, 1, \mathrm{R})$-multi-almost automorphic provided that there exists a real sequence (s_n) from R which contains only a finite number of even numbers; in particular, the function $f(\cdot)$ is not Weyl 1-almost automorphic. Towards this end, it suffices to show that, for every fixed real number $\omega \notin 2\mathbb{Z}$, we have

$$\lim_{l\rightarrow+\infty} \frac{1}{2l} \int_{0}^{l} \big| f(x+\omega) - f(x) \big| \, dx = +\infty, \tag{3.5}$$

where $\omega' \in 2\mathbb{Z}$ denotes the nearest even number to ω. Without loss of generality, we may assume that $\omega' < \omega$ so that $h := \omega - \omega' \in (0,1]$. Keeping in mind the triangle inequality and the estimate (3.4) with $t = 0$ and $n = \omega'$, we only need to show that

$$\lim_{l\rightarrow+\infty} \frac{1}{2l} \int_{0}^{l} \Big| f(x+\omega) - f\big(x+\omega'\big) \Big| \, dx = +\infty. \tag{3.6}$$

This follows from the following calculation ($l > 2 + |\omega'|$):

$$\frac{1}{2l}\int\limits_0^l \Big|f(x+\omega) - f\big(x+\omega'\big)\Big|\,dx$$

$$= \frac{1}{2l}\int\limits_{\omega'}^{\omega'+l} \big|f(x+h) - f(x)\big|\,dx$$

$$= 2\frac{\omega'+l}{l}\frac{1}{2(\omega'+l)}\int\limits_0^{\omega'+l}\big|f(x+h) - f(x)\big|\,dx - \frac{1}{2l}\int\limits_0^{\omega'}\big|f(x+h) - f(x)\big|\,dx$$

$$\geq 2\frac{\omega'+l}{l}\left[\frac{4}{3}h\sqrt{\Big\lfloor\frac{l+\omega'}{2}\Big\rfloor - 1}\right] - \frac{1}{2l}\int\limits_0^{\omega'}\big|f(x+h) - f(x)\big|\,dx \to +\infty, \quad l \to +\infty,$$

where the estimate used above follows by applying the estimate proved on [2, p. 149, l. 7-9], which is valid for $h \in (0,1]$ and $L \geq 2$. Using a similar argumentation involving the estimates (3.5)-(3.6), it follows that the function $f(\cdot)$ is not 1-$(1/l, \mathrm{R}, \{0\})$-multi-almost automorphic provided that there exists a sequence (s_n) from R satisfying that, for every its subsequence (s_{n_k}), there exist two arbitrarily large positive integers k' and k'' such that the difference $s_{n_{k'}} - s_{n_{k''}}$ is not an even number; in particular, the function $f(\cdot)$ is not Weyl 1-almost automorphic of type 2. It is also worth noting that the function $f(\cdot)$ is not Besicovitch 1-bounded, not Besicovitch 1-almost periodic and has no finite mean value (cf. also Theorem 2.10 above).

Example 3.5. (i) Similarly as in the one-dimensional case, it can be proved that for any compact set $K \subseteq \mathbb{R}^n$ and for any $p \in \mathcal{P}(\mathbb{R}^n)$ we have that the function $\chi_K(\cdot)$ belongs to any of the function spaces introduced in Definition 3.1 and Definition 3.2 with $\mathbb{F}(t, \mathbf{t}) \equiv l^{-\sigma}$, with $\sigma > 0$; see also [19, Example 2.7].

(ii) Suppose now that $F(\mathbf{t}) := \chi_{[0,\infty)^n}(\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^n$. Let R denote the collection of all sequences in $\mathbb{R}^n$, and let $1 \leq p < \infty$. Arguing as in [19, Example 2.8], we can prove that the function $F(\cdot)$ is Weyl-$(\mathbb{F}, p, \mathrm{R})$-multi-almost automorphic with $\mathbb{F}(l, \mathbf{t}) \equiv l^{-\sigma}$ if and only if $\sigma > (n-1)/p$, as well as that there is no $\sigma > 0$ such that $F(\cdot)$ is Weyl-$(\mathbb{F}, p, \mathrm{R})$-multi-almost automorphic with $\mathbb{F}(l, \mathbf{t}) \equiv l^{-\sigma}$. On the other hand, our analysis from Example 2.13 can be repeated with minor modifications in order to see that for each real number $\sigma > 0$ the function $F(\cdot)$ is Weyl p-$(\mathbb{F}, \mathrm{R}, \mathbb{R}^n)$-multi-almost automorphic with $\mathbb{F}(l, \mathbf{t}) \equiv l^{-\sigma}$. An insignificant modification of the construction established in the corresponding part of the proof of Theorem 2.12 shows that the function $F(\cdot)$ is also Weyl-$(\mathbb{F}, p, \mathrm{R})$-multi-almost automorphic of type 1 for any function $\mathbb{F}(l)$ satisfying that $\lim_{l\to+\infty}\mathbb{F}(l) = 0$.

Example 3.6. The following example is a simple modification of [9, Example 2.15(i)] and [19, Example 3.1]. Suppose that $1 \leq p < \infty$, the complex-valued mapping $t \mapsto g_j(t) \in Y$, $t \in \mathbb{R}$ is essentially bounded and jointly Weyl-$(\mathbb{F}_j, p, \mathrm{R}_j)$-almost automorphic, where R_j denotes the collection of all real sequences ($1 \leq j \leq n$). Define

$$F\big(t_1, \cdot\cdot\cdot, t_{2n}\big) := \prod_{j=1}^{n}\Big[g_j\big(t_{j+n}\big) - g_j\big(t_j\big)\Big], \text{ where } t_j \in \mathbb{R} \text{ for } 1 \leq j \leq 2n,$$

and $\Lambda' := \{(\tau,\tau) : \tau \in \mathbb{R}^n\}$. Then we know there exists a finite constant $M > 0$ such that

$$\Big\| F\big(t_1+\tau_1, \cdot\cdot\cdot, t_{2n}+\tau_{2n}\big) - F\big(t_1, \cdot\cdot\cdot, t_{2n}\big)\Big\|_Y$$
$$\leq M\Bigg\{ \Big\| g_1\big(t_{n+1}+\tau_1\big) - g_1\big(t_{n+1}\big)\Big\| + \Big\| g_1\big(t_1+\tau_1\big) - g_1\big(t_1\big)\Big\| + \cdots$$
$$+ \Big\| g_n\big(t_{2n}+\tau_n\big) - g_n\big(t_{2n}\big)\Big\| + \Big\| g_n\big(t_n+\tau_n\big) - g_n\big(t_n\big)\Big\| \Bigg\},$$

for any $(t_1, \cdot\cdot\cdot, t_{2n}) \in \mathbb{R}^{2n}$ and $(\tau_1, \cdot\cdot\cdot, \tau_{2n}) \in \Lambda'$. Suppose that $c \in (0,\infty)$, $\mathbb{F} : (0,\infty) \times \mathbb{R}^n \to Y$ and

$$\sum_{j=1}^{n}\Bigg[\frac{1}{F_j(l,t_j)} + \frac{1}{F_j(l,t_{j+n})}\Bigg] \leq \frac{c}{F(l,t_1,\cdot\cdot\cdot,t_{2n})}, \quad l > 0,\ \big(t_1, \cdot\cdot\cdot, t_{2n}\big) \in \mathbb{R}^{2n}.$$

Using the corresponding definition and the above estimates, it follows that the function $F(\cdot)$ is jointly Weyl-$(\mathbb{F}, p, \mathrm{R})$-almost automorphic, where R denotes the collection of all sequences in Λ'.

