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Characterization of some closed linear subspaces of Morrey spaces and approximation

Caractérisation de sous-espaces vectoriels fermés des espaces de Morrey et approximation

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ABSTRACT. Let $1 \leq q \leq \alpha < \infty$. $\left\{(L^q, l^p)^\alpha(\mathbb{R}^d) : \alpha \leq p \leq \infty\right\}$ is a nondecreasing family of Banach spaces such that the Lebesgue space $L^\alpha(\mathbb{R}^d)$ is its minimal element and the classical Morrey space $\mathcal{M}_q^\alpha(\mathbb{R}^d)$ is its maximal element. In this note we investigate some closed linear subspaces of $(L^q, l^p)^\alpha(\mathbb{R}^d)$. We give a characterization of the closure in $(L^q, l^p)^\alpha(\mathbb{R}^d)$ of the set of all its compactly supported elements and study the action of some classical operators on it. We also describe the closure in $(L^q, l^p)^\alpha(\mathbb{R}^d)$ of the set $\mathcal{C}_c^\infty(\mathbb{R}^d)$ of all infinitely differentiable and compactly supported functions on $\mathbb{R}^d$ as an intersection of other linear subspaces of $(L^q, l^p)^\alpha(\mathbb{R}^d)$ and obtain the weak density of $\mathcal{C}_c^\infty(\mathbb{R}^d)$ in some of these subspaces. We establish a necessary condition on a function f in order that its Riesz potential $I_\gamma(|f|)$ $(0 < \gamma < 1)$ be in a given Lebesgue space.

KEYWORDS. Morrey spaces, Closed linear subspaces, Approximation, Adams-Spanne type theorem, Riesz potential, Fractional maximal operator

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1. Introduction

Let d be a fixed positive integer and consider the d-dimensional Euclidean space $\mathbb{R}^d$ equipped with its Lebesgue measure $E \mapsto |E|$. The euclidean norm of an element x of $\mathbb{R}^d$ is denoted by $|x|$.

For $1 \leq \alpha \leq \infty$, L^α denotes the classical Lebesgue space on $\mathbb{R}^d$, endowed with its usual norm $\|\cdot\|_\alpha$.

For any element f of L^p $(1 \leq p \leq 2)$, we denote by $\widehat{f}$ its Fourier transform. In [2] Aguilera and Harboure have proved that if $1 < p < 2$ then, in order that a nonnegative function u on $\mathbb{R}$ satisfies the condition

$$\int_{\mathbb{R}} \left|\widehat{f}(x)\right|^p u(x)dx \leq K \int_{\mathbb{R}} |f(x)|^p dx \qquad , \quad f \in L^p$$

for some real constant K, it is

- necessary that

$$\left[\sum_{k\in\mathbb{Z}} \left(\int_{kr}^{(k+1)r} u(x)dx\right)^{\frac{2}{2-p}}\right]^{\frac{2-p}{2}} \leq C\, r^{p-1} \qquad , \quad r > 0$$

for some real constant C

- sufficient that

$$\sup_{\lambda>0} \lambda \left|\{x \in \mathbb{R} : u(x) > \lambda\}\right|^{\frac{2-p}{2}} < \infty.$$

In order to get a good insight in this result, Fofana has introduced in [14] a family of Banach spaces denoted by $(L^q, l^p)^\alpha$ $(1 \leq q, p, \alpha \leq \infty)$. Let us introduce some notations in order to recall their definitions.

Notation 1.1. *Let r be any positive real number.*

$$(i) \qquad I_k^r = \prod_{j=1}^{d} [k_j r, (k_j+1)r) \ , \quad k = (k_1, k_2, ..., k_d) \in \mathbb{Z}^d.$$

(ii) *If (q,p) is an element of $[1,\infty]^2$ and f belongs to the space L^1_{loc} of locally integrable functions on $\mathbb{R}^d$, then*

$$_r\|f\|_{q,p} = \begin{cases} \left(\sum\limits_{k\in\mathbb{Z}^d} \|f\chi_{I_k^r}\|_q^p\right)^{\frac{1}{p}} & \textit{if } \ p < \infty \\ \sup\limits_{k\in\mathbb{Z}^d} \|f\chi_{I_k^r}\|_q & \textit{if } \ p = \infty \end{cases}$$

where for any subset E of $\mathbb{R}^d$, χ_E stands for its characteristic function.

Definition 1.2. *Let (q,p,α) be an element of $[1,\infty]^3$.*

$$(L^q, l^p)^\alpha = \{f \in L^1_{\text{loc}} \ : \ \|f\|_{q,p,\alpha} < \infty\}$$

where

$$\|f\|_{q,p,\alpha} := \sup_{r>0} \ r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)} \ {}_r\|f\|_{q,p} \ .$$

Let us notice that, for (q,p,α) in $[1,\infty]^3$, $(L^q, l^p)^\alpha$ is a linear subspace of L^1_{loc} and when endowed with $f \mapsto \|f\|_{q,p,\alpha}$, a Banach space. It is also known that, for $1 \leq q \leq \alpha \leq p \leq \infty$, the space $(L^q, l^p)^\alpha$ is a linear subspace of the Morrey space $\mathcal{M}_q^\alpha$ which is defined as the set of all elements f of L^1_{loc} for which the norm

$$\|f\|_{\mathcal{M}_q^\alpha} := \sup_{x\in\mathbb{R}^d, \, r>0} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)} \left\|f\chi_{Q(x,r)}\right\|_q$$

is finite, where

$$Q(x,r) = \prod_{j=1}^{d} \left[x_j - \frac{r}{2}, x_j + \frac{r}{2}\right) \ , \qquad x = (x_1, x_2, ..., x_d) \in \mathbb{R}^d \ \text{ and } \ 0 < r < \infty.$$

Let us recall that the Morrey spaces were introduced in 1938 by C. B. Morrey (see [25]) and are used in connection with regularity problems of solutions to partial differential equations. In recent years, there is an increasing interest in applications and generalizations of these spaces (see [1, 27, 28] and the references therein).

It is worth noticing that the following continuous inclusion and equality relations hold :

$$L^\alpha = (L^q, l^\alpha)^\alpha \subset (L^q, l^{p_1})^\alpha \subset (L^q, l^{p_2})^\alpha \subset (L^q, l^\infty)^\alpha = \mathcal{M}_q^\alpha, \quad 1 \leq q \leq \alpha \leq p_1 \leq p_2 \leq \infty$$
$$(L^q, l^p)^\alpha = \{0\} \text{ if } \alpha \notin [q,p].$$

From now on we shall consider $(L^q, l^p)^\alpha$ only with $1 \leq q \leq \alpha \leq p \leq \infty$.

Many results in Fourier analysis (about the Fourier transform, fractional maximal operators, Riesz potential oprators and Calderon-Zygmund operators), well-known and widely used in Lebesgue or Morrey spaces, have been extended in the framework of the spaces $(L^q, l^p)^\alpha$ (see [8, 10, 11, 12, 13, 14, 15, 16, 23] and the references therein). However, some very useful properties of the Lebesgue spaces, linked to approximation by nice functions, are not shared by the spaces $(L^q, l^p)^\alpha$. For example it is known that:

1) if $1 \leq \alpha < \infty$ and f is an element of L^α then :

(i) $\lim\limits_{R\to\infty} \left\| f\chi_{\mathbb{R}^d \setminus Q(0,R)} \right\|_\alpha = 0$

(ii) $\lim\limits_{|E|\to 0} \| f\chi_E \|_\alpha = 0$

(iii) $u \mapsto \tau_u f = f(\cdot - u)$ is continuous from $\mathbb{R}^d$ to L^α

2) if $1 \leq q < \alpha < p \leq \infty$ then $(L^q, l^p)^\alpha$ has elements which do not satisfy properties similar to (i), (ii) and (iii).

Because of this fact various distinguished closed linear subspaces of $(L^q, l^p)^\alpha$ have been considered (see [4, 6, 22, 31, 32] for $p = \infty$ and [11, 14] when $\alpha \leq p \leq \infty$).

The purpose of the present paper is to contribute to a better understanding of some of these subspaces, mainly those defined below.

Definition 1.3.

$$(i) \quad (L^q, l^p)^\alpha_0 = \left\{ f \in (L^q, l^p)^\alpha \ : \ \lim_{R\to\infty} \| f\chi_{\mathbb{R}^d \setminus Q(0,R)} \|_{q,p,\alpha} = 0 \right\}$$
$$(ii) \quad (L^q, l^p)^\alpha_{\mathrm{c}} = \left\{ f \in (L^q, l^p)^\alpha \ : \ \lim_{u\to 0} \| f - \tau_u f \|_{q,p,\alpha} = 0 \right\}$$
$$(iii) \quad (L^q, l^p)^\alpha_{\mathrm{c},0} = (L^q, l^p)^\alpha_{\mathrm{c}} \cap (L^q, l^p)^\alpha_0.$$

Our main results may be summarized as follows :

1) a characterization of $(L^q, l^p)^\alpha_0$ from which we deduce that this subspace of $(L^q, l^p)^\alpha$ is invariant by the Hardy-Littlewood maximal operator and obtain for it an Adams and Spanne type theorem;

2) the description of $(L^q, l^p)^\alpha_{\mathrm{c}}$ as the closure in $(L^q, l^p)^\alpha$ of the set of all infinitely differentiable functions on $\mathbb{R}^d$ having all their derivatives in $L^p \cap (L^q, l^p)^\alpha$;

3) several descriptions of $(L^q, l^p)^\alpha_{\mathrm{c},0}$ as intersection of other subspaces of $(L^q, l^p)^\alpha$ and, as a consequence, the weak-density of the set $\mathcal{C}^\infty_{\mathrm{c}}$ (of all infinitely differentiable and compactly supported functions on $\mathbb{R}^d$) in a subspace of $(L^q, l^p)^\alpha$ larger than $(L^q, l^p)^\alpha_{\mathrm{c},0}$;

4) a proof of the inclusion in $(L^1, l^p)^\alpha_{\mathrm{c},0}$ of the set of all locally integrable functions f on $\mathbb{R}^d$ for which the Riesz potential $I_\gamma(|f|)$ $\left(\gamma = \frac{1}{\alpha} - \frac{1}{p} > 0\right)$ belongs to L^p.

It should be noted that the last point of our results provides a necessary condition on f for the solvability in $(L^p)^d$ of the equation div $u = f$ (see [26]).

The remainder of the paper is organized as follows. Section 2 contains a more detailed presentation of our results. In Section 3 we collect some basic results on Wiener amalgam spaces and $(L^q, l^p)^\alpha$. Sections 4, 5 and 6 deal with $(L^q, l^p)^\alpha_0$, $(L^q, l^p)^\alpha_{\mathrm{c}}$ and $(L^q, l^p)^\alpha_{\mathrm{c},0}$ respectively. Section 7 is devoted to the inverse image of L^p by the Riesz potential I_γ.

2. Statement of the main results

In this section we assume that $1 \leq q \leq \alpha \leq p \leq \infty$ unless otherwise specified.

Let us begin by noticing that $(L^q, l^p)^\alpha_0$ has been considered in [11] where it is proved that it is a closed linear subspace of $(L^q, l^p)^\alpha$.

Our first result reads as follows.

Theorem 2.1. *If $\alpha < \infty$ then an element f of $(L^q, l^p)^\alpha$ belongs to $(L^q, l^p)^\alpha_0$ if and only if*

$$\lim_{\rho\to\infty} \left\{ \sup_{r>0} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)} \left\| \left\| f\chi_{Q(\cdot\,,r)} \right\|_q \chi_{\mathbb{R}^d\setminus Q(0,\rho)} \right\|_p \right\} = 0\,. \tag{2.1}$$

The above characterization of $(L^q, l^p)^\alpha_0$ extends and improves on the following result obtained in [31] : if $q < \alpha < \infty$ then, for any f in $(L^q, l^\infty)^\alpha$,

$$f \in (L^q, l^\infty)^\alpha_0 \Longleftrightarrow \left\{ \begin{array}{l} \lim\limits_{\rho\to\infty} \left\{ \sup\limits_{r>0} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)} \left\| \left\| f\chi_{Q(\cdot\,,r)} \right\|_q \chi_{\mathbb{R}^d\setminus Q(0,\rho)} \right\|_\infty \right\} = 0 \\ \\ f \in V_\infty(L^q, l^\infty)^\alpha \end{array} \right. \tag{2.2}$$

where

$$V_\infty(L^q, l^p)^\alpha = \left\{ f \in (L^q, l^p)^\alpha \ : \ \lim_{r\to\infty} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)} {}_r\|f\|_{q,p} = 0 \right\}.$$

Notice that, in [31] and several subsequent papers, $(L^q, l^\infty)^\alpha_0$ is denoted by $\overset{*}{\mathcal{M}}{}^\alpha_q$. We recall also that $V_\infty(L^q, l^\infty)^\alpha$ has been introduced in [4] where it is denoted by $V_\infty L^{q,d\left(1-\frac{q}{\alpha}\right)}$.

The equivalence relation (2.2) implies that $(L^q, l^\infty)^\alpha_0$ is included in $V_\infty(L^q, l^\infty)^\alpha$. This inclusion is extended and refined by our next result which gives some relationships between $(L^q, l^p)^\alpha_0$ and other linear subspaces of $(L^q, l^p)^\alpha$.

Theorem 2.2. *Assume that $q < \alpha < \infty$. We have*

$$(L^q, l^p)^\alpha_0 \subset \overline{L^q \cap (L^q, l^p)^\alpha} \subset V_\infty(L^q, l^p)^\alpha \cap (L^q, l^p)^\alpha_\omega \tag{2.3}$$

where $\overline{L^q \cap (L^q, l^p)^\alpha}$ is the closure in $(L^q, l^p)^\alpha$ of $L^q \cap (L^q, l^p)^\alpha$ and

$$(L^q, l^p)^\alpha_\omega = \left\{ f \in (L^q, l^p)^\alpha \ : \ \lim_{m\to 0} \left\| f\chi_{\{|f|\leq m\}} \right\|_{q,p,\alpha} = 0 \right\}.$$

It is clear that $(L^q, l^p)^\alpha_0$ is the closure in $(L^q, l^p)^\alpha$ of the set of all elements of $(L^q, l^p)^\alpha$ whose supports are bounded (with respect to the Euclidean norm on $\mathbb{R}^d$). $(L^q, l^p)^\alpha_\omega$ may be described in a similar way, as follows.

Theorem 2.3. *$(L^q, l^p)^\alpha_\omega$ is the closure in $(L^q, l^p)^\alpha$ of the set*

$$(L^q, l^p)^\alpha_{\mathrm{FNS}} = \left\{ f \in (L^q, l^p)^\alpha \ : \ \left\| f\chi_{\mathbb{R}^d \setminus E} \right\|_{q,p,\alpha} = 0 \ \textit{for some subset } E \textit{ of } \mathbb{R}^d \textit{ satisfying } \|\chi_E\|_{q,p,\alpha} < \infty \right\}$$

of all elements of $(L^q, l^p)^\alpha$ with support of finite norm.

As an application of Theorem 2.1, we shall prove an Adams-Spanne type theorem for $(L^q, l^p)^\alpha_0$. Let us recall that the Riesz potential operator I_γ $(0 < \gamma < 1)$ is defined by

$$I_\gamma f(x) = \int_{\mathbb{R}^d} |x - y|^{d(\gamma - 1)} f(y) dy$$

when this integral makes sense. This operator is known to be closely related to the fractional maximal operator $\mathfrak{M}_{q,\beta}$ $(1 \leq q < \beta \leq \infty)$ defined on L^1_{loc} by

$$\mathfrak{M}_{q,\beta} f(x) = \sup_{r>0} r^{d\left(\frac{1}{\beta} - \frac{1}{q}\right)} \left\| f\chi_{Q(x,r)} \right\|_q \quad , \ x \in \mathbb{R}^d.$$

Notice that $\mathfrak{M}_{1,\infty}$ is the well known Hardy-Littlewood maximal operator.

