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Global existence of solutions to the spherically symmetric Einstein-Vlasov-Maxwell system

Existence globale de solutions au système sphériquement symétrique de Einstein-Vlasov-Maxwell

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ABSTRACT. We prove that the initial value problem with small data for the asymptotically flat spherically symmetric Einstein-Vlasov-Maxwell system admits the global in time solution in the case of the non-zero shift vector. This result extends the one already known for chargeless case.
KEYWORDS. Spherical symmetry, Shift vector, Fermi derivative, Malec's estimate.
MSC: 83C20, 83C22.

1. Introduction

The Einstein-Vlasov (EV) system describes the evolution of self-gravitating collisionless fast moving massive particles. This system models distribution of stars, galaxies or galaxy clusters and constitutes an accurate model for the large scale structure of spacetime. Recently a global existence theorem of the EV system was proved by D. Fajman in [6], L. Bigorgne in [2], H. Lindblad in [12] and M. Taylor in [22]. These authors used the vector field method for transport equations. Notice that these results need a strong knowledge in differential geometry, moreover the understanding and the proof are so heavy. To get a simpler proof on global existence theorem for that system, an assumption of spherical symmetry was taken into account. So, G. Rein and A. D. Rendall [17] established a global existence theorem for solutions to the EV system, for small initial data, and A. D. Rendall prove the same result in the case of non-diagonal metric [19]. We notice that all these results are concerned with the uncharged case.

The EV system coupled with the Maxwell equations models the time evolution of self-gravitating collisionless massive charged particles in the context of general relativity. The particles could be for instance the electrons in a plasma. Up to now, we are not aware that a global existence theorem has been already established for the Einstein-Vlasov-Maxwell (EVM) system without any symmetry assumptions. However, in [16], the author proved a global in time existence theorem for the spherically symmetric EVM system with initial data sufficiently small and with diagonal metric.

The present work is concerned with the EVM system in the spherical symmetry context, where the unknown spacetime is $(\mathbb{R}^4, g)$, and the isotropic metric ansatz g is not diagonal as in [16]. We try to extend to our new context, the global results obtained in the chargeless case. Contrary to what is done in [16], the general form of g forces us to use the $1+3$ formulation in writing down the EVM system; this changes the calculations considerably. Besides, in the chargeless case, trajectory of particles is a geodesic and the latter is deviated where introducing the electromagnetic field F that is reduced to its electric part e in the spherical symmetry setting. This forces us to use the concept of Fermi derivative that is necessary to control the bound of particle momentum on the support of the distribution function

f of particles and obtain a decay condition to the energy density ρ. That concept is not necessary in the previous works. On the other hand, the fundamental gauge fixing condition that allows us to obtain the global result not only depends on $\mathring{f}$ which is an initial datum of f as in [16] and [17] but also on $\mathring{A}$, $\mathring{K}$ and $\mathring{e}$ initial data of the metric function $A(t,r)$, the second fundamental form $K(t,r)$ and the electric field $e(t,r)$ respectively. This is a reason why the presence of the electric field gives rise to a new and interesting mathematical problem to be studied.

Here, we aim to establish a global existence theorem of solutions for the asymptotically flat spherically symmetric EVM system. To do so, we prove on the first hand, the continuous dependence of solution on the initial data $(\mathring{f}, \mathring{A}, \mathring{K}, \mathring{e})$ where $\mathring{A}$ (resp. $\mathring{K}$, $\mathring{e}$ and $\mathring{f}$) is the initial datum of the metric function $A(t,r)$ (resp. the second fundamental form $K(t,r)$, the electric field $e(t,r)$ and f). On the second hand, a global existence theorem of solutions is investigated. This generalizes the result obtained by A. D. Rendall in [19] in the chargeless case and the one of the second author in [16] where the shift function of ADM is equal to zero $(\beta = 0)$.

The paper proceeds as follows. In Section 2, we formulate the Cauchy problem. In Section 3, we show that if the charge is small then one can choose a set of initial data so that solutions depend continuously on the latter. In Section 4, we establish a global existence theorem for the solution corresponding to the initial boundary value problem and the obtained spacetime is found to be geodesically complete.

2. The Einstein-Vlasov-Maxwell system

We consider fast moving collisionless particles with charge q. The basic spacetime is $(\mathbb{R}^4, g)$, with g a Lorentzian metric with signature $(-,+,+,+)$. In what follows, we assume that Greek indices run from 0 to 3 and Latin indices from 1 to 3, unless otherwise specified. We also adopt the Einstein summation convention, both the gravitational constant and the speed of light are set to unity, ultimately, we assume that all particles have the identical rest mass equal to unity.