Immediately from the above definitions, we have the following simple proposition which can be clarified for all other classes of functions introduced above:

Proposition 3.7. *(i) Suppose that $c \in \mathbb{C}$ and $F(\cdot;\cdot)$ is Weyl-$(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphic. Then $cF(\cdot;\cdot)$ is Weyl-$(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphic.*

(ii) Suppose that $\tau \in \mathbb{R}^n$, $x_0 \in X$, the function $\mathbb{F}(\cdot,\cdot)$ does not depend on the second argument, and $F(\cdot;\cdot)$ is Weyl-$(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphic. Then $F(\cdot+\tau;\cdot+x_0)$ is Weyl-$(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphic, where $\mathcal{B}_{x_0} \equiv \{-x_0 + B : B \in \mathcal{B}\}$.

(iii) (a) Suppose that $c_2 \in \mathbb{C}\setminus\{0\}$, and $F(\cdot;\cdot)$ is Weyl-$(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphic. Then $F(\cdot;c_2\cdot)$ is Weyl-$(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphic, where $\mathcal{B}_{c_2} \equiv \{c_2^{-1}B : B \in \mathcal{B}\}$.

(b) Suppose that $c_1 \in \mathbb{C}\setminus\{0\}$, $c_2 \in \mathbb{C}\setminus\{0\}$, and $F(\cdot;\cdot)$ is Weyl-$(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphic, with some constant exponent $p \geq 1$. Then the function $F(c_1\cdot;c_2\cdot)$ is Weyl-$(\mathbb{F}_{c_1}, p, \mathrm{R}, \mathcal{B})$-multi-almost automorphic, where $\mathrm{R}_{c_1} \equiv \{c_1^{-1}\mathbf{b} : \mathbf{b} \in \mathrm{R}\}$ and $\mathbb{F}_{c_1}(l,\mathbf{t}) \equiv \mathbb{F}(l, c_1\mathbf{t})$.

We have the following simple result, which can be also clarified for all other classes of functions introduced in this section so far:

Proposition 3.8. *Suppose that $F(\cdot;\cdot)$ is Weyl-$(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphic and $A \in L(Y,Z)$. Then $(A \circ F)(\cdot;\cdot)$ is Weyl-$(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphic.*

Proof. Let $x \in X$, $\mathbf{t} \in \mathbb{R}^n$, $B \in \mathcal{B}$ and $(\mathbf{b}_k = (b_k^1, b_k^2, \cdot\cdot\cdot, b_k^n)) \in \mathrm{R}$ be fixed. Then we know that $F(\mathbf{t}+\mathbf{u};x) \in L^{p(\mathbf{u})}(l\Omega : Y)$, so that there exists a finite real number $\lambda > 0$ such that

$$\int\limits_{l\Omega} \varphi_{p(\mathbf{u})}\left(\frac{\|F(\mathbf{t}+\mathbf{u};x)\|_Y}{\lambda}\right) d\mathbf{u} \leq 1.$$

This easily implies that

$$\int_{l\Omega} \varphi_{p(\mathbf{u})}\left(\frac{\|A(F(\mathbf{t}+\mathbf{u};x))\|_Z}{\lambda'}\right) d\mathbf{u} \leq 1,$$

with $\lambda' = \|A\| \cdot \lambda$. Hence, $A(F(\mathbf{t}+\mathbf{u};x)) \in L^{p(\mathbf{u})}(l\Omega : Z)$. Further on, we know that there exist a subsequence $(\mathbf{b}_{k_l} = (b^1_{k_l}, b^2_{k_l}, \cdot\cdot\cdot, b^n_{k_l}))$ of $(\mathbf{b}_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that for each $x \in X$ and $\mathbf{t} \in \mathbb{R}^n$ we have $F^*(\mathbf{t}+\mathbf{u};x) \in L^{p(\mathbf{u})}(l\Omega : Y)$, as well as that (3.1)-(3.2) hold. By the foregoing, we have that $A(F^*(\mathbf{t}+\mathbf{u};x)) \in L^{p(\mathbf{u})}(l\Omega : Z)$. Using Lemma 1.2, it can be simply verified that (3.1)-(3.2) hold with the functions F and F^* replaced therein with the function $A \circ F$ and $A \circ F^*$, respectively, finishing the proof of the proposition. $\square$

3.1. *Weyl $p(\mathbf{u})$-$(\mathbb{F}, \mathrm{R}, \mathcal{B}, W)$-multi-almost automorphy of type 2 and jointly Weyl $p(\mathbf{u})$-$(\mathbb{F}, \mathrm{R}, \mathcal{B}, W)$-multi-almost automorphy*

In this subsection, we investigate the Weyl $p(\mathbf{u})$-$(\mathbb{F}, \mathrm{R}, \mathcal{B}, W)$-multi-almost automorphic functions of type 2 and jointly Weyl $p(\mathbf{u})$-$(\mathbb{F}, \mathrm{R}, \mathcal{B}, W)$-multi-almost automorphic functions, primarily from their invaluable importance in applications.

In Definition 3.2, we can also assume that the set W depends on l but the situation is more complicated then. In Subsection 2.1 we have $W = \{0\}$; case $W \neq \{0\}$ is also important to be analyzed, as the following result about the convolution invariance of Weyl $p(\mathbf{u})$-$(\mathrm{R}, \mathcal{B}, W)$-multi-almost automorphy of type 2 shows (the choice of sets $W_1 = (2\mathbb{Z}+1)^n$ and $W_2 \subseteq (2\mathbb{Z})^n$ strongly depends on the choice of region $\Omega = [-1,1]^n$ here; the things certainly can be arranged in a slightly different, generalized way, the reader may try to make this more precise):

Theorem 3.9. *Suppose that $h \in L^1(\mathbb{R}^n)$ and $F : \mathbb{R}^n \times X \to Y$ is Weyl $p(\mathbf{u})$-$(\mathbb{F}, \mathrm{R}, \mathcal{B}, (2\mathbb{Z}+1)^n)$-multi-almost automorphic of type 2. Let $p_1,\ q \in \mathcal{P}(\mathbb{R}^n)$, let $1/p(\mathbf{u}) + 1/q(\mathbf{u}) \equiv 1$, and let $\mathbb{F}_1 : (0,\infty) \times \mathbb{R}^n \to (0,\infty)$. Suppose that, for every $x \in X$, we have $\sup_{\mathbf{t}\in\mathbb{R}^n} \|F(\mathbf{t};x)\|_Y < \infty$, as well as that $\emptyset \neq W_2 \subseteq (2\mathbb{Z})^n$ and for every $\mathbf{t} \in \mathbb{R}^n$ there exists $l_1 > 0$ such that, for every $l \geq l_1$ and $w \in lW_2$, we have*

$$\int_{l\Omega} \varphi_{p_1(\mathbf{u})}\left(2\mathbb{F}_1(l, \mathbf{t}+w) \sum_{k\in l(2\mathbb{Z}+1)^n} \frac{\big\|h(\mathbf{u}+k-\mathbf{v})\big\|_{L^{q(\mathbf{v})}(l\Omega)}}{\mathbb{F}(l, \mathbf{t}-k+w)}\right) du \leq 1. \tag{3.7}$$

Then the function $h * F : \mathbb{R}^n \times X \to Y$, defined by

$$(h * F)(\mathbf{t};x) := \int_{\mathbb{R}^n} h(\sigma)F(\mathbf{t}-\sigma;x)\, d\sigma, \quad \mathbf{t} \in \mathbb{R}^n,\ x \in X, \tag{3.8}$$

is Weyl $p_1(\mathbf{u})$-$(\mathbb{F}_1, \mathrm{R}, \mathcal{B}, W_2)$-multi-almost automorphic of type 2.