It is known (see [7, 11, 13] and the references therein) that, for $1 < q$:

- $\mathfrak{M}_{1,\infty}$ is a bounded operator on $(L^q, l^p)^\alpha$
- under the hypotheses

$$0 < \gamma < \frac{1}{\alpha}, \ \ \frac{1}{\alpha^*} = \frac{1}{\alpha} - \gamma, \ \ \frac{1}{q^*} = \frac{1}{q} - \gamma, \ \ \frac{1}{\tilde{q}} = \frac{1 - \gamma\alpha}{q}, \ \ \frac{1}{\tilde{p}} = \frac{1 - \gamma\alpha}{p} \tag{2.4}$$

the Riesz potential I_γ maps $(L^q, l^p)^\alpha$ into :

(i) $(L^{\tilde{q}}, l^{\tilde{p}})^{\alpha^*}$ (Adams type theorem)

(ii) $(L^{q^*}, l^p)^{\alpha^*}$ if $\gamma < \frac{1}{\alpha} - \frac{1}{p}$ (Spanne type theorem).

Our result reads as follows.

Theorem 2.4. *Assume that $1 < q \leq \alpha < \infty$.*

1) The Hardy-Littlewood maximal operator $\mathfrak{M}_{1,\infty}$ maps $(L^q, l^p)^\alpha_0$ into itself.

2) Under the hypotheses (2.4) the Riesz potential I_γ maps $(L^q, l^p)^\alpha_0$ into :

- $(L^{\tilde{q}}, l^{\tilde{p}})^{\alpha^*}_0$
- $(L^{q^*}, l^p)^{\alpha^*}_0$ *if* $\gamma < \frac{1}{\alpha} - \frac{1}{p}$.

Let us mention that our proof of Theorem 2.4, with very slight modifications, shows that this result remains true with $(L^q, l^p)^\alpha_0$ replaced by either of $V_\infty(L^q, l^p)^\alpha$ and the so-called vanishing subspace $V_0(L^q, l^p)^\alpha$ defined by

$$V_0(L^q, l^p)^\alpha = \left\{ f \in (L^q, l^p)^\alpha \ : \ \lim_{r \to 0} r^{d\left(\frac{1}{\alpha} - \frac{1}{q}\right)} {}_r\|f\|_{q,p} = 0 \right\}.$$

We recall that these results, as far as, $V_0(L^q, l^\infty)^\alpha$ and $V_\infty(L^q, l^\infty)^\alpha$ are concerned, have been proved in [3].

In the statement of our results on $(L^q,l^p)^\alpha_{\mathrm{c}}$ we shall use another linear subspace of $(L^q,l^p)^\alpha$, namely

$$\mathrm{AC}(L^q,l^p)^\alpha = \left\{ f \in (L^q,l^p)^\alpha \ : \ \lim_{\|\chi_E\|_{q,p,\alpha}\to 0} \|f\chi_E\|_{q,p,\alpha} = 0 \right\}.$$

This subspace of $(L^q,l^p)^\alpha$ may be characterized as follows.

Theorem 2.5. *If $\alpha < \infty$ then $\mathrm{AC}(L^q,l^p)^\alpha$ is the closure in $(L^q,l^p)^\alpha$ of $L^\infty \cap (L^q,l^p)^\alpha$.*

Let us recall that the closure of L^∞ in a Morrey type space [larger than $(L^q,l^\infty)^\alpha$], has been considered in [29] in connection with the study of some partial differential equations. See also [20] where $\mathrm{AC}(L^q,l^\infty)^\alpha$ is denoted by $\overline{\mathcal{M}^\alpha_q}$.

A generalized form of $L^q \cap (L^q,l^\infty)^\alpha_{\mathrm{c}}$ has been introduced by Zorko in [32]. So, as done by some authors in the case where $p=\infty$, $(L^q,l^p)^\alpha_{\mathrm{c}}$ may be called the Zorko subspace of $(L^q,l^p)^\alpha$. However, in our best knowledge, $(L^q,l^p)^\alpha_{\mathrm{c}}$ itself has been considered first in [14] where it is denoted by $(L_q,l_p)_{\mathrm{c}}^{\frac{1}{q}-\frac{1}{\alpha}}$. In [22], $(L^q,l^\infty)^\alpha_{\mathrm{c}}$ is denoted by $\ddot{M}_{q,d\left(1-\frac{q}{\alpha}\right)}$.

It is known (see [11, Proposition 3.3]) that, if $q<\infty$, then $(L^q,l^p)^\alpha_{\mathrm{c}}$ is the closure in $(L^q,l^p)^\alpha$ of

$$\mathcal{C}^\infty_{(L^q,l^p)^\alpha} = \left\{ f \in \mathcal{C}^\infty \ : \ \partial^\beta f \in (L^q,l^p)^\alpha \quad \text{for any } \beta \text{ in } \mathbb{N}^d \right\}$$

where $\mathcal{C}^\infty$ denotes the set of all infinitely differentiable functions on $\mathbb{R}^d$, and for any β in $\mathbb{N}^d$, $\partial^\beta f$ stands for the derivative of order β of f.

The above characterization of $(L^q,l^p)^\alpha_{\mathrm{c}}$ shows that the space denoted by $\overset{\diamond}{\mathcal{M}}{}^\alpha_q$ in [31] is nothing else than $(L^q,l^\infty)^\alpha_{\mathrm{c}}$.

It is worth noticing the following result.

Theorem 2.6. *1) We have*

$$\mathcal{C}^\infty_{(L^q,l^p)^\alpha} \subset L^p \cap L^\infty. \tag{2.5}$$

2) If $p<\infty$ then $\mathcal{C}^\infty_{(L^q,l^p)^\alpha}$ is included in the set $\mathcal{C}_0$ of all continuous functions f on $\mathbb{R}^d$ such that $\lim\limits_{|x|\to\infty} |f(x)| = 0$.

As an obvious consequence of the above result we get the following description of $(L^q,l^p)^\alpha_{\mathrm{c}}$.

Theorem 2.7. *If $q<\infty$ then $(L^q,l^p)^\alpha_{\mathrm{c}}$ is the closure in $(L^q,l^p)^\alpha$ of*

- $\left\{ f \in \mathcal{C}^\infty \ : \ \partial^\beta f \in L^p \cap (L^q,l^p)^\alpha \cap \mathcal{C}_0 \ \textit{ for any } \beta \textit{ in } \mathbb{N}^d \right\} \quad \textit{if } p<\infty$
- $\left\{ f \in \mathcal{C}^\infty \ : \ \partial^\beta f \in L^\infty \cap (L^q,l^\infty)^\alpha \quad \textit{ for any } \beta \textit{ in } \mathbb{N}^d \right\} \quad \textit{if } p=\infty$

and therefore $(L^q,l^p)^\alpha_{\mathrm{c}}$ is included in the closure $\overline{L^p \cap (L^q,l^p)^\alpha}$ in $(L^q,l^p)^\alpha$ of $L^p \cap (L^q,l^p)^\alpha$.

Theorem 2.5 and Theorem 2.7 imply that $(L^q,l^\infty)^\alpha_{\mathrm{c}}$ is included in $\mathrm{AC}(L^q,l^\infty)^\alpha$ when $\alpha<\infty$. Actually, more can be said about the relationships between $(L^q,l^p)^\alpha_{\mathrm{c}}$ and other linear subspaces of $(L^q,l^p)^\alpha$.

Theorem 2.8. *1) If $\alpha < p$ then*

$$(L^q, l^p)^\alpha_{\mathrm{c}} \subset \overline{L^p \cap (L^q, l^p)^\alpha} \subset V_0(L^q, l^p)^\alpha \cap \mathrm{AC}(L^q, l^p)^\alpha. \tag{2.6}$$

In particular, if $\alpha < \infty$ then

$$(L^q, l^\infty)^\alpha_{\mathrm{c}} \subset \mathrm{AC}(L^q, l^\infty)^\alpha \subset V_0(L^q, l^\infty)^\alpha \tag{2.7}$$

and the inclusion of $(L^q, l^\infty)^\alpha_{\mathrm{c}}$ in $\mathrm{AC}(L^q, l^\infty)^\alpha$ is proper in general.

2)
$$V_0(L^q, l^\infty)^\alpha \cap V_\infty(L^q, l^\infty)^\alpha \cap \mathcal{C}_{\mathrm{u}} \subset (L^q, l^\infty)^\alpha_{\mathrm{c}} \tag{2.8}$$

where $\mathcal{C}_{\mathrm{u}}$ is the set of all uniformly continuous functions on $\mathbb{R}^d$.

Notice that Point 1) of Theorem 2.8 implies that $(L^q, l^\infty)^\alpha_{\mathrm{c}}$ is not equal to $V_0(L^q, l^\infty)^\alpha$, contrary to what is claimed in [31, Lemma 2.33].

As an easy consequence of Point 2) of Theorem 2.8 we obtain the following approximation result contained in [4].

Corollary 2.9. *Each element of $V_0(L^q, l^\infty)^\alpha \cap V_\infty(L^q, l^\infty)^\alpha \cap \mathcal{C}_u$ is the limit in $(L^q, l^\infty)^\alpha$ of a sequence of elements of $V_0(L^q, l^\infty)^\alpha \cap V_\infty(L^q, l^\infty)^\alpha \cap \mathcal{C}^\infty$.*

Among the various closed linear subspaces of $(L^q, l^p)^\alpha$, $(L^q, l^p)^\alpha_{\mathrm{c},0}$ is one of the most popular because it is the closure in $(L^q, l^p)^\alpha$ of the set $\mathcal{C}^\infty_{\mathrm{c}}$ (see the remark following [11, Proposition 2.6]). We have defined it as the intersection of $(L^q, l^p)^\alpha_0$ and $(L^q, l^p)^\alpha_{\mathrm{c}}$. It is worth noticing that it may be expressed in several manners as intersection of other distinguished subspaces of $(L^q, l^p)^\alpha$. Some of these expressions are contained in the theorem below.

Theorem 2.10. *1) We have*

$$(L^q, l^p)^\alpha_{\mathrm{c},0} = (L^q, l^p)^\alpha_0 \cap V_0(L^q, l^p)^\alpha = (L^q, l^p)^\alpha_0 \cap \mathrm{AC}(L^q, l^p)^\alpha. \tag{2.9}$$

2) If $p < \infty$ then

$$(L^q, l^p)^\alpha_{\mathrm{c},0} = (L^q, l^p)^\alpha_{\mathrm{c}} \cap V_\infty(L^q, l^p)^\alpha. \tag{2.10}$$

Using 2.9, we obtain another expression of $(L^q, l^p)^\alpha_{\mathrm{c},0}$ which is given below.

Theorem 2.11. *If $q < \alpha < p$ then*

$$(L^q, l^p)^\alpha_{\mathrm{c},0} = \begin{cases} V_0(L^q, l^p)^\alpha \cap V_\infty(L^q, l^p)^\alpha & \text{if } p < \infty \\ \\ V_0(L^q, l^\infty)^\alpha \cap V_\infty(L^q, l^\infty)^\alpha \cap (L^q, \mathfrak{c}_0) & \text{if } p = \infty \end{cases} \tag{2.11}$$

where

$$(L^q, \mathfrak{c}_0) = \left\{ f \in L^1_{\mathrm{loc}} \ : \ \sup_{x \in \mathbb{R}^d} \left\| f\chi_{Q(x,1)} \right\|_q < \infty \ \ \text{and} \ \lim_{|x| \to \infty} \left\| f\chi_{Q(x,1)} \right\|_q = 0 \right\}.$$

From Theorem 2.11 we obtain the following approximation result.

Theorem 2.12. *If $1 < q$ and $p < \infty$ then for any element f of $V_0(L^q, l^p)^\alpha$ or $V_0(L^q, l^\infty)^\alpha \cap (L^q, \mathfrak{c}_0)$ there is a sequence $(\varphi_n)_{n\geq 1}$ of elements of $\mathcal{C}_c^\infty$ converging weakly to f :*

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^d} \varphi_n(x)g(x)dx = \int\limits_{\mathbb{R}^d} f(x)g(x)dx \quad , \quad g \in \mathcal{C}_c^\infty.$$

Let us recall that, the set $I(\gamma, p) = \left\{f \in L^1_{\mathrm{loc}} \ : \ I_\gamma(|f|) \in L^p\right\}$ is strongly related to the problem of existence of solutions in $(L^p)^d$ of the equation div $u = f$ (see [26, Theorem 3.2]).

The well known theorem of Hardy-Littlewood-Sobolev states that L^α is included in $I(\gamma, p)$ for $0 < \gamma < \frac{1}{\alpha} < 1$ and $\frac{1}{p} = \frac{1}{\alpha} - \gamma$.

We shall prove what follows.

Theorem 2.13. *If $0 < \gamma < \frac{1}{\alpha} \leq 1$ and $\frac{1}{p} = \frac{1}{\alpha} - \gamma$ then $I(\gamma, p)$ is included in $(L^1, l^p)^\alpha_{c,0}$ [the closure in $(L^1, l^p)^\alpha$ of L^α].*

3. Preliminaries

Wiener amalgams and weak-Lebesgue spaces are known to be related to the spaces $(L^q, l^p)^\alpha$. We recall their definitions below.

Definition 3.1. *Let (q, p, α) be an element of $[1, \infty]^3$.*

a) The Wiener amalgam space (L^q, l^p) is defined by

$$(L^q, l^p) = \left\{f \in L^1_{\mathrm{loc}} \ : \ {}_1\|f\|_{q,p} < \infty\right\}.$$

b) For $\alpha < \infty$, the weak-Lebesgue space $L^{\alpha,\infty}$ is defined by

$$L^{\alpha,\infty} = \left\{f \in L^1_{\mathrm{loc}} \ : \ \|f\|^*_{\alpha,\infty} < \infty\right\}$$

where

$$\|f\|^*_{\alpha,\infty} = \sup_{\lambda>0} \lambda \left|\{x \in \mathbb{R}^d \ : \ |f(x)| > \lambda\}\right|^{\frac{1}{\alpha}}.$$

We shall use in the sequel the following basic properties of Wiener amalgams and $(L^q, l^p)^\alpha$ spaces.

Proposition 3.2. *[7, 10, 11, 13, 14, 15, 21, 23] Let $1 \leq q, p, \alpha \leq \infty$.*

1) a) (L^q, l^p) is a linear subspace of L^1_{loc} and, when endowed with $f \mapsto {}_1\|f\|_{q,p}$, a Banach space.

b) For any real number $r > 0$

- *$f \mapsto {}_r\|f\|_{q,p}$ is a norm on (L^q, l^p), equivalent to $f \mapsto {}_1\|f\|_{q,p}$*
- *if $q \leq p$ and m is the integer satisfying $2^m \leq r < 2^{m+1}$ then*

$${}_r\|f\|_{q,p} \leq 3^{d\left(\frac{1}{q}-\frac{1}{p}\right)} {}_{2^m}\|f\|_{q,p} \quad , \quad f \in L^1_{\mathrm{loc}}. \tag{3.1}$$

• $f \mapsto {}_r|||f|||_{q,p} = r^{-\frac{d}{p}} \left\| \|f\chi_{Q(\cdot\,,r)}\|_q \right\|_p$ *is a norm on* (L^q, l^p) *and there are two positive real numbers* A_1 *and* A_2*, depending only on* (d,q,p) *and such that*

$$A_1 \, {}_r|||f|||_{q,p} \leq {}_r\|f\|_{q,p} \leq A_2 \, {}_r|||f|||_{q,p} \qquad , \quad f \in L^1_{\mathrm{loc}}. \tag{3.2}$$

c) The set $(L^q, \mathfrak{c}_0)$ *defined in Theorem 2.11 satisfies*

$$(L^q, \mathfrak{c}_0) = \left\{ f \in (L^q, l^\infty) \ : \ \lim_{|k|\to\infty} \|f\chi_{I^1_k}\|_q = 0 \right\}.$$

d) There is a real number C *depending only on* (d,q,p) *such that for any vector* u *of* $\mathbb{R}^d$*,*

$${}_r\|\tau_u f\|_{q,p} \leq C \, {}_r\|f\|_{q,p} \qquad , \quad f \in L^1_{\mathrm{loc}}. \tag{3.3}$$

2) a) $(L^q, l^p)^\alpha$ *is a linear subspace of* (L^q, l^p) *and, when endowed with* $f \mapsto \|f\|_{q,p,\alpha}$ *, a Banach space. Moreover* $(L^q, l^p)^\alpha$ *is non trivial if and only if* $q \leq \alpha \leq p$*.*

b) Let f *be an element of* L^1_{loc}*.*

• *If* $q \leq \alpha \leq p$ *then*

$$\|f\|_{q,\infty,\alpha} \leq \|f\|_{q,p,\alpha} \leq \|f\|_\alpha \ , \ \|f\|_{\alpha,p,\alpha} \leq \|f\|_\alpha \leq 2^{d\left(\frac{1}{q}-\frac{1}{p}\right)} \|f\|_{\alpha,p,\alpha} \text{ and } \|f\|_{q,\alpha,\alpha} = \|f\|_\alpha \tag{3.4}$$

and therefore the following relations hold

$$\alpha \in \{q,p\} \Longrightarrow (L^q, l^p)^\alpha = L^\alpha. \tag{3.5}$$

• *If* $q < \alpha < p$ *then there is a real number* C *such that*

$$\|f\|_{q,p,\alpha} \leq C \|f\|^*_{\alpha,\infty} \tag{3.6}$$

and therefore the weak-Lebesgue space $L^{\alpha,\infty}$ *is continuously included in* $(L^q, l^p)^\alpha$*.*

• *We have*

$$A_1 \, |||f|||_{q,p,\alpha} \leq \|f\|_{q,p,\alpha} \leq A_2 \, |||f|||_{q,p,\alpha} \text{ and } |||f|||_{q,\infty,\alpha} = \|f\|_{\mathcal{M}^\alpha_q} \tag{3.7}$$

where

$$|||f|||_{q,p,\alpha} = \sup_{r>0} \, r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)} \, {}_r|||f|||_{q,p} = \sup_{r>0} \, r^{d\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)} \left\| \|f\chi_{Q(\cdot\,,r)}\|_q \right\|_p, \tag{3.8}$$

A_1 *and* A_2 *are the constants in (3.2).*

It follows from (3.4) and (3.7) that $(L^q, l^p)^\alpha$ is continuously included in the Morrey space $\mathcal{M}^\alpha_q$ and $(L^q, l^\infty)^\alpha = \mathcal{M}^\alpha_q$.