The metric g reads locally, in Cartesian coordinates $(x^\alpha) = (x^0, x^i) = (t, x)$

$$ds^2 = g_{\alpha\beta}\, dx^\alpha \otimes\, dx^\beta. \tag{1}$$

The Einstein-Vlasov-Maxwell system can be written

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi(T_{\alpha\beta} + \tau_{\alpha\beta}), \tag{2}$$

$$\mathcal{L}_{X(F)} f = 0, \tag{3}$$

$$\nabla_\alpha F^{\alpha\beta} = J^\beta \quad and \quad \nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = 0, \tag{4}$$

with

$$T_{\alpha\beta} = -\int_{\mathbb{R}^3} p_\alpha p_\beta f |g|^{\frac{1}{2}} \frac{dp^1 dp^2 dp^3}{p_0}; \quad \tau_{\alpha\beta} = -\frac{g_{\alpha\beta}}{4} F_{\lambda\sigma} F^{\lambda\sigma} + F_\alpha{}^\lambda F_{\beta\lambda};$$

$$J^\beta = q \int_{\mathbb{R}^3} f(x,p) \frac{p^\beta |g|^{\frac{1}{2}}}{p_0} dp^1 dp^2 dp^3; \quad X^\alpha(F) = (p^\alpha, -\Gamma^\alpha_{\lambda\mu} p^\lambda p^\mu - q p^\lambda F_\lambda{}^\alpha),$$

where $\Gamma^{\alpha}_{\lambda\mu}$ denote the Christoffel symbols, $\mathcal{L}_{X(F)}$ the Lie derivative, $T_{\alpha\beta}$ and $\tau_{\alpha\beta}$ are the energy-momentum and Maxwell tensors respectively and J^{β} denotes the Maxwell current. Here $x = (x^{\alpha})$ is the position and $p = (p^{\alpha})$ is the 4-momentum of the particles. In the expressions above, f stands for the distribution function of the charged particles, $F = (F_{\alpha\beta})$ stands for the electromagnetic field created by the charged particles. Here (2) are the Einstein equations for the metric tensor $g = (g_{\alpha\beta})$ with sources generated by both f and F, that appear in the stress-energy tensor $8\pi(T_{\alpha\beta} + \tau_{\alpha\beta})$. Equation (3) is the Vlasov equation for the distribution function f of the collisionless particles and (4) are the Maxwell equations for the electromagnetic field F with source current generated by f through $J = J(f)$. One verifies (using the normal coordinates) that the conservation law $\nabla_{\alpha}(T^{\alpha\beta} + \tau^{\alpha\beta}) = 0$ holds if f satisfies the Vlasov equation.

Now, using the assumption of spherical symmetry, the metric g can be written in the isotropic form (maximal-isotropic coordinates) [21].

$$ds^2 = -\alpha^2 dt^2 + A^2[(dr + \beta dt)^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)], \tag{5}$$

where $\alpha = \alpha(t, r)$, $\beta = \beta(t, r)$, $A = A(t, r)$; $t \in \mathbb{R}$, $r \in [0, +\infty[$, $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi]$. Here α and β are the lapse and shift functions respectively, and $(\beta^i = \beta\frac{x^i}{r})$ is the shift vector. One obtains that for the metric of the form (5) which satisfies the extra condition that the hypersurfaces of constant time are maximal, the EVM field equations and the coordinate freedom conditions can be written in the $1 + 3$ formulation, see [20] ($for\ details\ we\ \ refer\ to\ section$ $2.3,\ 3.2$ and $11.2.2$), as the following P.D.E system in A, α, K, β, e, f

$$\Big(r^2(A^{1/2})'\Big)' = -\frac{1}{8}r^2 A^{5/2}\Big(\frac{3}{2}K^2 + 16\pi\rho\Big), \tag{6}$$

$$\alpha'' + 2r^{-1}\alpha' + A^{-1}A'\alpha' = \alpha A^2\Big[\frac{3}{2}K^2 + 4\pi(\rho + trS)\Big], \tag{7}$$

$$K' + 3(A^{-1}A' + 1/r)K = 8\pi A\, j, \tag{8}$$

$$\beta' - r^{-1}\beta = \frac{3}{2}\alpha K, \tag{9}$$

$$\partial_t A = -\alpha AK + (\beta A)', \tag{10}$$

$$\begin{aligned}\partial_t K = &-A^{-2}\alpha'' + A^{-3}A'\alpha' + \alpha\big[-2A^{-3}A'' + 2A^{-4}A'^2 - 2A^{-3}A'/r - 8\pi S_r \\ &+ 4\pi trS - 4\pi\rho\big] + \beta K'\end{aligned} \tag{11}$$