Proof. It can be simply verified that the function $(h * F)(\cdot;\cdot)$ is well defined because we have assumed that, for every $x \in X$, we have $\sup_{\mathbf{t}\in\mathbb{R}^n} \|F(\mathbf{t};x)\|_Y < \infty$. By our assumption, for every $B \in \mathcal{B}$ and $(\mathbf{b}_k = (b^1_k, b^2_k, \cdot\cdot\cdot, b^n_k)) \in \mathrm{R}$, there exists a subsequence $(\mathbf{b}_{k_m} = (b^1_{k_m}, b^2_{k_m}, \cdot\cdot\cdot, b^n_{k_m}))$ of $(\mathbf{b}_k)$ such that for each $\epsilon > 0$, $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$ there exists $m_0 \in \mathbb{N}$ such that, for every $m,\ m' \in \mathbb{N}$ with $m \geq m_0$ and $m' \geq m_0$, there exists $l \geq l_0$ such that, for every $l \geq l_0$ and $w \in l(2\mathbb{Z}+1)^n$ (3.3) holds with

$W \equiv W_1 \equiv (2\mathbb{Z}+1)^n$. Let $l \geq \max(l_0, l_1)$ and $w \in lW_2$. Since the mapping $\varphi_{p_1(\mathbf{u})}(\cdot)$ is monotonically increasing, we have

$$\begin{aligned}
&\left\| (h * F)(\mathbf{t} + w + \mathbf{b}_{k_m} + \mathbf{u}; x) - (h * F)(\mathbf{t} + w + \mathbf{b}_{k_{m'}} + \mathbf{u}; x) \right\|_{L^{p_1(\mathbf{u})}(l\Omega : Y)} \\
&= \left\| \int_{\mathbb{R}^n} h(\mathbf{s}) \Big[F(\mathbf{t} + w + \mathbf{b}_{k_m} + \mathbf{u} - \mathbf{s}; x) - F(\mathbf{t} + w + \mathbf{b}_{k_{m'}} + \mathbf{u} - \mathbf{s}; x) \Big] d\mathbf{s} \right\|_{L^{p_1(\mathbf{u})}(l\Omega : Y)} \\
&\leq \left(\int_{\mathbb{R}^n} |h(\mathbf{s})| \cdot \left\| F(\mathbf{t} + w + \mathbf{b}_{k_m} + \mathbf{u} - \mathbf{s}; x) - F(\mathbf{t} + w + \mathbf{b}_{k_{m'}} + \mathbf{u} - \mathbf{s}; x) \right\|_Y d\mathbf{s} \right)_{L^{p_1(\mathbf{u})}(l\Omega)},
\end{aligned}$$

which is equal to the infimum of all positive real numbers $\lambda > 0$ such that

$$\int_{l\Omega} \varphi_{p_1(\mathbf{u})} \left(\frac{\int_{\mathbb{R}^n} |h(\mathbf{s})| \cdot \left\| F(\mathbf{t} + w + \mathbf{b}_{k_m} + \mathbf{u} - \mathbf{s}; x) - F(\mathbf{t} + w + \mathbf{b}_{k_{m'}} + \mathbf{u} - \mathbf{s}; x) \right\|_Y d\mathbf{s}}{\lambda} \right) d\mathbf{u} \leq 1.$$

By definition of norm in $L^{p_1(\mathbf{u})}(l\Omega)$, it suffices to show that the last estimate holds with $\lambda = \epsilon / \mathbb{F}_1(l, \mathbf{t} + w)$. This follows from the next computation involving the Hölder inequality (see Lemma 1.1(i)) as well as our assumptions $W_2 \subseteq (2\mathbb{Z})^n$ and (3.7):

$$\begin{aligned}
&\int_{l\Omega} \varphi_{p_1(\mathbf{u})} \left(\frac{\int_{\mathbb{R}^n} |h(\mathbf{s})| \cdot \left\| F(\mathbf{t} + w + \mathbf{b}_{k_m} + \mathbf{u} - \mathbf{s}; x) - F(\mathbf{t} + w + \mathbf{b}_{k_{m'}} + \mathbf{u} - \mathbf{s}; x) \right\|_Y d\mathbf{s}}{\epsilon / \mathbb{F}_1(l, \mathbf{t} + w)} \right) d\mathbf{u} \\
&= \int_{l\Omega} \varphi_{p_1(\mathbf{u})} \left(\frac{\int_{\mathbb{R}^n} |h(\mathbf{s} + \mathbf{u})| \cdot \left\| F(\mathbf{t} + w + \mathbf{b}_{k_m} - \mathbf{s}; x) - F(\mathbf{t} + w + \mathbf{b}_{k_{m'}} - \mathbf{s}; x) \right\|_Y d\mathbf{s}}{\epsilon / \mathbb{F}_1(l, \mathbf{t} + w)} \right) d\mathbf{u} \\
&= \int_{l\Omega} \varphi_{p_1(\mathbf{u})} \left(\sum_{k \in l(2\mathbb{Z}+1)^n} \frac{\int_{k - l\Omega} |h(\mathbf{s} + \mathbf{u})| \cdot \left\| F(\mathbf{t} + w + \mathbf{b}_{k_m} - \mathbf{s}; x) - F(\mathbf{t} + w + \mathbf{b}_{k_{m'}} - \mathbf{s}; x) \right\|_Y d\mathbf{s}}{\epsilon / \mathbb{F}_1(l, \mathbf{t} + w)} \right) d\mathbf{u} \\
&\leq \int_{l\Omega} \varphi_{p_1(\mathbf{u})} \left(\sum_{k \in lW_1} \int_{l\Omega} \frac{|h(\mathbf{u} - \mathbf{v} + k)| \cdot \left\| F(\mathbf{t} + w + \mathbf{b}_{k_m} + \mathbf{v} - k; x) - F(\mathbf{t} + w + \mathbf{b}_{k_{m'}} + \mathbf{v} - k; x) \right\|_Y d\mathbf{v}}{\epsilon / \mathbb{F}_1(l, \mathbf{t} + w)} \right) d\mathbf{u} \\
&\leq \int_{l\Omega} \varphi_{p_1(\mathbf{u})} \left(2 \sum_{k \in lW_1} \frac{\left\| h(\mathbf{u} + k - \mathbf{v}) \right\|_{L^{q(\mathbf{v})}(l\Omega)} \cdot \big(\epsilon / \mathbb{F}(l, \mathbf{t} - k + w) \big)}{\epsilon / \mathbb{F}_1(l, \mathbf{t} + w)} \right) d\mathbf{u} \\
&\leq \int_{l\Omega} \varphi_{p_1(\mathbf{u})} \left(2F_1(l, \mathbf{t} + w) \sum_{k \in l(2\mathbb{Z}+1)^n} \frac{\left\| h(\mathbf{u} + k - \mathbf{v}) \right\|_{L^{q(\mathbf{v})}(l\Omega)}}{\mathbb{F}(l, \mathbf{t} - k + w)} \right) d\mathbf{u} \leq 1.
\end{aligned}$$

$\square$

Remark 3.10. In contrast with the one-dimensional case, the set $\{l \geq l_0 : \mathbf{t} - l(2\mathbb{Z}+1)^n\}$ cannot be bounded in $\mathbb{R}^n$ for any $\mathbf{t} \in \mathbb{R}^n$, as easily approved ($n \geq 2$). But, even in the one-dimensional setting, the requirements of Theorem 3.9 do not imply the equi-Weyl-p-almost periodicity of function under our consideration (see Example 2.13). However, it is not clear whether the requirements of Theorem 3.9 imply the Weyl-p-almost periodicity of considered function or not.