We shall use the following result.

Proposition 3.3. *Let* $1 \leq q \leq \alpha \leq p \leq \infty$*. For any element* f *of* (L^q, l^p) *and any bounded Radon measure* μ *on* $\mathbb{R}^d$*, the convolution product* $\mu * f$ *is a well defined element of* (L^q, l^p) *and there is a real number* C*, not depending on* (f, μ) *and such that*

$${}_r\|\mu * f\|_{q,p} \leq C \, |\mu|(\mathbb{R}^d) \, {}_r\|f\|_{q,p} \qquad , \quad r > 0 \tag{3.9}$$

and therefore

$$\|\mu * f\|_{q,p,\alpha} \leq C\ |\mu|(\mathbb{R}^d)\ \|f\|_{q,p,\alpha} \tag{3.10}$$

where $|\mu|$ *stands for the total variation of* μ*.*

Proof. When $p < \infty$ the assertions follow from the proof of [23, Theorem 2.4] which may be adapted in an obvious manner to cover the case where $p = \infty$. ■

We shall use, in the sequel, $1 \leq q < \alpha < p \leq \infty$ unless otherwise specified.

4. The subspace $(L^q, l^p)^\alpha_0$ of $(L^q, l^p)^\alpha$

The following results are known.

Proposition 4.1. *[11, 23]* $(L^q, l^p)^\alpha_0$ *is a closed linear subspace of* $(L^q, l^p)^\alpha$ *which contains* L^α *and is invariant with respect to convolution product with bounded Radon measures on* $\mathbb{R}^d$*.*

It is not difficult to show that $(L^q, l^p)^\alpha_{\text{FNS}}$ is a linear subspace of $(L^q, l^p)^\alpha$.

Let us set, for any element f of L^1_{loc}, $E_f = \{|f| > 0\}$.

We notice that, for any element f of $(L^q, l^p)^\alpha$,

$$E_f \text{ bounded } \Longrightarrow \begin{cases} f \in L^q \\ |E_f| < \infty \Longrightarrow \|\chi_{E_f}\|_{q,p,\alpha} < \infty \Longrightarrow f \in (L^q, l^p)^\alpha_\omega. \end{cases} \tag{4.1}$$

The last implication stems from the inequalities below :

$$\left\|f\chi_{\{|f|\leq m\}}\right\|_{q,p,\alpha} \leq m\|\chi_{E_f}\|_{q,p,\alpha},\ m > 0 \text{ and } f \in L^1_{\text{loc}}.$$

From (4.1) we get

$$(L^q, l^p)^\alpha_0 \subset \overline{L^q \cap (L^q, l^p)^\alpha} \tag{4.2}$$

and

$$\begin{cases} (L^q, l^p)^\alpha_{\text{FNS}} \subset (L^q, l^p)^\alpha_\omega \\ \\ (L^q, l^p)^\alpha_0 \subset \overline{(L^q, l^p)^\alpha_{\text{FNS}}} \end{cases} \tag{4.3}$$

where, for any subset X of $(L^q, l^p)^\alpha$, $\overline{X}$ stands for its closure in $(L^q, l^p)^\alpha$.

We shall now prove Theorem 2.3.

Proof of Theorem 2.3

Let f be an element of $(L^q, l^p)^\alpha$.

a) Suppose that f is in $\overline{(L^q, l^p)^\alpha_{\text{FNS}}}$ and let ϵ be any positive real number.

There is an element g_ϵ of $(L^q, l^p)^\alpha_{\text{FNS}}$ such that

$$\|f - g_\epsilon\|_{q,p,\alpha} < \epsilon. \qquad (*)$$

By (4.3) there is a real number $m_\epsilon > 0$ such that

$$\left\|g_\epsilon \chi_{\{|g_\epsilon|\leq m\}}\right\|_{q,p,\alpha} < \epsilon \qquad , \quad m \in (0, 2m_\epsilon]\,. \qquad\qquad (**)$$

Furthermore, for any m in $(0, m_\epsilon)$

$$\begin{aligned}|f|\chi_{\{|f|\leq m\}} &\leq |f-g_\epsilon|\chi_{\{|f|\leq m\}} + |g_\epsilon|\chi_{\{|f|\leq m\}\cap\{|g_\epsilon|\leq 2m_\epsilon\}} + |g_\epsilon|\chi_{\{|f|\leq m\}\cap\{|g_\epsilon|> 2m_\epsilon\}}\\ &\leq |f-g_\epsilon| + |g_\epsilon|\chi_{\{|g_\epsilon|\leq 2m_\epsilon\}} + 2|g_\epsilon - f|. \qquad\qquad (***)\end{aligned}$$

By $(*)$, $(**)$ and $(***)$ we get

$$\left\|f\chi_{\{|f|\leq m\}}\right\|_{q,p,\alpha} \leq 4\,\epsilon \qquad , \quad m \in (0, m_\epsilon).$$

This shows that f belongs to $(L^q, l^p)^\alpha_\omega$.

b) Suppose that f is in $(L^q, l^p)^\alpha_\omega$ and let ϵ be any positive real number.

There is a real number $m_\epsilon > 0$ such that

$$\left\|f\chi_{\{|f|\leq m\}}\right\|_{q,p,\alpha} < \epsilon \qquad , \quad m \in (0, m_\epsilon].$$

It is clear that $f_\epsilon = f\chi_{\{|f|>m_\epsilon\}}$ belongs to $(L^q, l^p)^\alpha$, satisfies

$$E_{f_\epsilon} = \{|f_\epsilon| > m_\epsilon\}\ ,\ \left\|\chi_{E_{f_\epsilon}}\right\|_{q,p,\alpha} \leq \frac{1}{m_\epsilon}\left\|f\right\|_{q,p,\alpha} < \infty \ \text{ and } \left\|f - f_\epsilon\right\|_{q,p,\alpha} < \epsilon$$

and therefore

$$f_\epsilon \in (L^q, l^p)^\alpha_{\text{FNS}} \ \text{ and } \left\|f - f_\epsilon\right\|_{q,p,\alpha} < \epsilon.$$

This shows that f is in $\overline{(L^q, l^p)^\alpha_{\text{FNS}}}$.

c) The results obtained in Point a) and Point b) show that

$$\overline{(L^q, l^p)^\alpha_{\text{FNS}}} = (L^q, l^p)^\alpha_\omega. \quad \blacksquare$$

The following holds true.

Proposition 4.2. *$V_\infty(L^q, l^p)^\alpha$ is a closed linear subspace of $(L^q, l^p)^\alpha$ which is invariant with respect to the convolution product with bounded Radon measures on $\mathbb{R}^d$.*

Proof. a) It is easy to check that $V_\infty(L^q, l^p)^\alpha$ is a linear subspace of $(L^q, l^p)^\alpha$. Its invariance with respect to the convolution product with bounded Radon measures on $\mathbb{R}^d$ follows readily from Proposition 3.3.

b) For any element f in the closure in $(L^q, l^p)^\alpha$ of $V_\infty(L^q, l^p)^\alpha$ and any real number $\epsilon > 0$, there is an element (g_ϵ, r_ϵ) of $V_\infty(L^q, l^p)^\alpha \times (0, \infty)$ such that

$$\left\|f - g_\epsilon\right\|_{q,p,\alpha} < \epsilon \ \text{ and } \ r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}{}_r\left\|g_\epsilon\right\|_{q,p} \leq \epsilon \qquad , \quad r > r_\epsilon$$

and therefore

$$r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}{}_r\left\|f\right\|_{q,p} \leq 2\epsilon \qquad , \quad r > r_\epsilon.$$

This shows that $V_\infty(L^q,l^p)^\alpha$ is closed in $(L^q,l^p)^\alpha$. ■

Proof of Theorem 2.2

a) The first inclusion is just (4.2).

b) Let f be an element of $L^q \cap (L^q,l^p)^\alpha$.

• Since $q<p$, it follows from the definition of ${}_r\|\cdot\|_{q,p}$ that

$$r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}{}_r\|f\|_{q,p} \leq r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}\|f\|_q \qquad , \quad r>0$$

and so

$$\lim_{r\to\infty} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}{}_r\|f\|_{q,p} = 0 \ .$$

Thus f belongs to $V_\infty(L^q,l^p)^\alpha$.

• Let us consider a sequence $(m_n)_{n\geq 1}$ of elements of $(0,1)$ converging to 0 in $\mathbb{R}$ and set

$$f_n = |f|\chi_{\{|f|\leq m_n\}} \qquad , \qquad n\geq 1.$$

We have

$$0 \leq f_n \leq |f|\chi_{\{|f|\leq 1\}} \in L^q\cap L^\infty \subset L^\alpha \qquad , \quad n\geq 1.$$

Moreover, if $f(x)\neq 0$ then there are two positive integers n'_x and n''_x such that

$$m_n < \frac{1}{n'_x} < |f(x)| \qquad , \quad n\geq n''_x$$

and therefore

$$f_n(x)=0 \qquad , \quad n\geq n''_x.$$

So $\lim\limits_{n\to\infty} f_n = 0$ in L^1_{loc}.

Consequently, by the dominated convergence theorem, $(f_n)_{n\geq 1}$ converges to 0 in L^α and therefore in $(L^q,l^p)^\alpha$; that is : $\lim\limits_{n\to\infty}\left\|f\chi_{\{|f|\leq m_n\}}\right\|_{q,p,\alpha}=0$. This shows that $\lim\limits_{m\to 0}\left\|f\chi_{\{|f|\leq m\}}\right\|_{q,p,\alpha}=0$ and so f belongs to $(L^q,l^p)^\alpha_\omega$.

c) The results obtained in b) show that $L^q\cap(L^q,l^p)^\alpha$ is included in $V_\infty(L^q,l^p)^\alpha\cap(L^q,l^p)^\alpha_\omega$ which is closed in $(L^q,l^p)^\alpha$ (see Theorem 2.3 and Proposition 4.2) . Therefore $\overline{L^q\cap(L^q,l^p)^\alpha}$ is included in $V_\infty(L^q,l^p)^\alpha\cap(L^q,l^p)^\alpha_\omega$. ■

The following example shows that the inclusion of $(L^q,l^p)^\alpha_0$ in $V_\infty(L^q,l^p)^\alpha$ is proper.

Example 4.3. *Let us assume that $d=1$ and set $g=\sum\limits_{n\geq 1} 2^{\frac{n}{\alpha}}\chi_{[2^n,\,2^n+2^{-n})}$.*

a) For any real number $\lambda\geq 2^{\frac{1}{\alpha}}$ there is an integer $k\geq 1$ such that $2^{\frac{k}{\alpha}}\leq\lambda<2^{\frac{k+1}{\alpha}}$ and so

$$\lambda\,|\{x\in\mathbb{R}:|g(x)|>\lambda\}|^{\frac{1}{\alpha}} = \lambda\left|\bigcup_{n>k}\left[2^n,\,2^n+2^{-n}\right)\right|^{\frac{1}{\alpha}} = \lambda\left(\sum_{n>k}2^{-n}\right)^{\frac{1}{\alpha}} = \lambda\,2^{-\frac{k}{\alpha}} < 2^{\frac{1}{\alpha}}.$$

If $0<\lambda<2^{\frac{1}{\alpha}}$ *then*

$$\lambda\left|\{x\in\mathbb{R}:|g(x)|>\lambda\}\right|^{\frac{1}{\alpha}}=\lambda\left|\bigcup_{n\geq1}\left[2^n,\,2^n+2^{-n}\right)\right|^{\frac{1}{\alpha}}=\lambda\left(\sum_{n\geq1}2^{-n}\right)^{\frac{1}{\alpha}}=\lambda<2^{\frac{1}{\alpha}}.$$

This shows that g *belongs to* $L^{\alpha,\infty}$ *and therefore to* $(L^q,l^p)^\alpha$ *(see Proposition 3.2).*

b) Let us consider a real number $r>4$ *and the unique integer* $m>1$ *such that* $2^m\leq r<2^{m+1}$.