$$\frac{\partial}{\partial r}\Big(r^2 A^3 e\Big) = qr^2 A^3 M, \tag{12}$$

$$\frac{\partial}{\partial t}\Big(A^3 e\Big) = q\beta A^3 M - q\alpha A^2 N, \tag{13}$$

$$\begin{aligned}&\frac{\partial f}{\partial t} + \left(\alpha A^{-1}\frac{v}{\sqrt{1+|v|^2}} - \beta\frac{x}{r}\right)\cdot\frac{\partial f}{\partial x} + \Bigg[-A^{-1}\alpha'\sqrt{1+|v|^2}\frac{x}{r} - \frac{1}{2}\alpha K\Big(v - 3\frac{x\cdot v}{r}\frac{x}{r}\Big) \\ &-\alpha A^{-2}A'\Big(\frac{x\cdot v}{r}v - |v|^2\frac{x}{r}\Big)\frac{1}{\sqrt{1+|v|^2}} - qeA\left(-\alpha + \frac{\beta A}{\sqrt{1+|v|^2}}\frac{x\cdot v}{r}\right)\frac{x}{r}\Bigg]\cdot\frac{\partial f}{\partial v} = 0,\end{aligned} \tag{14}$$

where $A' = \frac{\partial A}{\partial r}$. Here equation (6) is the explicit form of the Hamiltonian constraint in this class of spacetime with this kind of coordinate. The maximal slicing condition is expressed by the lapse equation (7). The momentum constraint equation is given by (8). Equation (9) is a consequence of the coordinate condition chosen while (10) follows from the definition of the second fundamental form. The one non-trivial Einstein evolution equation in this class of spacetimes is (11), (12) is the Maxwell constraint equation for e, (13) is the Maxwell evolution equation. Finally, (14) is the Vlasov equation for f, f being the distribution function of the charged particles which is defined on the mass-shell. Here $x := (x^i)$ and v belong to $\mathbb{R}^3$, $r := |x|$, $x \cdot v$ denotes the usual scalar product of vectors in $\mathbb{R}^3$, and $|v|^2 = v \cdot v$. This form of the Einstein field equations had been used by S. L. Shapiro and S. A. Teukolsky for numerical calculations (see [21]). The distribution function f is assumed to be invariant under simultaneous rotations of x and v, hence the differents matter terms above can be regarded as functions of t and r. It is assumed that $f(t)$ has compact support for each fixed t. We also recall that in the above, we took $x^i = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)$ and considered the corresponding orthonormal frame $e_i = A^{-1}\partial/\partial x^i$. So the mass-shell is coordinatized by (t, x^i, v^i) in which the Vlasov equation is writing.

The matter quantities are given by

$$\rho(t,r) = \rho(t,x) := \int_{\mathbb{R}^3} \sqrt{1+|v|^2} f(t,x,v)dv \; + \; \frac{1}{2}A^2(t,x)e^2(t,x), \tag{15}$$

$$trS(t,r) = trS(t,x) := \int_{\mathbb{R}^3} |v|^2 f(t,x,v)\frac{dv}{\sqrt{1+|v|^2}} \; + \; \frac{1}{2}A^2(t,x)e^2(t,x), \tag{16}$$

$$j(t,r) = j(t,x) := \int_{\mathbb{R}^3} \left(\frac{x \cdot v}{r}\right) f(t,x,v)dv, \tag{17}$$

$$S_r(t,r) = S_r(t,x) := \int_{\mathbb{R}^3} \left(\frac{x \cdot v}{r}\right)^2 f(t,x,v)\frac{dv}{\sqrt{1+|v|^2}} \; - \; \frac{1}{2}A^2(t,x)e^2(t,x), \tag{18}$$

$$M(t,r) = M(t,x) := \int_{\mathbb{R}^3} f(t,x,v)dv, \tag{19}$$

$$N(t,r) = N(t,x) := \int_{\mathbb{R}^3} \left(\frac{x \cdot v}{r}\right) f \frac{dv}{\sqrt{1+|v|^2}}, \tag{20}$$

where if n^α is the unit future-pointing normal vector to the hypersurface of constant time, then $\rho = (T_{\alpha\beta} + \tau_{\alpha\beta})n^\alpha n^\beta$ is the energy density, $trS = (T_{\alpha\beta} + \tau_{\alpha\beta})(g^{\alpha\beta} + n^\alpha n^\beta)$ is the $3\times$ $isotropic\,''pressure''$, j is obtained by contracting $-(g^{i\alpha} + n^i n^\alpha)(T_{\alpha\beta} + \tau_{\alpha\beta})n^\beta$ with $\frac{x^i}{r}$, S_r is obtained by contracting $(g^{i\alpha} + n^i n^\alpha)(g^{j\beta} + n^j n^\beta)(T_{\alpha\beta} + \tau_{\alpha\beta})$ once with $\frac{x^i}{r}$ and once with $\frac{x^j}{r}$, M comes from the Maxwell current component J^0 and N follows from contracting the part of the Maxwell current J^i with $\frac{x^i}{r}$.