Concerning the invariance of Weyl $p_1(\mathbf{u})$-$(\mathbb{F}_1, \mathrm{R}, W)$-multi-almost automorphy of type 2 under the actions of infinite convolution products, we will only investigate the one-dimensional case for simplicity (the statements of Theorem 3.11 can be also formulated in the multi-dimensional setting, with minor complications; see [9]-[10] for more details):

Theorem 3.11. *Suppose that $p,\ q \in \mathcal{P}(\mathbb{R})$, $F : (0,\infty) \times X \to (0,\infty)$, $F_1 : (0,\infty) \times X \to (0,\infty)$, $\emptyset \neq W_2 \subseteq 2\mathbb{Z}$, $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family and $f : \mathbb{R} \to X$ is a measurable function such that*

$$\int_{-\infty}^{t} \|R(t-s)\| \cdot \|f(s)\| \, ds < \infty, \quad t \in \mathbb{R}.$$

If $f(\cdot)$ is Weyl $p(u)$-$(\mathbb{F}, \mathrm{R}, 2\mathbb{Z}+1)$-multi-almost automorphic of type 2 and condition

$$\int_{-l}^{l} \varphi_{p(u)}\Biggl(2\mathbb{F}_1(l,t-w)\Biggl[\sum_{m\in\mathbb{N}} \frac{\bigl\|R(-v+2ml-u)\bigr\|_{L^{q(v)}[-l,l]}}{\mathbb{F}(l,t-w-2ml)} + \frac{\bigl\|R(-v+u)\bigr\|_{L^{q(v)}[-l,l]}}{\mathbb{F}(l,t-w)}\Biggr]\Biggr) \, du \leq 1, \tag{3.9}$$

holds, then the function $F(\cdot)$, given by (2.7)*, is Weyl $p(u)$-$(\mathbb{F}_1, \mathrm{R}, W_2)$-multi-almost automorphic of type* 2.

Proof. It is clear that the function $F(\cdot)$, given by (2.7), is well defined since the integral defining this function is absolutely convergent. Let $(b_k) \in \mathrm{R}$. Then we know that there exists a subsequence (b_{k_m}) of (b_k) such that for each $\epsilon > 0$ and $t \in \mathbb{R}$ there exists $m_0 \in \mathbb{N}$ such that, for every $m,\ m' \in \mathbb{N}$ with $m \geq m_0$ and $m' \geq m_0$, there exists $l \geq l_0$ such that, for every $l \geq l_0$ and $w \in l(2\mathbb{Z}+1)$ (3.3) holds with $W \equiv W_1 \equiv (2\mathbb{Z}+1)$ and $n = 1$. Let $l \geq \max(l_0, l_1)$ and $w \in lW_2$. Then the final conclusion follows similarly as in Theorem 3.9, by using (3.9) and the next estimate:

$$\int_{-l}^{l} \varphi_{p(u)}\Biggl(\mathbb{F}_1(l,t-w) \int_{0}^{\infty} \|R(s)\| \times \bigl\|f(t+u+b_{k_m}-w-s) - f(t+u+b_{k_{m'}}-w-s)\bigr\|_Y \, ds\Biggr) \, du \leq 1.$$

In order to see the last estimate is valid, we first conclude that:

$$\begin{aligned}
&\int_{-l}^{l} \varphi_{p(u)}\Biggl(\mathbb{F}_1(l,t-w) \int_{0}^{\infty} \|R(s)\| \\
&\times \Bigl\|f(t+u+b_{k_m}-w-s) - f(t+u+b_{k_{m'}}-w-s)\Bigr\|_Y \, ds\Biggr) \, du \\
&= \int_{-l}^{l} \varphi_{p(u)}\Biggl(\mathbb{F}_1(l,t-w) \int_{-l}^{u} \|R(-s+u)\| \\
&\times \Bigl\|f(t+b_{k_m}-w+s) - f(t+b_{k_{m'}}-w+s)\Bigr\|_Y \, ds \\
&+ \sum_{m=1}^{\infty} \int_{-l}^{l} \bigl\|R(-v+2ml-u)\bigr\| \cdot \Bigl\|f(t+b_{k_m}-w-2ml+v) - f(t+b_{k_{m'}}-w-2ml+v)\Bigr\|_Y\Biggr) du.
\end{aligned}$$

After that, we can apply the Hölder inequality, our assumption on the function $f(\cdot)$ and the estimate (3.9). $\square$

It is worth noting that an analogue of Theorem 3.9 can be formulated for the following slight generalization of the class introduced in Definition 3.1(iii):

Definition 3.12. Suppose that $\emptyset \neq W \subseteq \mathbb{R}^n$ and $F : \mathbb{R}^n \times X \to Y$ satisfies that for each $x \in X$, $l > 0$ and $\mathbf{t} \in \mathbb{R}^n$ we have $F(\mathbf{t}+\mathbf{u}; x) \in L^{p(\mathbf{u})}(l\Omega : Y)$. Let for every $B \in \mathcal{B}$ and $(\mathbf{b}_k = (b_k^1, b_k^2, \cdot\cdot\cdot, b_k^n)) \in \mathrm{R}$ there exist a subsequence $(\mathbf{b}_{k_m} = (b_{k_m}^1, b_{k_m}^2, \cdot\cdot\cdot, b_{k_m}^n))$ of $(\mathbf{b}_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that for each $x \in B$, $l > 0$ and $\mathbf{t} \in \mathbb{R}^n$ we have $F^*(\mathbf{t}+\mathbf{u}; x) \in L^{p(\mathbf{u})}(l\Omega : Y)$, as well as for each $\epsilon > 0$, $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$, there exists $p > 0$ such that, for every $l \in [p, +\infty)$, $m \in \mathbb{N}$ with $m \geq p$ and $w \in lW$, we have

$$\mathbb{F}(l, \mathbf{t}-w)\Big\|F(\mathbf{t}+\mathbf{u}+(b_{k_m}^1, \cdot\cdot\cdot, b_{k_m}^n)-w; x) - F^*(\mathbf{t}+\mathbf{u}-w; x)\Big\|_{L^{p(\mathbf{u})}(l\Omega:Y)} < \epsilon \tag{3.10}$$

and

$$\mathbb{F}(l, \mathbf{t}-w)\Big\|F^*(\mathbf{t}+\mathbf{u}-(b_{k_m}^1, \cdot\cdot\cdot, b_{k_m}^n)-w; x) - F(\mathbf{t}+\mathbf{u}-w; x)\Big\|_{L^{p(\mathbf{u})}(l\Omega:Y)} < \epsilon, \tag{3.11}$$

then we say that the function $F(\cdot;\cdot)$ is jointly Weyl $(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, W)$-multi-almost automorphic.

It is clear that Lemma 1.1(ii) implies that any jointly Weyl $(\mathbb{F}_q, q(\mathbf{u}), \mathrm{R}, W)$-multi-almost automorphic function $F(\cdot;\cdot)$ is jointly Weyl $(\mathbb{F}_p, p(\mathbf{u}), \mathrm{R}, W)$-multi-almost automorphic, provided that $p,\ p' \in \mathcal{P}(\mathbb{R}^n)$, $1 \leq p \leq p'$ a.e. on $\mathbb{R}^n$ and $\mathbb{F}_p(l, \mathbf{t}) := (1+l^n)^{-1}\mathbb{F}_q(l, \mathbf{t})$ for $l > 0$ and $\mathbf{t} \in \mathbb{R}^n$. Furthermore, if we assume that for each sequence belonging to R any its subsequence belongs to R, then the jointly Weyl-$(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, W)$-multi-almost automorphic functions form a vector space with the usual operations (the same holds for all other classes of functions introduced in this section).