We have

$$\begin{aligned}{}_{2^m}\|g\|_{q,\infty}&=\max\left\{\left[\sum_{n=1}^{m-1}2^{n\left(\frac{q}{\alpha}-1\right)}\right]^{\frac{1}{q}},\sup_{n\geq m}2^{n\left(\frac{1}{\alpha}-\frac{1}{q}\right)}\right\}=\max\left\{2^{\left(\frac{1}{\alpha}-\frac{1}{q}\right)}\left[\frac{1-2^{(m-1)\left(\frac{q}{\alpha}-1\right)}}{1-2^{\frac{q}{\alpha}-1}}\right]^{\frac{1}{q}},\ 2^{m\left(\frac{1}{\alpha}-\frac{1}{q}\right)}\right\}\\&\leq\max\left\{\left[\frac{2^{\frac{q}{\alpha}-1}}{1-2^{\frac{q}{\alpha}-1}}\right]^{\frac{1}{q}},1\right\}=A(q,\alpha,\infty)<\infty.\end{aligned}$$

and for $p<\infty$

$$\begin{aligned}{}_{2^m}\|g\|_{q,p}&=\left\{\left[\sum_{n=1}^{m-1}2^{\frac{nq}{\alpha}}2^{-n}\right]^{\frac{p}{q}}+\sum_{n\geq m}\left(2^{\frac{nq}{\alpha}}2^{-n}\right)^{\frac{p}{q}}\right\}^{\frac{1}{p}}=\left\{2^{\left(\frac{1}{\alpha}-\frac{1}{q}\right)p}\left[\frac{1-2^{(m-1)\left(\frac{q}{\alpha}-1\right)}}{1-2^{\frac{q}{\alpha}-1}}\right]^{\frac{p}{q}}+\frac{2^{m\left(\frac{1}{\alpha}-\frac{1}{q}\right)p}}{1-2^{\left(\frac{1}{\alpha}-\frac{1}{q}\right)p}}\right\}^{\frac{1}{p}}\\&\leq\left\{\left[\frac{2^{\frac{q}{\alpha}-1}}{1-2^{\frac{q}{\alpha}-1}}\right]^{\frac{p}{q}}+\frac{1}{1-2^{\left(\frac{1}{\alpha}-\frac{1}{q}\right)p}}\right\}^{\frac{1}{p}}=A(q,\alpha,p)<\infty.\end{aligned}$$

Therefore, by (3.1) we have in all cases

$$r^{\frac{1}{\alpha}-\frac{1}{q}}\ {}_r\|g\|_{q,p}\leq3^{\frac{1}{q}-\frac{1}{p}}\ A(q,\alpha,p)\ r^{\frac{1}{\alpha}-\frac{1}{q}}.$$

This shows that g *belongs to* $V_\infty(L^q,l^p)^\alpha$.

c) Let us consider an integer $m\geq1$*. We have*

$$2^{-m\left(\frac{1}{\alpha}-\frac{1}{q}\right)}{}_{2^{-m}}\left\|g\chi_{[2^m,\infty)}\right\|_{q,\infty}\geq2^{-m\left(\frac{1}{\alpha}-\frac{1}{q}\right)}\left\|g\chi_{[2^m,2^m+2^{-m})}\right\|_q=2^{-m\left(\frac{1}{\alpha}-\frac{1}{q}\right)}\ 2^{\frac{m}{\alpha}}2^{-\frac{m}{q}}=1$$

and for $p<\infty$

$$\begin{aligned}2^{-m\left(\frac{1}{\alpha}-\frac{1}{q}\right)}{}_{2^{-m}}\left\|g\chi_{[2^m,\infty)}\right\|_{q,p}&=2^{-m\left(\frac{1}{\alpha}-\frac{1}{q}\right)}\left[\sum_{n\geq m}\left\|g\chi_{[2^n,2^n+2^{-n})}\right\|_q^p\right]^{\frac{1}{p}}=2^{-m\left(\frac{1}{\alpha}-\frac{1}{q}\right)}\left[\sum_{n\geq m}\left(2^{\frac{n}{\alpha}}2^{-\frac{n}{q}}\right)^p\right]^{\frac{1}{p}}\\&=2^{-m\left(\frac{1}{\alpha}-\frac{1}{q}\right)}\left[2^{\left(\frac{1}{\alpha}-\frac{1}{q}\right)mp}\frac{1}{1-2^{\left(\frac{1}{\alpha}-\frac{1}{q}\right)p}}\right]^{\frac{1}{p}}=\frac{1}{\left(1-2^{\left(\frac{1}{\alpha}-\frac{1}{q}\right)p}\right)^{\frac{1}{p}}}>1.\end{aligned}$$

Therefore, in all cases

$$\left\|g\chi_{[2^m,\infty)}\right\|_{q,p,\alpha}\geq1$$

and consequently g *does not belong to* $(L^q,l^p)_0^\alpha$.

It is easy to see that $(L^q, l^p)^\alpha$ is included in $(L^q, \mathfrak{c}_0)$ when $p < \infty$. But in the case where $p = \infty$, this inclusion is not true. Actually we have

$$[V_\infty(L^q, l^\infty)^\alpha \cap V_0(L^q, l^\infty)^\alpha] \setminus [(L^q, \mathfrak{c}_0) \cap (L^q, l^\infty)^\alpha] \neq \varnothing.$$

(See [4, Theorem 4.1], where $(L^q, \mathfrak{c}_0) \cap (L^q, l^\infty)^\alpha$ is denoted by $V^{(*)}L^{q,d\left(1-\frac{q}{\alpha}\right)}$).

However we have what follows.

Proposition 4.4. *$(L^q, l^\infty)^\alpha_0$ is embedded in $(L^q, \mathfrak{c}_0)$.*

Proof. Let us consider an element (f, r, ϵ) of $(L^q, l^\infty)^\alpha_0 \times (0, \infty) \times (0, \infty)$.

There exists a real number $\rho_{r,\epsilon}$ such that

$$\left\| f\chi_{\mathbb{R}^d \setminus Q(0,\rho_{r,\epsilon})} \right\|_{q,\infty,\alpha} < \epsilon\, r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}.$$

For any element x of $\mathbb{R}^d$ satisfying $|x| > \rho_{r,\epsilon} + \frac{\sqrt{d}}{2}r$, we have $Q(x,r) \subset \mathbb{R}^d \setminus Q\left(0, \rho_{r,\epsilon}\right)$ and so

$$\left\| f\chi_{Q(x,r)} \right\|_q = \left\| \left(f\chi_{\mathbb{R}^d \setminus Q(0,\rho_{r,\epsilon})}\right) \chi_{Q(x,r)} \right\|_q \leq {}_r \left\| f\chi_{\mathbb{R}^d \setminus Q(0,\rho_{r,\epsilon})} \right\|_{q,\infty} \leq r^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)} \left\| f\chi_{\mathbb{R}^d \setminus Q(0,\rho_{r,\epsilon})} \right\|_{q,\infty,\alpha} < \epsilon.$$

Thus

$$\lim_{|x|\to\infty} \left\| f\chi_{Q(x,r)} \right\|_q = 0\ .$$

Hence f belongs to $(L^q, \mathfrak{c}_0)$. ■

Theorem 2.1 is a consequence of the following two propositions.

Proposition 4.5. *For any element f of $(L^q, l^p)^\alpha_0$, we have*

$$\lim_{\rho\to\infty} \left\{ \sup_{r>0} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)} \left\| \left\| f\chi_{Q(\cdot\,,\,r)} \right\|_q \chi_{\mathbb{R}^d \setminus Q(0,\rho)} \right\|_p \right\} = 0\,.$$

Proof. Suppose that f is an element of $(L^q, l^p)^\alpha_0$ and consider a positive real number ϵ.

By (2.3), f belongs also to $V_\infty(L^q, l^p)^\alpha$. Therefore there are two positive real numbers R_ϵ and r_ϵ such that

$$\begin{cases} \|f\chi_{\mathbb{R}^d \setminus Q(0,R_\epsilon)}\|_{q,p,\alpha} < \epsilon & (*) \\ \\ r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}{}_r\|f\|_{q,p} < \epsilon \qquad , \quad r \geq r_\epsilon\ . & (**) \end{cases}$$

Set $\rho_\epsilon = \sqrt{d}\,(R_\epsilon + r_\epsilon)$ and consider $\rho \geq \rho_\epsilon$.

a) Suppose that $0 < r < r_\epsilon$.

We notice that

$$|x| > \rho \Longrightarrow Q(x,r) \subset Q(x,r_\epsilon) \subset \mathbb{R}^d \setminus Q(0,R_\epsilon) \Longrightarrow f\chi_{Q(x,r)} = \left(f\chi_{\mathbb{R}^d \setminus Q(0,R_\epsilon)}\right) \chi_{Q(x,r)}.$$

From this, $(*)$ and (3.7) we get

$$r^{d\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)}\left\|\left\|f\chi_{Q(\cdot\,,r)}\right\|_q\chi_{\mathbb{R}^d\backslash Q(0,\rho)}\right\|_p\leq r^{d\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)}\left\|\left\|\left(f\chi_{\mathbb{R}^d\backslash Q(0,R_\epsilon)}\right)\chi_{Q(\cdot\,,r)}\right\|_q\right\|_p$$
$$\leq |||f\chi_{\mathbb{R}^d\backslash Q(0,R_\epsilon)}|||_{q,p,\alpha}\leq A_1^{-1}\,\epsilon\,.$$

b) Suppose that $r\geq r_\epsilon$.

From $(**)$ and (3.2) we obtain

$$r^{d\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)}\left\|\left\|f\chi_{Q(\cdot\,,r)}\right\|_q\chi_{\mathbb{R}^d\backslash Q(0,\rho)}\right\|_p\leq r^{d\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)}\left\|\left\|f\chi_{Q(\cdot\,,r)}\right\|_q\right\|_p\leq A_1^{-1}\,r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}\,{}_r\|f\|_{q,p}\leq A_1^{-1}\,\epsilon.$$

The claim follows from the above observations. ■

The following proposition is the converse of Proposition 4.5

Proposition 4.6. *Any element f of $(L^q,l^p)^\alpha$ satisfying the condition*

$$\lim_{\rho\to\infty}\left\{\sup_{r>0}r^{d\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)}\left\|\left\|f\chi_{Q(\cdot\,,r)}\right\|_q\chi_{\mathbb{R}^d\backslash Q(0,\rho)}\right\|_p\right\}=0$$

belongs to $(L^q,l^p)^\alpha_0$.

Proof. Let f be an element of $(L^q,l^p)^\alpha$ satisfying the condition

$$\lim_{\rho\to\infty}\left\{\sup_{r>0}r^{d\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)}\left\|\left\|f\chi_{Q(\cdot\,,r)}\right\|_q\chi_{\mathbb{R}^d\backslash Q(0,\rho)}\right\|_p\right\}=0$$

and assume that $p<\infty$.

1) Let ϵ be a positive real number.

By hypothesis there is a real number $\rho_\epsilon>0$ such that

$$r^{d\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)}\left(\int\limits_{|x|_\infty>\rho_\epsilon}\left\|f\chi_{Q(x,r)}\right\|_q^p dx\right)^{\frac{1}{p}}<\epsilon\qquad,\quad 0<r<\infty$$

where

$$|x|_\infty=\max\left(|x_1|,|x_1|,...,|x_d|\right)\qquad,\quad x=(x_1,x_2,...,x_d)\in\mathbb{R}^d.$$

• Suppose that $r>2\,\rho_\epsilon$.

Fix an element j of $\{1,2,...,d\}$ and set

$$Q_j^+=\left\{x\in\mathbb{R}^d\backslash\{0\}\ :\ |x|_\infty=x_j\leq\rho_\epsilon\right\},\ Q_j^-=\left\{x\in\mathbb{R}^d\backslash\{0\}\ :\ |x|_\infty=-x_j\leq\rho_\epsilon\right\},$$
$$T_j(x)=\begin{cases}x+\rho_\epsilon e_j & \text{if } x\in Q_j^+\\ x-\rho_\epsilon e_j & \text{if } x\in Q_j^-\end{cases}$$

where e_j stands for the j-th element of the canonical basis of $\mathbb{R}^d$.

It is easy to see that, whenever x is in $Q_j^+ \cup Q_j^-$,

$$|T_j(x)|_\infty = |x|_\infty + \rho_\epsilon > \rho_\epsilon \text{ and } Q(x,r) \subset \overline{Q}(T_j(x), r+2\rho_\epsilon) = \left\{ y \in \mathbb{R}^d : \ |y - T_j(x)|_\infty \leq \frac{r}{2} + \rho_\epsilon \right\}$$

and therefore, if Q stands for Q_j^+ or Q_j^- then

$$\int_Q \left\| f\chi_{Q(x,r)} \right\|_q^p dx \leq \int_Q \left\| f\chi_{Q(T_j(x),r+2\rho_\epsilon)} \right\|_q^p dx = \int_{T_j(Q)} \left\| f\chi_{Q(y,r+2\rho_\epsilon)} \right\|_q^p dy \leq \int_{|y|_\infty > \rho_\epsilon} \left\| f\chi_{Q(y,r+2\rho_\epsilon)} \right\|_q^p dy.$$

Since

$$\left\{ x \in \mathbb{R}^d \ : \ 0 < |x|_\infty \leq \rho_\epsilon \right\} = \bigcup_{j=1}^{d} \left(Q_j^+ \cup Q_j^- \right),$$

the above observations imply that

$$\begin{aligned}
& r^{d\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)} \left(\int_{\mathbb{R}^d} \left\| f\chi_{Q(x,r)} \right\|_q^p dx \right)^{\frac{1}{p}} \\
&= \left\{ r^{d\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)p} \int_{|x|_\infty \leq \rho_\epsilon} \left\| f\chi_{Q(x,r)} \right\|_q^p dx + r^{d\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)p} \int_{|x|_\infty > \rho_\epsilon} \left\| f\chi_{Q(x,r)} \right\|_q^p dx \right\}^{\frac{1}{p}} \\
&\leq \left\{ 2d\, 2^{d\left(\frac{1}{q}+\frac{1}{p}-\frac{1}{\alpha}\right)p} (r+2\rho_\epsilon)^{d\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)p} \int_{|x|_\infty > \rho_\epsilon} \left\| f\chi_{Q(x,r+2\rho_\epsilon)} \right\|_q^p dx + r^{d\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)p} \int_{|x|_\infty > \rho_\epsilon} \left\| f\chi_{Q(x,r)} \right\|_q^p dx \right\}^{\frac{1}{p}} \\
&\leq \left\{ (2d)\ 2^{d\left(\frac{1}{q}+\frac{1}{p}-\frac{1}{\alpha}\right)p} + 1 \right\}^{\frac{1}{p}} \epsilon \qquad (\text{ by the choice of } \rho_\epsilon \text{ }).
\end{aligned}$$

Therefore, by (3.2), there is a real number $A_1 = A_1(d,q,p)$ such that

$$r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}{}_r\|f\|_{q,p} \leq A_1\, \epsilon \qquad , \quad r > 2\rho_\epsilon \ . \qquad (*)$$

• Suppose that $0 < r \leq 2\rho_\epsilon$ and consider a real number $R > 3\rho_\epsilon$.

We notice that for any x in $\mathbb{R}^d$

$$|x|_\infty \leq \rho_\epsilon \Longrightarrow Q(x,r) \subset Q(0,R) \Longrightarrow \left(f\chi_{\mathbb{R}^d \setminus Q(0,R)} \right) \chi_{Q(x,r)} = 0.$$

This implies that

$$\begin{aligned}
r^{d\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)} \left(\int_{\mathbb{R}^d} \left\| f\chi_{\mathbb{R}^d \setminus Q(0,R)} \chi_{Q(x,r)} \right\|_q^p dx \right)^{\frac{1}{p}} &= r^{d\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)} \left(\int_{|x|_\infty > \rho_\epsilon} \left\| f\chi_{\mathbb{R}^d \setminus Q(0,R)} \chi_{Q(x,r)} \right\|_q^p dx \right)^{\frac{1}{p}} \\
&\leq r^{d\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)} \left(\int_{|x|_\infty > \rho_\epsilon} \left\| f\chi_{Q(x,r)} \right\|_q^p dx \right)^{\frac{1}{p}} \\
&< \epsilon \qquad (\text{by the choice of } \rho_\epsilon).
\end{aligned}$$

Therefore, by (3.2), there is a real number $A_2 = A_2(d,q,p)$ such that

$$r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}{}_r\|f\chi_{\mathbb{R}^d\setminus Q(0,R)}\|_{q,p} < A_2\,\epsilon \qquad 0 < r \leq 2\rho_\epsilon \quad \text{and} \quad R > 3\rho_\epsilon\,. \qquad (**)$$

From $(*)$ and $(**)$ we get

$$\|f\chi_{\mathbb{R}^d\setminus Q(0,R)}\|_{q,p,\alpha} < \max\left(A_1, A_2\right)\epsilon \qquad , \quad R > 3\rho_\epsilon.$$

This shows that f is in $(L^q, l^p)_0^\alpha$.

In the case where $p = \infty$ the claim is proved by a similar argumentation. ■

In order to prove Theorem 2.4, let us recall the following result of Guliyev.

Proposition 4.7. *[19] Let $1 < q < \infty$. Then there is a real number $C > 0$ such that for any f in L^q_{loc}*

1) $$\left\|(\mathfrak{M}_{1,\infty}f)\,\chi_{Q(x,t)}\right\|_q \leq C\,t^{\frac{d}{q}}\int_t^\infty r^{-1-\frac{d}{q}}\left\|f\chi_{Q(x,r)}\right\|_q dr \qquad , \quad t > 0 \quad \textit{and} \quad x \in \mathbb{R}^d \qquad (4.4)$$

2) if $0 < \gamma < \frac{1}{q}$ and $\frac{1}{q^*} = \frac{1}{q} - \gamma$ then

$$\left\|(I_\gamma f)\,\chi_{Q(x,t)}\right\|_{q^*} \leq C\,t^{\frac{d}{q^*}}\int_t^\infty r^{-1-\frac{d}{q^*}}\left\|f\chi_{Q(x,r)}\right\|_q dr \qquad , \quad t > 0 \quad \textit{and} \quad x \in \mathbb{R}^d. \qquad (4.5)$$

Proof of Theorem 2.4

Let f be an element of $(L^q, l^p)_0^\alpha$ and ρ be a positive real number.