In actual fact, we have the following result:

Theorem 3.13. *Suppose that $h \in L^1(\mathbb{R}^n)$ and $F : \mathbb{R}^n \times X \to Y$ is jointly Weyl $p(\mathbf{u})$-$(\mathbb{F}, \mathrm{R}, \mathcal{B}, \mathbb{Z}^n)$-multi-almost automorphic. Let $p_1,\ q \in \mathcal{P}(\mathbb{R}^n)$, let $1/p(\mathbf{u}) + 1/q(\mathbf{u}) \equiv 1$, and let $\mathbb{F}_1 : (0,\infty) \times \mathbb{R}^n \to (0,\infty)$. Suppose that, for every $x \in X$, we have $\sup_{\mathbf{t}\in\mathbb{R}^n} \|F(\mathbf{t}; x)\|_Y < \infty$, as well as that for every $\mathbf{t} \in \mathbb{R}^n$ there exists $l_1 > 0$ such that, for every $l \geq l_1$ and $w \in l\mathbb{Z}^n$, the estimate* (3.7) *holds. If, for every compact set $K \subseteq \mathbb{R}^n$, $x \in X$ and $l > 0$, there exists a finite real constant $c > 0$ such that*

$$\big\|h(\mathbf{u}-\mathbf{v})\big\|_{L^{q(\mathbf{v})}(l\Omega)} \leq \left(\mathbb{F}(l,0)^{-1} + \sup_{\mathbf{t}\in\mathbb{R}^n}\big\|F(\mathbf{t}; x)\big\|_Y \cdot \big\|1\big\|_{L^{p(\mathbf{u})}(l\Omega)}\right)^{-1}, \quad \mathbf{u} \in K, \tag{3.12}$$

then $h * F : \mathbb{R}^n \times X \to Y$ (cf. (3.8)*), is a well defined, jointly Weyl $p_1(\mathbf{u})$-$(\mathbb{F}_1, \mathrm{R}, \mathcal{B}, \mathbb{Z}^n)$-multi-almost automorphic function.*

Proof. The proof of theorem is very similar to the proof of Theorem 3.9 and we will only emphasize the most important differences. First of all, it is clear that the function $h * F : \mathbb{R}^n \times X \to Y$ is well defined. Fix $B \in \mathcal{B}$ and $(\mathbf{b}_k = (b_k^1, b_k^2, \cdot\cdot\cdot, b_k^n)) \in \mathrm{R}$. Then there exists a subsequence $(\mathbf{b}_{k_m} = (b_{k_m}^1, b_{k_m}^2, \cdot\cdot\cdot, b_{k_m}^n))$ of $(\mathbf{b}_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that for each $x \in B$, $l > 0$ and $\mathbf{t} \in \mathbb{R}^n$ we have $F^*(\mathbf{t}+\mathbf{u}; x) \in L^{p(\mathbf{u})}(l\Omega : Y)$, as well as for each $\epsilon > 0$, $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$, there exists $p > 0$ such that, for every $l \in [p, +\infty)$, $m \in \mathbb{N}$ with $m \geq p$ and $w \in lW$, we have (3.10) and (3.11). Let $\mathbf{t} \in \mathbb{R}^n$ and $x \in B$ be fixed; we will prove that the value $(h * F^*)(\mathbf{t}; x)$ is well defined. It suffices to prove that

$$\int_{\mathbb{R}^n} |h(\mathbf{t}-\mathbf{s})| \big\| F^*(\mathbf{s}; x) \big\|_Y \, d\mathbf{s} := \lim_{l \to +\infty} \int_{l\Omega} |h(\mathbf{t}-\mathbf{s})| \big\| F^*(\mathbf{s}; x) \big\|_Y \, d\mathbf{s} < +\infty.$$

Since the mapping $l \mapsto \int_{l\Omega} |h(\mathbf{t}-\mathbf{s})| \|F^*(\mathbf{s}; x)\|_Y \, d\mathbf{s}$, $l > 0$ is monotonically increasing, it suffices to show its boundedness for $l > 0$. This follows from the fact that $F^*(\mathbf{u}; x) \in L^{p(\mathbf{u})}(l\Omega : Y)$ for all $l > 0$ (this is a consequence of (3.10) with $\mathbf{t} = \omega = 0$ and our assumption that, for every $x \in X$, we have $\sup_{\mathbf{t} \in \mathbb{R}^n} \|F(\mathbf{t}; x)\|_Y < \infty$), the Hölder inequality and the assumption that, for every $\mathbf{u} \in \mathbb{R}^n$ and $l > 0$, there exists a finite real constant $c > 0$ such that (3.12) holds. The remainder of proof can be given by copying the corresponding part of proof of Theorem 3.9. □

In order to relax our exposition, we will only note that an analogue of Theorem 3.11 can be formulated for jointly Weyl $p(u)$-$(\mathbb{F}, \mathrm{R}, W)$-multi-almost automorphic functions following the method proposed in the proofs of Theorem 3.11 and Theorem 3.13, by assuming condition of type (3.12) for the resolvent family $(R(t))_{t>0} \subseteq L(X, Y)$ under consideration. Details can be left to the readers.

Concerning the pointwise products of Weyl $p(\mathbf{u})$-$(\mathbb{F}, \mathrm{R}, \mathcal{B}, W)$-multi-almost automorphic functions of type 2 and jointly Weyl $p(\mathbf{u})$-$(\mathbb{F}, \mathrm{R}, \mathcal{B}, W)$-multi-almost automorphic functions with the scalar-valued functions of the same type, we will clarify the following result, only:

Proposition 3.14. *Assume that for each sequence belonging to* R *any its subsequence belongs to* R.

(i) Suppose that $\emptyset \neq W \subseteq \mathbb{R}^n$, $p \in \mathcal{P}(\mathbb{R}^n)$, $g : (0, \infty) \times \mathbb{R}^n \to (0, \infty)$, $\mathbb{F} : (0, \infty) \times \mathbb{R}^n \to (0, \infty)$, $f : \mathbb{R}^n \to \mathbb{C}$ *is essentially bounded and Weyl* $p(\mathbf{u})$-(g, R, W)*-multi-almost automorphic of type* 2*,* $F : \mathbb{R}^n \times X \to Y$ *is Weyl* $p(\mathbf{u})$-$(\mathbb{F}, \mathrm{R}, \mathcal{B}, W)$*-multi-almost automorphic of type* 2 *and for each* $x \in X$ *we have* $\sup_{\mathbf{t} \in \mathbb{R}^n} \|F(\mathbf{t}; x)\|_Y < \infty$. *Suppose that* $\mathbb{F}_1 : (0, \infty) \times \mathbb{R}^n \to (0, \infty)$ *satisfies that there exist real numbers* $c > 0$ *and* $l_0 > 0$ *such that*

$$\frac{1}{\mathbb{F}(l, \mathbf{t})} + \frac{1}{g(l, \mathbf{t})} \leq \frac{c}{\mathbb{F}_1(l, \mathbf{t})}, \quad l \geq l_0, \ \mathbf{t} \in \mathbb{R}^n. \tag{3.13}$$

Then the function $F_1(\mathbf{t}; x) := f(\mathbf{t}) \cdot F(\mathbf{t}; x), \quad \mathbf{t} \in \mathbb{R}^n, \ x \in X$ *is Weyl* $p(\mathbf{u})$-$(\mathbb{F}_1, \mathrm{R}, \mathcal{B}, W)$*-multi-almost automorphic of type* 2.

(ii) Suppose that $\emptyset \neq W \subseteq \mathbb{R}^n$, $p, \ q \in \mathcal{P}(\mathbb{R}^n)$, $1/p(\mathbf{u}) + 1/q(\mathbf{u}) \equiv 1$, $g : (0, \infty) \times \mathbb{R}^n \to (0, \infty)$, $\mathbb{F} : (0, \infty) \times \mathbb{R}^n \to (0, \infty)$, $f : \mathbb{R}^n \to \mathbb{C}$ *is essentially bounded and jointly Weyl* $p(\mathbf{u})$-(g, R, W)*-multi-almost automorphic,* $F : \mathbb{R}^n \times X \to Y$ *is jointly Weyl* $q(\mathbf{u})$-$(\mathbb{F}, \mathrm{R}, \mathcal{B}, W)$*-multi-almost automorphic and for each* $x \in X$ *we have* $\sup_{\mathbf{t} \in \mathbb{R}^n} \|F(\mathbf{t}; x)\|_Y < \infty$. *Suppose that* $\mathbb{F}_1 : (0, \infty) \times \mathbb{R}^n \to (0, \infty)$ *satisfies that for each* $x \in X$ *and* $\mathbf{t} \in \mathbb{R}^n$ *there exist real numbers* $c > 0$ *and* $l_0 > 0$ *such that, for every* $l \geq l_0$*, we have:*

$$\mathbb{F}_1(l, \mathbf{t}-w) \Bigg[\frac{1}{g(l, \mathbf{t}-w)} \sup_{\mathbf{t} \in \mathbb{R}^n} \big\| F(\mathbf{t}; x) \big\|_Y \big\| 1 \big\|_{L^{q(\mathbf{u})}(l\Omega)} + \frac{1}{\mathbb{F}(l, \mathbf{t}-w)} \Bigg(\frac{1}{g(l, \mathbf{t}-w)} + \|f\|_\infty \cdot \big\| 1 \big\|_{L^{p(\mathbf{u})}(l\Omega)} \Bigg) \Bigg] \leq c. \tag{3.14}$$

Then the function $F_1(\mathbf{t}; x) := f(\mathbf{t}) \cdot F(\mathbf{t}; x), \quad \mathbf{t} \in \mathbb{R}^n, \ x \in X$ *is jointlyWeyl* 1-$(\mathbb{F}_1, \mathrm{R}, \mathcal{B}, W)$*-multi-almost automorphic.*

Proof. Let $B \in \mathcal{B}$ and $(\mathbf{b}_k = (b_k^1, b_k^2, \cdot\cdot\cdot, b_k^n)) \in \mathrm{R}$ be given. Since we have assumed that, for every sequence which belongs to R, any its subsequence also belongs to R, by the corresponding definition we get the existence of a subsequence $(\mathbf{b}_{k_m} = (b_{k_m}^1, b_{k_m}^2, \cdot\cdot\cdot, b_{k_m}^n))$ of $(\mathbf{b}_k)$ such that for each $\epsilon > 0$, $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$ there exists $m_0 \in \mathbb{N}$ such that, for every $m,\ m' \in \mathbb{N}$ with $m \geq m_0$ and $m' \geq m_0$, there exists $l \geq l_0$ such that, for every $l \geq l_0$ and $w \in lW$, we have

$$\Big\| f\Big(\mathbf{t} + \mathbf{u} + (b_{k_m}^1, \cdot\cdot\cdot, b_{k_m}^n) - w\Big) - f\Big(\mathbf{t} + \mathbf{u} + (b_{k_{m'}}^1, \cdot\cdot\cdot, b_{k_{m'}}^n) - w\Big)\Big\|_{L^{p(\mathbf{u})}(l\Omega:Y)} < \epsilon/g(l, \mathbf{t} - w), \tag{3.15}$$

and (3.3). Since we have assumed that the function $f(\cdot)$ is essentially bounded as well as that for each $x \in X$ we have $\sup_{\mathbf{t}\in\mathbb{R}^n} \|F(\mathbf{t}; x)\|_Y < \infty$, the estimates (3.3), (3.13)-(3.15) and the decomposition

$$\begin{aligned}
&F_1\Big(\mathbf{t} + \mathbf{u} + (b_{k_m}^1, \cdot\cdot\cdot, b_{k_m}^n) - w; x\Big) - F_1\Big(\mathbf{t} + \mathbf{u} + (b_{k_{m'}}^1, \cdot\cdot\cdot, b_{k_{m'}}^n) - w\Big) \\
&= f\Big(\mathbf{t} + \mathbf{u} + (b_{k_m}^1, \cdot\cdot\cdot, b_{k_m}^n) - w\Big) \\
&\times \Big[F\Big(\mathbf{t} + \mathbf{u} + (b_{k_m}^1, \cdot\cdot\cdot, b_{k_m}^n) - w; x\Big) - F\Big(\mathbf{t} + \mathbf{u} + (b_{k_{m'}}^1, \cdot\cdot\cdot, b_{k_{m'}}^n) - w; x\Big)\Big] \\
&+ F\Big(\mathbf{t} + \mathbf{u} + (b_{k_{m'}}^1, \cdot\cdot\cdot, b_{k_{m'}}^n) - w; x\Big) \\
&\times \Big[f\Big(\mathbf{t} + \mathbf{u} + (b_{k_m}^1, \cdot\cdot\cdot, b_{k_m}^n) - w\Big) - f\Big(\mathbf{t} + \mathbf{u} + (b_{k_{m'}}^1, \cdot\cdot\cdot, b_{k_{m'}}^n) - w\Big)\Big]
\end{aligned}$$

simply imply that $F_1(\cdot;\cdot)$ is Weyl $p(\mathbf{u})$-$(\mathbb{F}_1, \mathrm{R}, \mathcal{B}, W)$-multi-almost automorphic of type 2. The second part of proposition follows from a similar decomposition with limit functions, by applying the Hölder inequality, the estimate (3.14) and a simple estimate for the function $f^*(\cdot)$ obtained from (3.10). □

The interested reader may try to reformulate the statement of [19, Theorem 3.4] for Weyl $p(\mathbf{u})$-$(\mathbb{F}, \mathrm{R}, \mathcal{B}, W)$-multi-almost automorphic functions of type 2 and jointly Weyl $p(\mathbf{u})$-$(\mathbb{F}, \mathrm{R}, \mathcal{B}, W)$-multi-almost automorphic functions.

Finally, we would like to note that the inclusions presented in Section 2 and this section continue to hold in the multi-dimensional setting, for Stepanov $p(\mathbf{u})$-$(\mathrm{R}, \mathcal{B})$-multi-almost automorphic functions ([32], work-in-progress) and various classes of multi-dimensional Weyl almost automorphic functions considered here; details will appear in [25].

4. Applications to the abstract Volterra integro-differential equations

In this section, we will present some applications of the obtained theoretical results in the qualitative analysis of solutions for various classes of the abstract Volterra integro-differential equations in Banach spaces.