1) Assume that $1 < q$.

By (4.4) and the Minkowski inequality, there is a real number $C > 0$, not depending on f and ρ, such that

$$t^{d\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)}\left\|\left\|(\mathfrak{M}_{1,\infty}f)\,\chi_{Q(\cdot\,,t)}\right\|_q \chi_{\mathbb{R}^d\setminus Q(0,\rho)}\right\|_p$$
$$\leq C\int_1^\infty s^{-1-d\left(\frac{1}{\alpha}-\frac{1}{p}\right)}\left[(ts)^{d\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)}\left\|\left\|f\chi_{Q(\cdot\,,ts)}\right\|_q \chi_{\mathbb{R}^d\setminus Q(0,\rho)}\right\|_p\right]ds \qquad , \quad t > 0. \qquad (4.6)$$

Therefore

$$\sup_{t>0} t^{d\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)}\left\|\left\|(\mathfrak{M}_{1,\infty}f)\,\chi_{Q(\cdot\,,t)}\right\|_q \chi_{\mathbb{R}^d\setminus Q(0,\rho)}\right\|_p$$
$$\leq C\,K_{d,\alpha,p}\left[\sup_{r>0} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)}\left\|\left\|f\chi_{Q(\cdot\,,r)}\right\|_q \chi_{\mathbb{R}^d\setminus Q(0,\rho)}\right\|_p\right] \qquad (**) \qquad (4.7)$$

with

$$K_{d,\alpha,p} = \int_1^\infty s^{-1-d\left(\frac{1}{\alpha}-\frac{1}{p}\right)}ds < \infty.$$

Inequality (4.7) and Theorem 2.1 infer that $\mathfrak{M}_{1,\infty}f$ belongs to $(L^q, l^p)_0^\alpha$. 2) Assume that the hypotheses (2.4) hold.

• It is known (see the proof of [13, Theorem 4.3]) that there is a real number D, not depending on f and such that

$$|I_\gamma f| \leq D \left\|f\right\|_{q,\infty,\alpha}^{\alpha\gamma} \left(\mathfrak{M}_{1,\infty} f\right)^{1-\alpha\gamma}.$$

This implies that, for any real number $t > 0$

$$\left\|(I_\gamma f)\, \chi_{Q(x,t)}\right\|_{\tilde{q}} \leq D \left\|f\right\|_{q,\infty,\alpha}^{\alpha\gamma} \left\|(\mathfrak{M}_{1,\infty} f)\, \chi_{Q(x,t)}\right\|_{q}^{1-\alpha\gamma} \qquad , \quad x \in \mathbb{R}^d$$

and therefore

$$\begin{aligned} & t^{d\left(\frac{1}{\alpha^*}-\frac{1}{\tilde{q}}-\frac{1}{\tilde{p}}\right)} \left\|\left\|(I_\gamma f)\, \chi_{Q(\cdot\,,t)}\right\|_{\tilde{q}} \chi_{\mathbb{R}^d\setminus Q(0,\rho)}\right\|_{\tilde{p}} \\ & \leq D\|f\|_{q,\infty,\alpha}^{\alpha\gamma} \left\{t^{d\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)} \left\|\left\|(\mathfrak{M}_{1,\infty} f)\, \chi_{Q(\cdot\,,t)}\right\|_{q} \chi_{\mathbb{R}^d\setminus Q(0,\rho)}\right\|_{p}\right\}^{1-\alpha\gamma}. \end{aligned} \tag{4.8}$$

By (4.6) and (4.8), there is a real number $C > 0$, not depending on (f, ρ) and such that, for any real number $t > 0$

$$\begin{aligned} & t^{d\left(\frac{1}{\alpha^*}-\frac{1}{\tilde{q}}-\frac{1}{\tilde{p}}\right)} \left\|\left\|(I_\gamma f)\, \chi_{Q(\cdot\,,t)}\right\|_{\tilde{q}} \chi_{\mathbb{R}^d\setminus Q(0,\rho)}\right\|_{\tilde{p}} \\ & \leq C \left\|f\right\|_{q,\infty,\alpha}^{\alpha\gamma} \left\{\int_1^\infty s^{-1-d\left(\frac{1}{\alpha}-\frac{1}{p}\right)} \left[(ts)^{d\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)} \left\|\left\|f\chi_{Q(\cdot\,,ts)}\right\|_{q} \chi_{\mathbb{R}^d\setminus Q(0,\rho)}\right\|_{p}\right] ds\right\}^{1-\alpha\gamma}. \end{aligned}$$

Therefore

$$\begin{aligned} & \sup_{t>0} t^{d\left(\frac{1}{\alpha^*}-\frac{1}{\tilde{q}}-\frac{1}{\tilde{p}}\right)} \left\|\left\|(I_\gamma f)\, \chi_{Q(\cdot\,,t)}\right\|_{\tilde{q}} \chi_{\mathbb{R}^d\setminus Q(0,\rho)}\right\|_{\tilde{p}} \\ & \leq C K_{d,\alpha,p}^{1-\alpha\gamma} \left\|f\right\|_{q,\infty,\alpha}^{\alpha\gamma} \left[\sup_{r>0} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)} \left\|\left\|f\chi_{Q(\cdot\,,r)}\right\|_{q} \chi_{\mathbb{R}^d\setminus Q(0,\rho)}\right\|_{p}\right]^{1-\alpha\gamma}. \end{aligned} \tag{4.9}$$

Inequality (4.9) and Theorem 2.1 imply that $I_\gamma f$ belongs to $(L^{\tilde{q}}, l^{\tilde{p}})_0^{\alpha^*}$.

• Suppose that $\gamma < \frac{1}{\alpha} - \frac{1}{p}$.

By (4.5) and the Minkowski inequality, there is a real number $C > 0$, not depending on (f, ρ), such that for any real number $t > 0$

$$\begin{aligned} & t^{d\left(\frac{1}{\alpha^*}-\frac{1}{q^*}-\frac{1}{p}\right)} \left\|\left\|(I_\gamma f)\, \chi_{Q(\cdot\,,t)}\right\|_{q^*} \chi_{\mathbb{R}^d\setminus Q(0,\rho)}\right\|_{p} \\ & \leq C \int_1^\infty s^{-1-d\left(\frac{1}{\alpha}-\frac{1}{p}-\gamma\right)} \left[(ts)^{d\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)} \left\|\left\|f\chi_{Q(\cdot\,,ts)}\right\|_{q} \chi_{\mathbb{R}^d\setminus Q(0,\rho)}\right\|_{p}\right] ds. \end{aligned}$$

Therefore

$$\begin{aligned} & \sup_{t>0} t^{d\left(\frac{1}{\alpha^*}-\frac{1}{q^*}-\frac{1}{p}\right)} \left\|\left\|(I_\gamma f)\, \chi_{Q(\cdot\,,t)}\right\|_{q^*} \chi_{\mathbb{R}^d\setminus Q(0,\rho)}\right\|_{p} \\ & \leq C\, K_{d,\alpha,p,\gamma} \left[\sup_{r>0} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)} \left\|\left\|f\chi_{Q(\cdot\,,r)}\right\|_{q} \chi_{\mathbb{R}^d\setminus Q(0,\rho)}\right\|_{p}\right] \end{aligned} \tag{4.10}$$

with

$$K_{d,\alpha,p,\gamma} = \int_1^\infty s^{-1-d\left(\frac{1}{\alpha}-\frac{1}{p}-\gamma\right)} ds < \infty.$$

Inequality (4.10) and an application of Theorem 2.1 show that $I_\gamma f$ belongs to $(L^{q^*}, l^p)_0^{\alpha^*}$. ■

5. The subspace $(L^q, l^p)^\alpha_{\mathrm{c}}$ of $(L^q, l^p)^\alpha$

The following results are known.

Proposition 5.1. *[11] $(L^q, l^p)^\alpha_{\mathrm{c}}$ is a closed linear subspace of $(L^q, l^p)^\alpha$ which contains L^α and :*

a) satisfies

$$(L^q, l^p)^\alpha_{\mathrm{c}} = L^1 * (L^q, l^p)^\alpha_{\mathrm{c}} = L^1 * (L^q, l^p)^\alpha,$$

where $*$ stands for the usual convolution product

b) is equal to :

- $\left\{ f \in (L^q, l^p)^\alpha \ : \ \lim_{n\to\infty} \|f - \rho_n * f\|_{q,p,\alpha} = 0 \right\}$ *whenever ρ is a nonnegative element of L^1*

such that $\|\rho\|_1 = 1$ and $\rho_n = n^d \rho(n\,\cdot)$ for any integer $n \geq 1$

- *the closure in $(L^q, l^p)^\alpha$ of the set $\mathcal{C}^\infty_{(L^q,l^p)^\alpha} = \left\{ f \in \mathcal{C}^\infty \ : \ \partial^\beta f \in (L^q, l^p)^\alpha \ \ \text{for any } \beta \text{ in } \mathbb{N}^d \right\}$.*

By an application of the Sobolev embedding theorem we shall prove Theorem 2.6.

<u>Proof of Theorem 2.6</u>

a) Let f be an element of $\mathcal{C}^\infty_{(L^q,l^p)^\alpha}$. Let us consider an element (x, ω) of $\mathbb{R}^d \times \mathcal{C}^\infty_c$ such that $\chi_{\overline{B}(0,\frac{1}{2})} \leq \omega \leq \chi_{B(0,1)}$ and set $g_x = f\omega(\cdot - x)$.

We notice that g_x is an element of $\mathcal{C}^\infty_{(L^q,l^p)^\alpha}$ such that for any $\theta \in \mathbb{N}^d$

$$\begin{cases} \partial^\theta g_x = \sum_{\beta\leq\theta} C^\beta_\theta \partial^\beta f \left(\partial^{\theta-\beta}\omega\right)(\cdot - x) \\ \partial^\theta g_x = \partial^\theta f \ \text{ on } \ B(x, \frac{1}{2}) \ \text{ and } \ \partial^\theta g_x = 0 \ \text{ on } \ \mathbb{R}^d \backslash B(x, 1) \\ |\partial^\theta g_x| \leq M_\theta \max_{\beta\leq\theta} |\partial^\beta f| \chi_{B(x,1)} \end{cases}$$

where M_θ is a real number depending only on d, θ and the partials derivatives $\partial^\beta \omega$ $(\beta \leq \theta)$ of ω.

Let us consider an integer $m \geq 1$ such that $m > \frac{d}{q}$.

Since g_x belongs to the Sobolev space $W^{m,q}$ and $mq > d$, there is a real constant A, depending only on (d, m, q) and such that

$$\|g_x\|_\infty \leq A\|g_x\|_{W^{m,q}} \leq A \sum_{|\theta|\leq m} \|\partial^\theta g_x\|_q.$$

Therefore

$$|f(x)| \leq A \left(\sum_{|\theta|\leq m} M_\theta \right) \max_{\theta\leq m} \left\| (\partial^\theta f)\chi_{B(x,1)} \right\|_q.$$

b) From the above observations and (3.2) we get

$$\|f\|_s \leq A \left(\sum_{|\theta|\leq m} M_\theta \right) \max_{\theta\leq m} \left\| \left\| (\partial^\theta f)\chi_{B(\cdot,1)} \right\|_q \right\|_s \leq C \max_{|\theta|\leq m} {}_1 \left\| \partial^\theta f \right\|_{q,s} \quad , \quad s \in \{p, \infty\}$$

where C is a real constant not depending on f.

c) Assume that $p < \infty$.

Since any derivative of f is in $\mathcal{C}^\infty_{(L^q,l^p)^\alpha}$, the result obtained above shows that $|\nabla f|$ is bounded and therefore f is uniformly continuous on $\mathbb{R}^d$.

Consider a real number $\epsilon > 0$.

There is a real number $\delta > 0$ such that

$$|x - y| < \sqrt{d}\,\delta \Longrightarrow |f(x) - f(y)| < \frac{\epsilon}{2} \qquad , \quad x, y \in \mathbb{R}^d$$

and therefore

$$\sup_{x \in I_k^\delta} |f(x)| \leq \inf_{x \in I_k^\delta} |f(x)| + \frac{\epsilon}{2} \qquad , \quad k \in \mathbb{Z}^d \, . \qquad\qquad (*)$$

Since

$$\sum_{k \in \mathbb{Z}^d} \|f\chi_{I_k^\delta}\|_q^p \leq \left(\delta^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)} \|f\|_{q,p,\alpha}\right)^p < \infty$$

there is a positive integer K such that

$$\|f\chi_{I_k^\delta}\|_q \leq \frac{\epsilon}{2}\,\delta^{\frac{d}{q}} \qquad , \quad k \in \mathbb{Z}^d \text{ with } |k| > K$$

and therefore, by $(*)$,

$$|f(x)| \leq \epsilon \qquad , \quad x \in \mathbb{R}^d \backslash \left(\bigcup_{k \in \mathbb{Z}^d,\, |k| > K} I_k^\delta \right) \, .$$

This ends the proof. ■

Remark 5.2. *Theorem 2.7 follows readily from Theorem 2.6 and Point b) of Proposition 5.1.*

Before the proof of Theorem 2.8, let us give some properties of $V_0(L^q, l^p)^\alpha$ and $\mathrm{AC}(L^q, l^p)^\alpha$.

Proposition 5.3. *$V_0(L^q, l^p)^\alpha$ is a closed linear subspace of $(L^q, l^p)^\alpha$ which is invariant with respect to the convolution product with bounded Radon measures on $\mathbb{R}^d$.*

Proof. The claim is proved by an argumentation similar to the proof of Proposition 4.2. ■

The following proposition contains Theorem 2.5.

Proposition 5.4. *$\mathrm{AC}(L^q, l^p)^\alpha$ is the closure in $(L^q, l^p)^\alpha$ of $L^\infty \cap (L^q, l^p)^\alpha$ and therefore a closed linear subspace of $(L^q, l^p)^\alpha$.*

Proof. 1) Let f be an element of $(L^q, l^p)^\alpha$.

• Suppose that : f is in $\overline{L^\infty \cap (L^q, l^p)^\alpha}$, E is a measurable subset of $\mathbb{R}^d$ and ϵ is a positive real number.

There is an element f_ϵ of $L^\infty \cap (L^q, l^p)^\alpha$ such that

$$\|f - f_\epsilon\|_{q,p,\alpha} < \frac{\epsilon}{2}$$

and therefore

$$\|f\chi_E\|_{q,p,\alpha} \leq \|(f-f_\epsilon)\chi_E\|_{q,p,\alpha} + \|f_\epsilon\chi_E\|_{q,p,\alpha} < \frac{\epsilon}{2} + \|f_\epsilon\|_\infty \|\chi_E\|_{q,p,\alpha}.$$

So there is a positive real number t_ϵ such that

$$\sup\left\{\|f\chi_E\|_{q,p,\alpha} \ : \ E \text{ is a measurable subset of } \mathbb{R}^d \text{ and } \|\chi_E\|_{q,p,\alpha} < t_\epsilon\right\} \leq \epsilon.$$

This shows that f is in $\mathrm{AC}(L^q, l^p)^\alpha$.

• Suppose that f is in $\mathrm{AC}(L^q, l^p)^\alpha$ and consider a real number $\epsilon > 0$.

Let us set

$$E_k = \{|f| \geq k\} \quad \text{and} \quad f_k = f\,(1-\chi_{E_k}) \qquad , \quad k > 0.$$

Notice that, for any real number $k > 0$, f_k belongs to $L^\infty \cap (L^q, l^p)^\alpha$ and

$$\|\chi_{E_k}\|_{q,p,\alpha} \leq \frac{1}{k}\,\|f\chi_{E_k}\|_{q,p,\alpha} \leq \frac{1}{k}\,\|f\|_{q,p,\alpha}. \qquad (*)$$

Since f is in $\mathrm{AC}(L^q, l^p)^\alpha$, there is a real number $t_\epsilon > 0$ such that, for any measurable subset E of $\mathbb{R}^d$

$$\|\chi_E\|_{q,p,\alpha} < t_\epsilon \implies \|f\chi_E\|_{q,p,\alpha} < \epsilon. \qquad (**)$$

From $(*)$ and $(**)$ it follows that

$$\|f - f_k\|_{q,p,\alpha} = \|f\chi_{E_k}\|_{q,p,\alpha} < \epsilon \qquad , \quad k > t_\epsilon^{-1}\left(\|f\|_{q,p,\alpha} + 1\right).$$

This shows that f is in $\overline{L^\infty \cap (L^q, l^p)^\alpha}$.

2) By the observations in Point 1), $\mathrm{AC}(L^q, l^p)^\alpha = \overline{L^\infty \cap (L^q, l^p)^\alpha}$.

Moreover, $L^\infty \cap (L^q, l^p)^\alpha$ is a linear subspace of $(L^q, l^p)^\alpha$. Therefore $\mathrm{AC}(L^q, l^p)^\alpha$ is a closed linear subspace of $(L^q, l^p)^\alpha$. ■

Proof of Theorem 2.8

1) a) It follows from Theorem 2.7 that $(L^q, l^p)^\alpha_{\mathrm{c}}$ is included in $\overline{L^p \cap (L^q, l^p)^\alpha}$.

b) Let f be an element of $L^p \cap (L^q, l^p)^\alpha$.

• We have

$$r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}{}_r\|f\|_{q,p} \leq r^{d\left(\frac{1}{\alpha}-\frac{1}{p}\right)}\|f\|_p \qquad , \ r > 0$$

and therefore, since $\alpha < p$,

$$\lim_{r\to 0} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}{}_r\|f\|_{q,p} = 0 \quad \text{and so } f \in V_0(L^q, l^p)^\alpha.$$

• 1^{st} case : $p < \infty$. Let (ϵ, m) be an element of $(0,\infty)\times(0,\infty)$ and set $f_m = f\chi_{\{|f|>m\}}$.

By the result obtained in a), there is a real number $r_\epsilon > 0$ not depending on m and such that

$$r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}{}_r\,\|f_m\|_{q,p} < \epsilon \qquad , \quad 0 < r \leq r_\epsilon. \qquad (*)$$

For any real number $r \geq r_\epsilon$, we have

$$r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}{}_r \left\|f_m\right\|_{q,p} \leq r_\epsilon^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)} \left[\sum_{k\in\mathbb{Z}^d} \left\|f_m \chi_{I_k^r}\right\|_p^p \ |\{|f|>m\} \cap I_k^r|^{\frac{p}{q}-1}\right]^{\frac{1}{p}}$$
$$\leq r_\epsilon^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)} \|f_m\|_p \left(m^{-p}\|f_m\|_p^p\right)^{\frac{1}{q}-\frac{1}{p}} \leq m^{-\frac{p}{q}+1} r_\epsilon^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)} \|f\|_p^{\frac{p}{q}}.$$

Therefore, since $q < \alpha < p$ and $\|f\|_p < \infty$, there is a real number $m_\epsilon > 0$ such that

$$r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}{}_r \left\|f_m\right\|_{q,p} < \epsilon \qquad , \ m \geq m_\epsilon \text{ and } r \geq r_\epsilon . \qquad (**)$$

From $(*)$ and $(**)$ we get

$$\left\|f - f\chi_{\{|f|\leq m\}}\right\|_{q,p,\alpha} = \|f_m\|_{q,p,\alpha} < \epsilon \qquad , \ m \geq m_\epsilon. \qquad (***)$$

Notice that for any $m > 0$, $f\chi_{\{|f|\leq m\}}$ belongs to $L^\infty \cap (L^q, l^p)^\alpha$. Thus, it follows from $(***)$ that f is in $\overline{L^\infty \cap (L^q, l^p)^\alpha}$ and so in $\mathrm{AC}(L^q, l^p)^\alpha$ (see Proposition 5.4).

2^{nd} case : $p = \infty$. Proposition 5.4 implies that $L^\infty \cap (L^q, l^\infty)^\alpha$ is included in $\mathrm{AC}(L^q, l^\infty)^\alpha$.

Hence, we have

$$L^p \cap (L^q, l^p)^\alpha \subset V_0(L^q, l^p)^\alpha \cap \mathrm{AC}(L^q, l^p)^\alpha.$$

Since $V_0(L^q, l^p)^\alpha \cap \mathrm{AC}(L^q, l^p)^\alpha$ is closed in $(L^q, l^p)^\alpha$ (see Proposition 5.3 and Proposition 5.4), the previous inclusion shows that

$$\overline{L^p \cap (L^q, l^p)^\alpha} \subset V_0(L^q, l^p)^\alpha \cap \mathrm{AC}(L^q, l^p)^\alpha.$$

c) By the results obtained in Point a) and Point b), we get

$$(L^q, l^p)^\alpha_{\mathrm{c}} \subset \overline{L^p \cap (L^q, l^p)^\alpha} \subset V_0(L^q, l^p)^\alpha \cap \mathrm{AC}(L^q, l^p)^\alpha.$$

In particular, taking $p = \infty$ and using Proposition 5.4, we obtain

$$(L^q, l^\infty)^\alpha_{\mathrm{c}} \subset \mathrm{AC}(L^q, l^\infty)^\alpha \subset V_0(L^q, l^\infty)^\alpha.$$

d) It is known (see [22, Example 3.4]) that the function f defined on $\mathbb{R}$ by

$$f(x) = \sum_{n\geq 1} \varphi_n(x - 2^n) \quad \text{with} \quad \varphi_n(x) = [\sin(2\pi n x)]\, \chi_{(0,1)}(x)$$

is in $V_0(L^q, l^\infty)^\alpha(\mathbb{R}) \backslash (L^q, l^\infty)^\alpha_{\mathrm{c}}(\mathbb{R})$. Moreover it is clearly in $L^\infty(\mathbb{R})$ and therefore in $\mathrm{AC}(L^q, l^\infty)^\alpha(\mathbb{R})$.

This shows that the inclusion of $(L^q, l^\infty)^\alpha_{\mathrm{c}}(\mathbb{R})$ in $\mathrm{AC}(L^q, l^\infty)^\alpha(\mathbb{R})$ is proper.

2) Let f be an element of $V_0(L^q, l^\infty)^\alpha \cap V_\infty(L^q, l^\infty)^\alpha \cap \mathcal{C}_u$ and ϵ be any positive real number.

Let us consider a vector u in $\mathbb{R}^d$.

Since f is in $V_0(L^q, l^\infty)^\alpha \cap V_\infty(L^q, l^\infty)^\alpha$, there are two real numbers r_ϵ and R_ϵ such that $0 < r_\epsilon < R_\epsilon$ and

$$r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}{}_r\|f\|_{q,\infty} < \frac{\epsilon}{1+2^{\frac{d}{q}}}, \qquad , \ r \in (0, r_\epsilon) \cup (R_\epsilon, \infty). \qquad (*)$$

From (3.3) it follows that

$$r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}{}_r\|\tau_u f - f\|_{q,\infty} \leq \left(1+2^{\frac{d}{q}}\right) r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}{}_r\|f\|_{q,\infty} \qquad , \ r > 0$$

and therefore, by $(*)$,

$$r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}{}_r\|\tau_u f - f\|_{q,\infty} < \epsilon \qquad , \ r \in (0, r_\epsilon) \cup (0, R_\epsilon). \qquad (**)$$

Notice that for any (r, k) in $[r_\epsilon, R_\epsilon] \times \mathbb{Z}^d$,

$$r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}\left\|(\tau_u f - f)\,\chi_{I_k^r}\right\|_q \leq r_\epsilon^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)} R_\epsilon^{\frac{d}{q}} \sup_{|x-y|=|u|} |f(x) - f(y)|.$$

Therefore, since f is uniformly continuous, there is a real number $\delta_\epsilon > 0$ such that

$$r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}{}_r\|\tau_u f - f\|_{q,\infty} < \epsilon \qquad , \ r \in [r_\epsilon, R_\epsilon] \ \text{ and } \ |u| < \delta_\epsilon. \qquad (***)$$

From $(**)$ and $(***)$ it follows that

$$\|\tau_u f - f\|_{q,\infty,\alpha} < \epsilon \qquad , \ |u| < \delta_\epsilon.$$

This shows that f belongs to $(L^q, l^\infty)_{\mathrm{c}}^\alpha$. ■

6. The subspace $(L^q, l^p)_{\mathrm{c},0}^\alpha$ of $(L^q, l^p)^\alpha$

We begin by recalling some known properties of $(L^q, l^p)_{\mathrm{c},0}^\alpha$.

Proposition 6.1. *[10, 11] $(L^q, l^p)_{\mathrm{c},0}^\alpha$ is a closed linear subspace of $(L^q, l^p)^\alpha$ which is equal to the :*

- *closure in $(L^q, l^p)^\alpha$ of $\mathcal{C}_{\mathrm{c}}^\infty$ [and of L^α]*
- *set of all elements f of $(L^q, l^p)^\alpha$ of absolutely continuous norm [that is, for which, $\lim_{n\to\infty} \|f\chi_{E_n}\|_{q,p,\alpha} = 0$ whenever $(E_n)_{n\geq 1}$ is a nonincreasing sequence of measurable subsets of $\mathbb{R}^d$ satisfying $\left|\bigcap_{n\geq 1} E_n\right| = 0$].*

In the proof of Theorem 2.10 we shall use the following two propositions.

Proposition 6.2. *$(L^q, l^p)_{\mathrm{c}}^\alpha$ contains $(L^q, l^p)_0^\alpha \cap V_0(L^q, l^p)^\alpha$ and $(L^q, l^p)_0^\alpha \cap \mathrm{AC}(L^q, l^p)^\alpha$.*

Proof. Let f be an element of $(L^q, l^p)_0^\alpha$.

a) Assume that f is in $V_0(L^q, l^p)^\alpha$ and consider any vector u in $\mathbb{R}^d$.

Notice that, for any positive real numbers ρ and R

$$\begin{aligned}\left\|\tau_u f - f\right\|_{q,p,\alpha} &\leq \sup_{0<r<\rho} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}\ {}_r\left\|\tau_u\left(f\chi_{Q(0,R)}\right) - f\chi_{Q(0,R)}\right\|_{q,p} \\ &+ \rho^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}\sup_{r\geq\rho}\ {}_r\left\|\tau_u\left(f\chi_{Q(0,R)}\right) - f\chi_{Q(0,R)}\right\|_{q,p} \\ &+ \left\|\tau_u\left(f\chi_{\mathbb{R}^d\setminus Q(0,R)}\right) - f\chi_{\mathbb{R}^d\setminus Q(0,R)}\right\|_{q,p,\alpha} \\ &\leq (1+C)\left[\sup_{0<r<\rho} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}{}_r\left\|f\chi_{Q(0,R)}\right\|_{q,p} + \left\|f\chi_{\mathbb{R}^d\setminus Q(0,R)}\right\|_{q,p,\alpha}\right] \\ &+ \rho^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}\left\|\tau_u\left(f\chi_{Q(0,R)}\right) - f\chi_{Q(0,R)}\right\|_q \qquad (\text{ by } (3.3)).\end{aligned}$$

Since f is in $(L^q, l^p)_0^\alpha \cap V_0(L^q, l^p)^\alpha$ and $q < \infty$, for any real number $\epsilon > 0$ there are three positive real numbers ρ_ϵ, R_ϵ and δ_ϵ such that

$$\begin{cases} \sup_{0<r<\rho_\epsilon} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}\ {}_r\left\|f\chi_{Q(0,R_\epsilon)}\right\|_{q,p} + \left\|f\chi_{\mathbb{R}^d\setminus Q(0,R_\epsilon)}\right\|_{q,p,\alpha} < \epsilon \\ \left\|\tau_u\left(f\chi_{Q(0,R_\epsilon)}\right) - f\chi_{Q(0,R_\epsilon)}\right\|_q < \epsilon\,\rho_\epsilon^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)} \qquad , \ |u| < \delta_\epsilon \end{cases}$$

and therefore

$$\left\|\tau_u f - f\right\|_{q,p,\alpha} < (2+C)\,\epsilon \qquad , \ |u| < \delta_\epsilon.$$

This shows that f is in $(L^q, l^p)_{\mathrm{c}}^\alpha$.

b) Assume that f is in $\mathrm{AC}(L^q, l^p)^\alpha$ and consider any vector u in $\mathbb{R}^d$.

Notice that $\left\{f\chi_{Q(0,n)\cap\{|f|\leq n\}}\right\}_{n\geq 1}$ is a sequence of elements of L^α and therefore, of $(L^q, l^p)_{\mathrm{c}}^\alpha$. Furthermore, for any integer $n \geq 1$

$$\left\|f - f\chi_{Q(0,n)\cap\{|f|\leq n\}}\right\|_{q,p,\alpha} \leq \left\|f\chi_{\mathbb{R}^d\setminus Q(0,n)}\right\|_{q,p,\alpha} + \left\|f\chi_{\{|f|>n\}}\right\|_{q,p,\alpha}.$$

The hypotheses on f and the closedness of $(L^q, l^p)_{\mathrm{c}}^\alpha$ in $(L^q, l^p)^\alpha$ imply that f is in $(L^q, l^p)_{\mathrm{c}}^\alpha$. ■

Proposition 6.3. *Suppose that* $p < \infty$. *Then* $(L^q, l^p)_{\mathrm{c}}^\alpha \cap V_\infty(L^q, l^p)^\alpha$ *is included in* $(L^q, l^p)_0^\alpha$.

Proof. Let f be an element of $(L^q, l^p)_c^\alpha \cap V_\infty(L^q, l^p)^\alpha$ and ϵ be any positive real number.

Consider a nonnegative element ρ of $\mathcal{C}_{\mathrm{c}}^\infty$ such that $\|\rho\|_1 = 1$ and set

$$f_n = f * \rho_n \quad \text{with} \quad \rho_n = n^d\,\rho(n\,\cdot) \qquad , \qquad n \in \mathbb{N}.$$

By Proposition 5.1 and Proposition 4.2, $(f_n)_{n\geq 1}$ is a sequence of elements of $\mathcal{C}^\infty_{(L^q,l^p)^\alpha} \cap V_\infty(L^q, l^p)^\alpha$ converging to f in $(L^q, l^p)^\alpha$. Moreover, Theorem 2.6 asserts that each $f_n (n \geq 1)$ is in L^p.