1. Besides many other applications, we would like to note that Proposition 2.8 takes effect in the qualitative analysis of jointly Weyl-1-almost automorphic solutions of the following fractional Poisson heat equation $D_{t,+}^{\gamma}[m(x)v(t,x)] = (\Delta - b)v(t,x) + f(t,x)$, $t \in \mathbb{R}$, $x \in \Omega$; $v(t,x) = 0$, $v(t,x) \in [0,\infty) \times \partial\Omega$ in the space $X := L^p(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$, $b > 0$, $m(x) \geq 0$ a.e. $x \in \Omega$, $m \in L^\infty(\Omega)$, $\gamma \in (0,1)$ and $1 < p < \infty$; in the case of consideration of general exponent $p \in \mathcal{P}(\mathbb{R})$, we can also apply Theorem 3.11. See [24] for more details.

2. Let Y be one of the spaces $L^p(\mathbb{R}^n)$, $C_0(\mathbb{R}^n)$ or $BUC(\mathbb{R}^n)$, where $1 \leq p < \infty$. It is well known that the Gaussian semigroup

$$(G(t)F)(x) := \bigl(4\pi t\bigr)^{-(n/2)} \int\limits_{\mathbb{R}^n} F(x-y)e^{-\frac{|y|^2}{4t}}\, dy, \quad t>0,\ f \in Y,\ x \in \mathbb{R}^n,$$

can be extended to a bounded analytic C_0-semigroup of angle $\pi/2$, generated by the Laplacian Δ_Y acting with its maximal distributional domain in Y; see [4, Example 3.7.6] for more details. Suppose now that the number $t_0 > 0$ is fixed as well as that $F : \mathbb{R}^n \rightarrow \mathbb{C}$ is both essentially bounded and Weyl p-$(\mathbb{F}, \mathrm{R}, (2\mathbb{Z}+1)^n)$-multi-almost automorphic of type 2, where $p(\mathbf{u}) \equiv p \in [1,\infty)$ and $\mathbb{F} : (0,\infty) \rightarrow (0,\infty)$. Let $p_1 \in [1,\infty)$, let $1/p + 1/q = 1$, and let $\mathbb{F}_1 : (0,\infty) \rightarrow (0,\infty)$. Then the function $x \mapsto (G(t_0)F)(x)$, $x \in \mathbb{R}^n$ is essentially bounded. Suppose, further, that $\emptyset \neq W_2 \subseteq (2\mathbb{Z})^n$ and for every $\mathbf{t} \in \mathbb{R}^n$ there exists $l_1 > 0$ such that, for every $l \geq l_1$ and $w \in lW_2$, we have

$$\bigl(4\pi t_0\bigr)^{-(n/2)} \left(\frac{\mathbb{F}_1(l)}{\mathbb{F}(l)}\right)^{p_1} \int\limits_{l\Omega} \left[\sum_{k \in l(2\mathbb{Z}+1)^n} \left(\int\limits_{l\Omega} e^{-\frac{q|(\mathbf{u}+k-\mathbf{v})|^2}{4t_0}}\, d\mathbf{v} \right)^{1/q} \right]^{p_1} d\mathbf{u} \leq 1. \tag{4.1}$$

Then Theorem 3.9 implies that the function $x \mapsto (G(t_0)F)(x)$, $x \in \mathbb{R}^n$ is Weyl p_1-$(\mathbb{F}_1, \mathrm{R}, W_2)$-multi-almost automorphic of type 2. Here we only want to note that the series in (4.1) converges because, for every $\mathbf{u} \in l\Omega$, $\mathbf{v} \in l\Omega$ and $k \in l(2\mathbb{Z}+1)^n$, we have $|\mathbf{u}+k-\mathbf{v}| \geq |k - 2l\sqrt{n}|$. Note that Theorem 3.13 is also applicable here.

3. (cf. also the third application in [19, Section 4]) Suppose that $Y := L^r(\mathbb{R}^n)$ for some $r \in [1,\infty)$ and $A(t) := \Delta + a(t)I$, $t \geq 0$, where Δ is the Dirichlet Laplacian on $L^r(\mathbb{R}^n)$, I is the identity operator on $L^r(\mathbb{R}^n)$ and $a \in L^\infty([0,\infty))$. Then the evolution system $(U(t,s))_{t \geq s \geq 0} \subseteq L(Y)$ generated by the family $(A(t))_{t\geq 0}$ exists and is given by $U(t,t) := I$ for all $t \geq 0$ and

$$[U(t,s)F](\mathbf{u}) := \int\limits_{\mathbb{R}^n} K(t,s,\mathbf{u},\mathbf{v})F(\mathbf{v})\, d\mathbf{v}, \quad F \in L^r(\mathbb{R}^n), \quad t > s \geq 0,$$

where

$$K(t,s,\mathbf{u},\mathbf{v}) := (4\pi(t-s))^{-\frac{n}{2}} e^{\int_s^t a(\tau)\, d\tau} \exp\left(-\frac{|\mathbf{u}-\mathbf{v}|^2}{4(t-s)}\right), \quad t>s,\ \mathbf{u},\ \mathbf{v} \in \mathbb{R}^n.$$

Under certain assumptions, a unique mild solution of the abstract Cauchy problem $(\partial/\partial t)u(t,x) = A(t)u(t,x)$, $t>0$; $u(0,x) = F(x)$ is given by $u(t,x) := [U(t,0)F](x)$, $t \geq 0$, $x \in \mathbb{R}^n$. Suppose now that $F \in L^r(\mathbb{R}^n)$ and $F(\cdot)$ is Weyl p-$(\mathbb{F}, \mathrm{R}, (2\mathbb{Z}+1)^n)$-multi-almost automorphic of type 2, where $1 \leq p < \infty$ and the function $\mathbb{F}(l,\mathbf{t}) \equiv \mathbb{F}(l)$ does not depend on $\mathbf{t}$. Let $1/p+1/q=1$, let $\epsilon > 0$ be given, and let $(\mathbf{b}_k = (b_k^1, b_k^2, \cdots, b_k^n)) \in \mathrm{R}$. Then we know that there exists a subsequence $(\mathbf{b}_{k_m} = (b_{k_m}^1, b_{k_m}^2, \cdots, b_{k_m}^n))$ of $(\mathbf{b}_k)$ such that for each $\epsilon > 0$ and $\mathbf{t} \in \mathbb{R}^n$ there exists $m_0 \in \mathbb{N}$ such that, for every $m,\ m' \in \mathbb{N}$ with $m \geq m_0$ and $m' \geq m_0$, there exists $l_0 > 0$ such that, for every $l \geq l_0$ and $w \in l(2\mathbb{Z})^n$, we have (3.3). Let a number $t_0 > 0$ be fixed. Arguing as in [19], we get that there exists a finite constant $c_{t_0} > 0$ such that:

$$\begin{aligned} &\left| u\Bigl(t_0, \mathbf{t}+\mathbf{u}+(b_{k_m}^1, \cdots, b_{k_m}^n) - w\Bigr) - u\Bigl(t_0, \mathbf{t}+\mathbf{u}+(b_{k_{m'}}^1, \cdots, b_{k_{m'}}^n) - w\Bigr) \right| \\ &\leq c_{t_0} \int\limits_{\mathbb{R}^n} e^{-\frac{|\mathbf{u}-\mathbf{v}|^2}{4t_0}} \Bigl| F\bigl(\mathbf{v}+\mathbf{t}+(b_{k_m}^1, \cdots, b_{k_m}^n) - w\bigr) - F\bigl(\mathbf{v}+\mathbf{t}+(b_{k_{m'}}^1, \cdots, b_{k_{m'}}^n) - w\bigr) \Bigr|\, d\mathbf{v} \end{aligned}$$