Notice that for any (n, ρ, R) in $\mathbb{N} \times (0, \infty) \times (0, \infty)$

$$\begin{aligned}\left\|f\chi_{\mathbb{R}^d\setminus Q(0,R)}\right\|_{q,p,\alpha} &\leq \|f - f_n\|_{q,p,\alpha} + \left\|f_n\chi_{\mathbb{R}^d\setminus Q(0,R)}\right\|_{q,p,\alpha} \\ &\leq \|f - f_n\|_{q,p,\alpha} + \max\left\{\sup_{0<r<\rho} r^{d\left(\frac{1}{\alpha}-\frac{1}{p}\right)}\left\|f_n\chi_{\mathbb{R}^d\setminus Q(0,R)}\right\|_p, \sup_{r\geq\rho} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}{}_r\|f_n\|_{q,p}\right\} \\ &\leq \|f - f_n\|_{q,p,\alpha} + \max\left\{\rho^{d\left(\frac{1}{\alpha}-\frac{1}{p}\right)}\left\|f_n\chi_{\mathbb{R}^d\setminus Q(0,R)}\right\|_p, \sup_{r\geq\rho} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}{}_r\|f_n\|_{q,p}\right\}. \qquad (*)\end{aligned}$$

Moreover, by the properties of $(f_n)_{n\geq 1}$, we may choose successively an integer $n_\epsilon \geq 1$ and two positive real numbers ρ_ϵ and R_ϵ such that

$$\|f - f_{n_\epsilon}\|_{q,p,\alpha} < \epsilon \ , \ \sup_{r>\rho_\epsilon} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}{}_r\,\|f_{n_\epsilon}\|_{q,p} < \epsilon \ , \ \left\|f_{n_\epsilon}\chi_{\mathbb{R}^d\setminus Q(0,R)}\right\|_p < \epsilon\rho_\epsilon^{d\left(\frac{1}{p}-\frac{1}{\alpha}\right)} \quad \text{for} \quad R \geq R_\epsilon. \qquad (**)$$

By $(*)$ and $(**)$, there is a real number $R_\epsilon > 0$ such that

$$\left\|f\chi_{\mathbb{R}^d\setminus Q(0,R)}\right\|_{q,p,\alpha} < 2\epsilon \qquad , \quad R \geq R_\epsilon.$$

This ends the proof. ■

Proof of Theorem 2.10

1) From (2.6) and Proposition 6.2 we get

• $(L^q,l^p)^\alpha_{\rm c} \cap (L^q,l^p)^\alpha_0 \subset V_0(L^q,l^p)^\alpha \cap (L^q,l^p)^\alpha_0 \subset (L^q,l^p)^\alpha_{\rm c} \cap (L^q,l^p)^\alpha_0$

and therefore

$$(L^q,l^p)^\alpha_{{\rm c},0} = (L^q,l^p)^\alpha_0 \cap V_0(L^q,l^p)^\alpha$$

• $(L^q,l^p)^\alpha_{\rm c} \cap (L^q,l^p)^\alpha_0 \subset \mathrm{AC}(L^q,l^p)^\alpha \cap (L^q,l^p)^\alpha_0 \subset (L^q,l^p)^\alpha_{\rm c} \cap (L^q,l^p)^\alpha_0$

and therefore

$$(L^q,l^p)^\alpha_{{\rm c},0} = (L^q,l^p)^\alpha_0 \cap \mathrm{AC}(L^q,l^p)^\alpha.$$

2) If $p < \infty$ then from (2.3) and Proposition 6.3 we get

$$(L^q,l^p)^\alpha_{\rm c} \cap (L^q,l^p)^\alpha_0 \subset (L^q,l^p)^\alpha_{\rm c} \cap V_\infty(L^q,l^p)^\alpha \subset (L^q,l^p)^\alpha_{\rm c} \cap (L^q,l^p)^\alpha_0$$

and therefore

$$(L^q,l^p)^\alpha_{{\rm c},0} = (L^q,l^p)^\alpha_{\rm c} \cap V_\infty(L^q,l^p)^\alpha. \quad ■$$

Proof of Theorem 2.11

We set

$$X = \begin{cases} V_0(L^q,l^p)^\alpha \cap V_\infty(L^q,l^p)^\alpha & \text{if } \ p < \infty \\ \\ V_0(L^q,l^\infty)^\alpha \cap V_\infty(L^q,l^\infty)^\alpha \cap (L^q,\mathfrak{c}_0) & \text{if } \ p = \infty. \end{cases}$$

• Let us consider an element f of X and a real number $\epsilon > 0$.

Since f is in $V_0(L^q,l^p)^\alpha \cap V_\infty(L^q,l^p)^\alpha$, there are two real numbers r_0 and r_1 such that :

$$\begin{cases} 0 < r_0 < r_1 < \infty \\ r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}{}_r\|f\|_{q,p} < \epsilon \qquad \text{for} \quad 0 < r < r_0 \ \text{ or } r_1 < r < \infty \end{cases} \qquad (*)$$

1^{st} case : $p = \infty$

Since f belongs to $(L^q,\mathfrak{c}_0)$, there exists a real number $\rho'_\epsilon > 0$ such that

$$\sup_{|x|>\rho'_\epsilon} \left\|f\chi_{Q(x,r_1)}\right\|_q < r_0^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)}\epsilon \qquad , \ \rho \geq \rho'_\epsilon$$

and therefore

$$\sup_{r_0<r<r_1} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)} \left\| \left\| f\chi_{Q(\cdot,r)} \right\|_q \chi_{\mathbb{R}^d\setminus Q(0,\rho)} \right\|_\infty < \epsilon \qquad , \; \rho \geq \rho'_\epsilon \,. \qquad (**)$$

<u>2^{nd} case</u> : $p<\infty$

Since f is in $(L^q, l^p)^\alpha$, (3.2) asserts that $x \mapsto \left\| f\chi_{Q(x,r_1)} \right\|_q$ belongs to L^p and therefore there is a real number $\rho''_\epsilon > 0$ such that

$$\left\| \left\| f\chi_{Q(\cdot,r_1)} \right\|_q \chi_{\mathbb{R}^d\setminus Q(0,\rho)} \right\|_p < \epsilon \, r_0^{d\left(\frac{1}{q}+\frac{1}{p}-\frac{1}{\alpha}\right)} \qquad , \; \rho \geq \rho''_\epsilon \,.$$

This implies that

$$\sup_{r_0<r<r_1} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)} \left\| \left\| f\chi_{Q(\cdot,r)} \right\|_q \chi_{\mathbb{R}^d\setminus Q(0,\rho)} \right\|_p < \epsilon \qquad , \; \rho \geq \rho''_\epsilon \,. \qquad (**\text{ bis})$$

By (2.2), $(*)$, $(**)$ and $(**\text{ bis})$, there is a real number $\rho_\epsilon > 0$ such that

$$\sup_{r>0} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)} \left\| \left\| f\chi_{Q(\cdot,r)} \right\|_q \chi_{\mathbb{R}^d\setminus Q(0,\rho)} \right\|_p < \epsilon \qquad , \; \rho \geq \rho_\epsilon \,.$$

This shows that f belongs to $(L^q, l^p)_0^\alpha$ (by Theorem 2.1) and therefore to $(L^q, l^p)_{\mathrm{c},0}^\alpha$ (by (2.9)). So X is included in $(L^q, l^p)_{\mathrm{c},0}^\alpha$.

- From (2.3) and (2.6), we have

$$(L^q, l^p)_{\mathrm{c},0}^\alpha = (L^q, l^p)_{\mathrm{c}}^\alpha \cap (L^q, l^p)_0^\alpha \subset V_0(L^q, l^p)^\alpha \cap V_\infty(L^q, l^p)^\alpha.$$

The above inclusion and Proposition 4.4 imply that

$$(L^q, l^\infty)_{\mathrm{c},0}^\alpha \subset V_0(L^q, l^\infty)^\alpha \cap V_\infty(L^q, l^\infty)^\alpha \cap (L^q, l^\infty)_0^\alpha \subset V_0(L^q, l^\infty)^\alpha \cap V_\infty(L^q, l^\infty)^\alpha \cap (L^q, \mathfrak{c}_0).$$

So $(L^q, l^p)_{\mathrm{c},0}^\alpha$ is included in X. ■

<u>Proof of Theorem 2.12</u>

Let us recall that, if $p<\infty$ then $(L^q, l^p)^\alpha$ is included in $(L^q, \mathfrak{c}_0)$ and so

$$V_0(L^q, l^p)^\alpha \cap (L^q, \mathfrak{c}_0) = V_0(L^q, l^p)^\alpha.$$

Assume that $1<q$ and let f be an element of $V_0(L^q, l^p)^\alpha \cap (L^q, \mathfrak{c}_0)$.

a) Fix R in $(0,\infty)$. It is clear that, for any real number $r>R$

$${}_r\left\| f\chi_{Q(0,R)} \right\|_{q,p} = \begin{cases} \left[\displaystyle\sum_{k\in\{-1,0\}^d} \left\| f\chi_{I_k^R} \right\|_q^p \right]^{\frac{1}{p}} & \text{if } \; p<\infty \\ \displaystyle\sup_{k\in\{-1,0\}^d} \left\| f\chi_{I_k^R} \right\|_q & \text{if } \; p=\infty \end{cases}$$

and so

$$r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)} {}_r\left\| f\chi_{Q(0,R)} \right\|_{q,p} \leq \left(R\, r^{-1}\right)^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)} \left\| f \right\|_{q,p,\alpha} .$$

This shows that $f\chi_{Q(0,R)}$ is in $V_\infty(L^q,l^p)^\alpha$ and therefore in $(L^q,l^p)^\alpha_{c,0}$ (by the hypothesis on f and Theorem 2.11).

b) It is obvious that $\left(f\chi_{Q(0,m)}\right)_{m\geq 1}$ is a bounded sequence of elements of $(L^q,l^p)^\alpha$.

Furthermore, $(L^q,l^p)^\alpha$ represents the dual space of the Banach space $\mathcal{H}(q',p',\alpha')$ which is separable and contains the set $X=\mathcal{C}_c^\infty\cup\left\{\chi_E : E \text{ is a bounded measurable subset of } \mathbb{R}^d\right\}$ (see [10, 12]). Therefore, there are an increasing sequence $(m_n)_{n\geq 1}$ of elements of $\mathbb{N}$ and an element $\widetilde{f}$ of $(L^q,l^p)^\alpha$ such that

$$\lim_{n\to\infty}\int_{\mathbb{R}^d} f(x)\chi_{Q(0,m_n)}(x)g(x)dx=\int_{\mathbb{R}^d}\widetilde{f}(x)g(x)dx \quad , \; g\in X \qquad (*)$$

(see [5, Corollary 3.30]).

From $(*)$ and the dominated convergence theorem it follows that, for any bounded measurable subset E of $\mathbb{R}^d$,

$$\int_E f(x)dx=\int_E\widetilde{f}(x)dx. \qquad (**)$$

This implies that $\widetilde{f}=f$ in L^1_{loc} (see [17, Theorem 2.23]) and therefore we have, by $(*)$,

$$\lim_{n\to\infty}\int_{\mathbb{R}^d} f(x)\chi_{Q(0,m_n)}(x)g(x)dx=\int_{\mathbb{R}^d}f(x)g(x)dx \quad , \; g\in\mathcal{C}_c^\infty. \qquad (***)$$

Since $f\chi_{Q(0,m_n)}$ $(n\geq 1)$ belongs to $(L^q,l^p)^\alpha_{c,0}$, there is a sequence $(\varphi_n)_{n\geq 1}$ of elements of $\mathcal{C}_c^\infty$ such that

$$\left\|\varphi_n-f\chi_{Q(0,m_n)}\right\|_{q,p,\alpha}<\frac{1}{n} \quad , \; n\geq 1 \qquad (****)$$

The claim follows from $(***)$ and $(****)$. ■

7. Proof of Theorem 2.13

In our proof we shall use the following link between the fractional maximal and the Riesz potential operators.

Proposition 7.1. *[13] Let $0<\gamma<1$ and $1\leq q\leq\alpha\leq p\leq+\infty$. Then there is a real number $C>0$ such that*

$$C^{-1}\left\|\mathfrak{M}_{1,\frac{1}{\gamma}}f\right\|_{q,p,\alpha}\leq\|I_\gamma(|f|)\|_{q,p,\alpha}\leq C\left\|\mathfrak{M}_{1,\frac{1}{\gamma}}f\right\|_{q,p,\alpha} \quad , \; f\in L^1_{\mathrm{loc}}. \tag{7.1}$$

In particular, for any element p of $[1;+\infty]$, there is a real number $C>0$ such that

$$C^{-1}\left\|\mathfrak{M}_{1,\frac{1}{\gamma}}f\right\|_{p}\leq\|I_\gamma(|f|)\|_{p}\leq C\left\|\mathfrak{M}_{1,\frac{1}{\gamma}}f\right\|_{p} \quad , \; f\in L^1_{\mathrm{loc}}. \tag{7.2}$$

In view of Proposition 7.1 the following results are clearly related to Theorem 2.13.

Proposition 7.2. *Let $1\leq q\leq\alpha\leq\beta\leq\infty$ and $\frac{1}{s}=\frac{1}{\alpha}-\frac{1}{\beta}$.*
1) If $s\leq p$ then

$$\|f\|_{q,p,\alpha}\leq 2^{d\left(\frac{1}{q}-\frac{1}{\beta}\right)}\|\mathfrak{M}_{q,\beta}f\|_{1,p,s} \quad , \; f\in L^q_{\mathrm{loc}}. \tag{7.3}$$

In particular, if $\frac{1}{p} = \frac{1}{\alpha} - \frac{1}{\beta}$ *then*

$$\|f\|_{q,p,\alpha} \leq 2^{d\left(\frac{1}{q}-\frac{1}{\beta}\right)} \|\mathfrak{M}_{q,\beta} f\|_p \quad , \quad f \in L^q_{\mathrm{loc}}. \tag{7.4}$$

2) If $\frac{1}{p_0} = \frac{1}{q} - \frac{1}{\beta} \leq \frac{1}{\alpha}$ *then there is a real number* $C = C(d, q, \alpha, \beta, p_0) > 0$ *such that*

$$\|\mathfrak{M}_{q,\beta} f\|^*_{s,\infty} \leq C \|f\|^{\frac{p_0}{s}}_{q,p_0,\alpha} \|f\|^{1-\frac{p_0}{s}}_{q,\infty,\alpha} \leq C \|f\|_{q,p_0,\alpha} \quad , \quad f \in L^q_{\mathrm{loc}}. \tag{7.5}$$

These results appear in [16] (a paper written by one of the authors) without a full proof of (7.3) and with a weighted form of (7.5). For the reader's convenience we give a full and simple proof of these results.

Proof. Let f be any element of L^q_{loc}.
1) a) For any element (k, r) of $\mathbb{Z}^d \times (0, \infty)$ we have

$$\mathfrak{M}_{q,\beta} f(y) \geq (2r)^{d\left(\frac{1}{\beta}-\frac{1}{q}\right)} \left\|f \chi_{Q(y,2r)}\right\|_q \geq 2^{d\left(\frac{1}{\beta}-\frac{1}{q}\right)} r^{d\left(\frac{1}{\beta}-\frac{1}{q}\right)} \left\|f \chi_{I^r_k}\right\|_q \quad , \quad y \in I^r_k$$

$$2^{d\left(\frac{1}{q}-\frac{1}{\beta}\right)} r^{d\left(\frac{1}{s}-1\right)} \left\|(\mathfrak{M}_{q,\beta} f)\, \chi_{I^r_k}\right\|_1 \geq r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)} \left\|f \chi_{I^r_k}\right\|_q .$$

This yields (7.3).

b) By (3.4), (7.4) is a particular case of (7.3).

2) By hypothesis, $0 \leq \frac{1}{\alpha} - \frac{1}{\beta} = \frac{1}{s} \leq \frac{1}{q} - \frac{1}{\beta} = \frac{1}{p_0} \leq \frac{1}{\alpha}$ and therefore $1 \leq q \leq \alpha \leq p_0 \leq s \leq \infty$.

Let f be in L^q_{loc}.

<u>1^{st} case</u> : $s = \infty$; that is $\alpha = \beta$.

From the definitions and (3.4) we get

$$\mathfrak{M}_{q,\beta} f(x) = \mathfrak{M}_{q,\alpha} f(x) \leq 2^d \|f\|_{q,\infty,\alpha} \leq 2^d \|f\|_{q,p_0,\alpha} \quad , \quad x \in \mathbb{R}^d$$

and therefore

$$\|\mathfrak{M}_{q,\beta} f\|_\infty \leq 2^d \|f\|_{q,p_0,\alpha}.$$

<u>2^{nd} case</u> : $s < \infty$ and therefore $p_0 < \infty$.

We may assume that f is a non nul element of $(L^q, l^{p_0})^\alpha$ (otherwise the claim is obvious).

Let λ be any positive real number and set $E_\lambda = \left\{x \in \mathbb{R}^d \ : \ \mathfrak{M}_{q,\beta} f(x) > \lambda\right\}$.

For any element x of E_λ, there is a positive real number $r(x)$ such that

$$\lambda < |Q(x, r(x))|^{\frac{1}{\beta}-\frac{1}{q}} \left\|f \chi_{Q(x,r(x))}\right\|_q .$$

This implies that

$$1 < \lambda^{-1} |Q(x, r(x))|^{\frac{1}{\beta}-\frac{1}{q}} \left\|f \chi_{Q(x,r(x))}\right\|_q \leq 2^d \lambda^{-1} \|f\|_{q,\infty,\alpha}\, r(x)^{-\frac{d}{s}} \quad , \quad x \in E_\lambda$$

and as a consequence,

$$r = \sup_{x \in E_\lambda} r(x) \leq \left(2^d \lambda^{-1} \|f\|_{q,\infty,\alpha}\right)^{\frac{s}{d}} < \infty . \tag{$*$}$$

Therefore, by Theorem 1.1 of Chap.I in [9] and the remarks following it, there is a sequence $(Q_l)_{l\geq 1}$ of elements of $\{Q(x,r(x))\ :\ x\in E_\lambda\}$ such that

- $E_\lambda \subset \bigcup_{l\geq 1}\overline{Q_l}$
- $\sum_{l\geq 1}\chi_{Q_l}\leq \theta_d$

where θ_d is a positive real number depending only on d. This implies that

$$|E_\lambda|^{\frac{1}{p_0}}\leq\left[\sum_{l\geq 1}|Q_l|\right]^{\frac{1}{p_0}}\leq\left[\sum_{l\geq 1}\left(\lambda^{-1}|Q_l|^{\frac{1}{\beta}-\frac{1}{q}}\left\|f\chi_{Q_l}\right\|_q\right)^{p_0}|Q_l|\right]^{\frac{1}{p_0}}=\lambda^{-1}\left[\sum_{l\geq 1}\left\|f\chi_{Q_l}\right\|_q^{p_0}\right]^{\frac{1}{p_0}}. \qquad (**)$$

Let us set, for any element k of $\mathbb{Z}^d$, $L_k=\{l\geq 1\ \ :\ \ |I_k^r\cap Q_l|>0\}$. We have

$$\begin{aligned}\sum_{l\geq 1}\left\|f\chi_{Q_l}\right\|_q^{p_0}&=\sum_{l\geq 1}\left[\sum_{\{k\in\mathbb{Z}^d\,:\,l\in L_k\}}\int_{I_k^r\cap Q_l}|f(x)|^q dx\right]^{\frac{p_0}{q}}\\&\leq\sum_{l\geq 1}\left\{2^{d\left(1-\frac{q}{p_0}\right)}\left[\sum_{\{k\in\mathbb{Z}^d\,:\,l\in L_k\}}\left(\int_{I_k^r\cap Q_l}|f(x)|^q dx\right)^{\frac{p_0}{q}}\right]^{\frac{q}{p_0}}\right\}^{\frac{p_0}{q}}\end{aligned}$$

by the Hölder inequality and the fact that the number of elements of $\left\{k\in\mathbb{Z}^d\ :\ l\in L_k\right\}$ does not exceed 2^d. Therefore

$$\begin{aligned}\sum_{l\geq 1}\left\|f\chi_{Q_l}\right\|_q^{p_0}&\leq 2^{d\left(\frac{p_0}{q}-1\right)}\sum_{k\in\mathbb{Z}^d}\sum_{l\geq 1}\left(\int_{I_k^r\cap Q_l}|f(x)|^q dx\right)^{\frac{p_0}{q}}\leq 2^{d\left(\frac{p_0}{q}-1\right)}\sum_{k\in\mathbb{Z}^d}\left(\sum_{l\geq 1}\int_{I_k^r\cap Q_l}|f(x)|^q dx\right)^{\frac{p_0}{q}}\\&=2^{d\left(\frac{p_0}{q}-1\right)}\sum_{k\in\mathbb{Z}^d}\left(\int_{\mathbb{R}^d}|f(x)|^q\sum_{l\geq 1}\chi_{I_k^r\cap Q_l}(x)dx\right)^{\frac{p_0}{q}}\leq 2^{d\left(\frac{p_0}{q}-1\right)}\,\theta_d^{\frac{p_0}{q}}\sum_{k\in\mathbb{Z}^d}\left\|f\chi_{I_k^r}\right\|_q^{p_0}.\end{aligned}$$

From this inequality, $(*)$ and $(**)$ we get

$$\begin{aligned}|E_\lambda|^{\frac{1}{p_0}}&\leq\lambda^{-1}\,2^{d\left(\frac{1}{q}-\frac{1}{p_0}\right)}\theta_d^{\frac{1}{q}}\,{}_r\|f\|_{q,p_0}\leq\lambda^{-1}\,2^{d\left(\frac{1}{q}-\frac{1}{p_0}\right)}\theta_d^{\frac{1}{q}}\,\|f\|_{q,p_0,\alpha}\,r^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)}\\&\leq\lambda^{-1}\,2^{d\left(\frac{1}{q}-\frac{1}{p_0}\right)}\theta_d^{\frac{1}{q}}\,\|f\|_{q,p_0,\alpha}\left(2^d\,\lambda^{-1}\|f\|_{q,\infty,\alpha}\right)^{s\left(\frac{1}{q}-\frac{1}{\alpha}\right)}\leq A\,\lambda^{-\frac{s}{p_0}}\|f\|_{q,p_0,\alpha}\,\|f\|_{q,\infty,\alpha}^{\frac{s}{p_0}-1}\end{aligned}$$

where $A=\theta_d^{\frac{1}{q}}\,2^{d\left[\frac{1}{q}-\frac{1}{p_0}+s\left(\frac{1}{q}-\frac{1}{\alpha}\right)\right]}$. Thus

$$\lambda\,\left|E_\lambda\right|^{\frac{1}{s}}\leq A^{\frac{p_0}{s}}\,\|f\|_{q,p_0,\alpha}^{\frac{p_0}{s}}\,\|f\|_{q,\infty,\alpha}^{1-\frac{p_0}{s}}\leq C\,\|f\|_{q,p_0,\alpha}$$

where C is a real number not depending on f and λ. This ends the proof. ■

We notice that (7.4) implies that : given q,β and p in $[1,\infty]$ such that $\frac{1}{p}+\frac{1}{\beta}\leq\frac{1}{q}$, the set $\{f\in L^q_{\mathrm{loc}}\ :\ \mathfrak{M}_{q,\beta}f\in L^p\}$ is included in $(L^q,l^p)^\alpha$ with $\frac{1}{\alpha}=\frac{1}{p}+\frac{1}{\beta}$. This result may be improved as follows.

Proposition 7.3. *Let* $1 \leq q < \beta < \infty,\ 0 < \frac{1}{p} \leq \frac{1}{q} - \frac{1}{\beta},\ \frac{1}{\alpha} = \frac{1}{p} + \frac{1}{\beta}$ *and* f *an element of* L^q_{loc} *such that* $\mathfrak{M}_{q,\beta} f$ *belongs to* L^p. *Then* f *is in* $(L^q, l^p)^\alpha_{\mathrm{c},0}$.

Proof. a) By (7.3), f belongs to $(L^q, l^p)^\alpha$.

b) • We have

$$\mathfrak{M}_{q,\beta} f(x) = \left(\mathfrak{M}_{1,\frac{\beta}{q}}\left(|f|^q\right)(x)\right)^{\frac{1}{q}} \qquad , \quad x \in \mathbb{R}^d. \qquad (*)$$

Therefore, by the hypothesis on $\mathfrak{M}_{q,\beta} f$ and Proposition 7.1, $I_{\frac{q}{\beta}}\left(|f|^q\right)$ belongs to $L^{\frac{p}{q}}$.

• Let $(E_n)_{n\geq 1}$ be a decreasing sequence of measurable subsets of $\mathbb{R}^d$ such that $\left|\bigcap\limits_{n\geq 1} E_n\right| = 0$.

Set for any integer $n \geq 1$, $f_n = f\chi_{E_n}$.

We have in L^1_{loc}

$$\begin{cases} |f_n| \leq |f| \qquad , \quad n \geq 1 \\ \\ \lim\limits_{n\to\infty} f_n = 0\ . \end{cases}$$

Furthermore, since $I_{\frac{q}{\beta}}(|f|^q)$ belongs to $L^{\frac{p}{q}}$, there is a measurable subset N of $\mathbb{R}^d$ such that $|N| = 0$ and

$$\int\limits_{\mathbb{R}^d} |x-y|^{d(\frac{q}{\beta}-1)} |f|^q(y)dy = I_{\frac{q}{\beta}}(|f|^q)(x) < \infty \qquad , \quad x \in \mathbb{R}^d \backslash N.$$

Thus, for any element x of $\mathbb{R}^d \backslash N$, we have

$$\begin{cases} \bullet\ y \mapsto |x-y|^{d(\frac{q}{\beta}-1)}|f(y)|^q \quad \text{belongs to } L^1 \\ \\ \bullet\ \text{for almost every element } y \text{ of } \mathbb{R}^d \\ \\ \quad \begin{cases} \bullet\ 0 \leq |x-y|^{d(\frac{q}{\beta}-1)}|f_n(y)|^q \leq |x-y|^{d(\frac{q}{\beta}-1)}|f(y)|^q \ , \quad n \geq 1 \\ \\ \bullet\ \lim\limits_{n\to\infty} |x-y|^{d(\frac{q}{\beta}-1)}|f_n(y)|^q = 0. \end{cases} \end{cases}$$

Consequently, by the dominated convergence theorem, we obtain

$$\lim_{n\to\infty} I_{\frac{q}{\beta}}(|f_n|^q)(x) = 0 \qquad , \quad x \in \mathbb{R}^d \backslash N.$$

Therefore we have

$$\begin{cases} \bullet\ \left(I_{\frac{q}{\beta}}(|f|^q)\right)^{\frac{p}{q}} \in L^1 \\ \\ \bullet\ \text{for almost every element } x \text{ of } \mathbb{R}^d \\ \\ \quad \begin{cases} \bullet\ 0 \leq \left(I_{\frac{q}{\beta}}\left(|f_n|^q\right)\right)^{\frac{p}{q}}(x) \leq \left(I_{\frac{q}{\beta}}|f|^q\right)^{\frac{p}{q}}(x) \quad , \quad n \geq 1 \\ \\ \bullet\ \lim\limits_{n\to\infty} \left(I_{\frac{q}{\beta}}(|f_n|^q)\right)^{\frac{p}{q}}(x) = 0\ . \end{cases} \end{cases}$$

Another application of the dominated convergence theorem gives $\lim_{n\to\infty}\left\|I_{\frac{q}{\beta}}\left(|f_n|^q\right)\right\|_{\frac{p}{q}} = 0.$

From this we obtain

$$\lim_{n\to\infty}\left\|\mathfrak{M}_{q,\beta}f_n\right\|_p = 0 \qquad (\text{ by } (*) \text{ and Proposition } 7.1)$$

and therefore $\lim_{n\to\infty}\|f_n\|_{q,p,\alpha} = 0 \quad (\text{ by } (7.4))$.

Thus f is of absolutely continuous norm in $(L^q, l^p)^\alpha$ and so it belongs to $(L^q, l^p)^\alpha_{c,0}$ (see Proposition 6.1). ■

Remark 7.4. *Theorem 2.13 follows readily from (7.2) and Propostion 7.3.*

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Bibliography

[1] Adams, D. R.: *Morrey Spaces*; Lecture Notes in Applied and Numerical Harmonic Analysis, Birkhäuser/ Springer, Cham (2015).

[2] Aguilera, N. E. and Harboure, E. O.: *On the search of weighted norm inequalities for the Fourier transform*; Pacific J. Math., 104 (1), 1-14 (1983).

[3] Alabalik, A., Almeida, A. and Samko, S.: *On the invariance of certain vanishing subspaces of Morrey spaces with respect to some classical operators*; Banach J. Math. Anal. (2020) https://doi.org/10.1007/s43037-019-00049-7.

[4] Almeida, A. and Samko, S.: *Approximation in Morrey spaces*; J. Funct. Anal., 272, 2392–2411 (2017).

[5] Brezis H.: Functional Analysis, Sobolev Spaces and Partial Differential Equations; Springer New York Dordrecht Heidelberg London (2011).

[6] Chiarenza, F. and Franciosi, M.: *A generalization of a theorem by C.Miranda*; Ann. Mat. Pura Appl., 161, 285-297 (1992).

[7] Chiarenza, F. and Frasca, M.: *Morrey spaces and Hardy-Littlewood maximal function*; Rend. Math., 7 , 273–279 (1987).

[8] Coulibaly, S. and Fofana, I.: *On Lebesgue integrability of Fourier transforms in amalgam spaces*; J. Fourier Anal. Appl., 25, 184–209 (2019).

[9] De Guzmán, M. : *Differentiation of Integrals in* $\mathbb{R}^n$; Lecture Notes in Mathematics (481); Springer-Verlag; Berlin Heidelberg-New York (1975).

[10] Diarra, N. and Fofana, I.: *On preduals and Köthe duals of some subspaces of Morrey spaces*; J. Math. Anal. Appl., 496 (2021), https://DOI.org/10.1016/j.jmaa.2020.124842.

[11] Dosso, M., Fofana, I. and Sanogo, M.: *On some subspaces of Morrey-Sobolev spaces and boundedness of Riesz integrals*; Ann. Pol. Math., 108, 133-153 (2013).

[12] Feichtinger, H. G. and Feuto, J.: *Pre-dual of Fofana's spaces*; MDPI Mathematics, 7, 528 (2019), DOI:10.3390/math7060528.

[13] Feuto, J.: *Norm inequalities in some subspaces of Morrey space*; Ann. Math. Blaise Pascal 21 (2), 21-37 (2014).

[14] Fofana, I.: *Étude d'une classe d'espaces de fonctions contenant les espaces de Lorentz*; Afr. Mat., 1 , 29-50 (1988).

[15] Fofana, I.: *Espace de fonctions à moyenne fractionnaire intégrable*. Thèse de Doctorat d'Etat es-Sciences, numéro d'ordre 224, Université de Cocody, Abidjan R.C.I, (1995) http://greenstone.refer.bf/collect/thef/index/assoc/HASH6a39.dir/CS_02767.pdf

[16] Fofana, I.: *Espace* $(L^q, l^p)^\alpha(\mathbb{R}^d)$ *et continuité de l'opérateur maximal fractionnaire de Hardy-Littlewood*; Afrika Mat., 12, 23-37 (2001).

[17] Folland, G. B.: *Real Analysis: Modern Techniques and Their Applications*; Second Edition, John Wiley and Sons, Inc, (1999).

[18] Fournier, J. and Stewart, J.: *Amalgams of* L^p *and* L^q; Bull. Am. Math. Soc., 13, 1–21 (1985).

[19] Guliyev S.: *Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces*; J. Inequal. Appl., (2009), DOI:10.1155/2009/503948.

[20] Hakim, D. I. and Sawano, Y.: *Calderón's first and second complex interpolations of closed subspaces of Morrey spaces*; J. Fourier Anal. Appl., 23 (5), 1195-1226 (2017).

[21] Holland, F. : *Harmonic analysis on amalgams of* L^p *and* l^q; J. Lond. Math. Soc., 10, 295–305 (1975).

[22] Kato, T.: *Strong solutions of the Navier-Stokes equations in Morrey spaces*; Bol. Soc. Bras. Mat., 22, 127–155 (1992).

[23] Kpata, B. A. and Fofana, I.: *Isomorphism between Sobolev spaces and Bessel potential spaces in the setting of Wiener amalgams spaces*; Com. Math. Ana., 16 (2) , 57–73 (2014).

[24] Muckenhoupt, B.: *Weighted norm inequalities for classical operators*; Proc. Symp. in Pure Math., 35 (1), 69–83 (1979).

[25] Morrey, C. B.: *On the solutions of quasi-linear elliptic partial differential equations*; Trans. Am. Math. Soc., 43, 126–166 (1938).

[26] Phuc, N. C. and Torrès, M.: *Characterizations of the existence and removable singularities of divergence-measure vector fields*; Indiana Univ. Math. J., 57 (4), 1573-1597 (2008).

[27] Rafeiro, H., Samko, N. and Samko, S.: *Morrey-Campanato spaces: an overview. In: Karlovich, Yi., Rodino, L., Silbermann, B., Spitkovsky, IM. (eds.) Operator Theory, Pseudo-Differential Equations, and Mathematical Physics*; Advances and Applications, Springer Basel, 228, 293–323 (2013).

[28] Sawano, Y.: *A thought on generalized Morrey spaces*; J. Indones. Math. Soc., 25 (3), 210-281 (2019).

[29] Transirico, M., Troisi, M. and Vitolo, A.: *Spaces of Morrey type and elliptic equations in divergence form on unbounded domains*; Bolletino U.M.I., 9B (1), 153-174 (1995).

[30] Wiener, N.: *On the representation of functions by trigonometrical integrals*; Math. Z., 24, 575-616 (1926).

[31] Yuan, W., Sickel, W. and Yang, D.: *Interpolation of Morrey–Campanato and related smoothness spaces*; Sci. Math. China., 58 (9), 1835–1908 (2015).

[32] Zorko, C. T.: *Morrey space*; Proc. Am. Math. Soc., 98, 586–592 (1986).