$$
\begin{aligned}
&= c_{t_0} \sum_{k\in l(2\mathbb{Z}+1)^n} \int_{k+l[-1,1]^n} e^{-\frac{|\mathbf{u}-\mathbf{v}|^2}{4t_0}} \\
&\times \Big| F\big(\mathbf{v}+\mathbf{t}+(b^1_{k_m},\cdot\cdot\cdot,b^n_{k_m})-w\big) - F\big(\mathbf{v}+\mathbf{t}+(b^1_{k_{m'}},\cdot\cdot\cdot,b^n_{k_{m'}})-w\big)\Big|\, d\mathbf{v} \\
&\leq c_{t_0} \sum_{k\in l\mathbb{Z}^n} \Big\| e^{-\frac{|\mathbf{u}-\cdot|^2}{4t_0}} \Big\|_{L^q(k+l[-1,1]^n)} \\
&\times \Big\| F\big(\cdot+\mathbf{t}+(b^1_{k_m},\cdot\cdot\cdot,b^n_{k_m})-w-k\big) - F\big(\cdot+\mathbf{t}+(b^1_{k_{m'}},\cdot\cdot\cdot,b^n_{k_{m'}})-w-k\big)\Big\|_{L^p(l[-1,1]^n)} \\
&\leq c_{t_0} \frac{\epsilon}{\mathbb{F}(l)} \sum_{k\in l\mathbb{Z}^n} \Big\| e^{-\frac{|\mathbf{u}-\cdot|^2}{4t_0}} \Big\|_{L^q(k+l[-1,1]^n)} := c_{t_0} \frac{\epsilon}{\mathbb{F}(l)} G(l,\mathbf{u}).
\end{aligned}
$$

Let $1 \leq p' < \infty$. Define the function $\mathbb{F}_1(\cdot)$ by

$$
\mathbb{F}_1(l,\mathbf{t}) := \frac{\mathbb{F}(l)}{\Big(\int_{l[-1,1]^n} G(l,\mathbf{u})^{p'}\, d\mathbf{u}\Big)^{1/p'}}, \quad l > 0.
$$

By the foregoing, we get that the mapping $x \mapsto u(t_0,x)$, $x \in \mathbb{R}^n$ is Weyl p'-$(\mathbb{F}_1, \mathrm{R}, (2\mathbb{Z})^n)$-multi-almost automorphic of type 2.

5. Conclusions and the final remarks

In this paper, we have reconsidered the notion of Weyl p-almost automorphy introduced by S. Abbas [1] in 2012 and proposed several new concepts of Weyl p-almost automorphy: the Weyl p-almost automorphy of type 1, the Weyl p-almost automorphy of type 2 and the joint Weyl p-almost automorphy $(1 \leq p < \infty)$. Furthermore, we have introduced and analyzed the multi-dimensional analogues of these concepts by using the definitions and results from the theory of Lebesgue spaces with variable exponents. Several illustrative examples, open problems and applications are exhibited, as well; Question 2.3, Question 2.9, Question 2.16, Question 2.17 and the following

Question 5.1. *Let $I = \mathbb{R}$ and $p \geq 1$. Does there exist a Weyl p-almost automorphic function of type 1 which is not Weyl p-almost automorphic?*

Remain open after this study. The results of Section 3, which are formulated for the functions of the form $F : \mathbb{R}^n \times X \to Y$, will be the basis of our further investigations of composition principles for Weyl almost automorphic type functions and related abstract semilinear Cauchy problems.

Concerning the invariance of generalized almost periodicity and automorphy under the action of infinite convolution product (2.7), we would like to note that the notion of equi-Weyl-p-normality (see Subsection 2.1) can be also modified following the approach obeyed in this paper; for example, in the one-dimensional setting, we can analyze the following notion:

1. A p-locally integrable function $f : \mathbb{R} \to X$ is said to be equi-Weyl-p-normal of type 1 if and only if for any real sequence (s_n) there exist a subsequence (s_{n_k}) of (s_n) and a p-locally integrable function $f^* : \mathbb{R} \to X$ such that

$$
\lim_{l\to+\infty} \lim_{k\to+\infty} \sup_{t\in\mathbb{R}} \frac{1}{2l} \int_{-l}^{l} \big\| f\big(t+x+s_{n_k}\big) - f^*(t+x)\big\|^p\, dx = 0.
$$

2. A p-locally integrable function $f : \mathbb{R} \to X$ is said to be jointly equi-Weyl-p-normal if and only if for any real sequence (s_n) there exist a subsequence (s_{n_k}) of (s_n) and a p-locally integrable function $f^* : \mathbb{R} \to X$ such that

$$\lim_{(k,l)\to+\infty} \sup_{t\in\mathbb{R}} \frac{1}{2l} \int_{-l}^{l} \left\| f\left(t + x + s_{n_k}\right) - f^*(t + x) \right\|^p dx = 0.$$

Then it is possible to state some results about the invariance of Weyl p-almost normality and jointly Weyl p-almost normality under the actions of convolution products, like [27, Proposition 7]. It is also worth noting that the characteristic function of any fixed compact subset of $\mathbb{R}$ is jointly equi-Weyl-p-normal, with the limit function $f^* \equiv 0$.

The class of Besicovitch almost automorphic functions has been analyzed by F. Bedouhene, N. Challali, O. Mellah, P. Raynaud de Fitte and M. Smaali in [5]. This class, which extends the class of Weyl p-almost automorphic functions, is defined in the following way: Let $p \geq 1$. Then we say that a function $f \in L^p_{loc}(\mathbb{R} : X)$ is Besicovitch p-almost automorphic if and only if for every real sequence (s_n), there exist a subsequence (s_{n_k}) and a function $f^* \in L^p_{loc}(\mathbb{R} : X)$ such that

$$\lim_{k\to\infty} \limsup_{l\to+\infty} \frac{1}{2l} \int_{-l}^{l} \left\| f\left(t + s_{n_k} + x\right) - f^*(t + x) \right\|^p dx = 0$$

and

$$\lim_{k\to\infty} \limsup_{l\to+\infty} \frac{1}{2l} \int_{-l}^{l} \left\| f^*\left(t - s_{n_k} + x\right) - f(t + x) \right\|^p dx = 0$$

for each $t \in \mathbb{R}$. Without going into any further details, we would like to note that the above notion can be further extended by replacing the $\limsup \cdot$ in the above definition with $\liminf \cdot$:

Definition 5.2. Let $p \geq 1$. Then we say that a function $f \in L^p_{loc}(\mathbb{R} : X)$ is weakly Besicovitch p-almost automorphic if and only if for every real sequence (s_n), there exist a subsequence (s_{n_k}) and a function $f^* \in L^p_{loc}(\mathbb{R} : X)$ such that

$$\lim_{k\to\infty} \liminf_{l\to+\infty} \frac{1}{2l} \int_{-l}^{l} \left\| f\left(t + s_{n_k} + x\right) - f^*(t + x) \right\|^p dx = 0$$

and

$$\lim_{k\to\infty} \liminf_{l\to+\infty} \frac{1}{2l} \int_{-l}^{l} \left\| f^*\left(t - s_{n_k} + x\right) - f(t + x) \right\|^p dx = 0$$

for each $t \in \mathbb{R}$.

The interested reader may try to illustrate the hierarchy of the functions introduced by means of a diagram or a figure, allowing to specify the link between the functions introduced and those existing in the literature, and by exploiting the examples of this article. Multi-dimensional analogues of (weak)

Besicovitch almost automorphic type functions can be also introduced and analyzed but it will be done somewhere else. For more details about multi-dimensional almost automorphic functions, Stepanov multi-dimensional almost automorphic functions and their applications, we refer the reader to the forthcoming research monograph [25].

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