The boundary conditions in case of asymptotically flat spacetime in spatial and time coordinate with regular center are given by

$$\lim_{r\to\infty} A(t,r) = \lim_{r\to\infty} \alpha(t,r) = 1, \tag{21}$$

$$\lim_{r\to\infty} \beta(t,r) = \beta(t,0) = \lim_{r\to\infty} K(t,r) = K(t,0) = \lim_{r\to\infty} e(t,r) = e(t,0) = 0, \tag{22}$$

$$\lim_{t\to\infty} A(t,r) = \lim_{t\to\infty} \alpha(t,r) = 1 \;\; and \;\; \lim_{t\to\infty} \beta(t,r) = 0. \tag{23}$$

Now, define the initial data by

$$\begin{cases} f(0,x,v) = \mathring{f}(x,v), \quad A(0,x) = \mathring{A}(x) = \mathring{A}(r) \\ K(0,x) = \mathring{K}(x) = \mathring{K}(r), \quad \alpha(0,x) = \mathring{\alpha}(x) = \mathring{\alpha}(r) \\ \beta(0,x) = \mathring{\beta}(x) = \mathring{\beta}(r), \quad e(0,x) = \mathring{e}(x) = \mathring{e}(r) \end{cases} \tag{24}$$

with $\mathring{f}$ being a C^1 function with compact support, which is non-negative and spherically symmetric, i.e.,

$$\forall\, A \in SO(3),\ \ \forall\,(x,v) \in \mathbb{R}^6, \quad \mathring{f}(Ax,Av) = \mathring{f}(x,v).$$

We have to solve the boundary initial value problem $(6)-(9)$, (12), (14), (21), (22) and (24).

3. Preliminary results

In this section, we first prove that some unknown functions can be bounded and we introduce the notion of ADM mass. Secondly, we recall the local existence and continuation criterion theorem on which our present result relies. Finally, we state an important theorem which proves that the difference between two solutions can be bounded by a constant time multiple of $d = \|\mathring{f} - \mathring{g}\|_\infty$. All these results will be applied in the proof of global existence.

Definition 3.1. *A solution (A,α,K,β,e,f) of $(6)-(9)$, (12) and (14) is said to be regular if*
$i)$ f is non-negative, spherically symmetric, C^1, and $f(t)$ is compactly supported for all t,
$ii)$ A, α, A', α', K, β and e are C^1 with A, α, K, β and e satisfying the boundary conditions (21) and (22).

It is easy to see that (A,α,K,β) satisfies the following inequalities.

Proposition 3.1. *Consider a solution (f,α,β,A,K,e) of the EVM system defined on some interval $[0,T[$ satisfying the boundary conditions, then we have the following estimates for $t \in [0,T[$ and $r \geq 0$.*

$$\begin{aligned} &A(t,r) \geq 1, \quad \alpha(t,r) \leq 1, \quad |\beta(t,r)| \leq 3, \\ &|K(t,r)| \leq 2r^{-1} \quad \textit{and} \quad |A^{-1}A'(t,r)| \leq 3r^{-1} \end{aligned} \tag{25}$$

Proof: It suffices to rearrange equations (6), (7) and (9), use the boundary conditions and an estimate of E. Malec and N. O. Muchadha $[13]$ which gives the expansions of the null geodesics.

We also notice that the ADM mass defined by

$$m_{ADM} = \frac{1}{8}\int_0^{+\infty} r^2\left[A^{5/2}\Big(\frac{3}{2}K^2 + 16\pi\rho\Big)\right](t,r)\,dr, \tag{26}$$

is time independent. Denote by $m(t,r)$ the mass function defined as follows

$$m(t,r) = \frac{1}{8}\int_0^{r} s^2\left[A^{5/2}\Big(\frac{3}{2}K^2 + 16\pi\rho\Big)\right](t,s)\,ds. \tag{27}$$

Theorem 3.1. (Local existence, continuation criterion). *Let $\mathring{f} \in C^\infty(\mathbb{R}^6)$ be a non-negative, compactly supported and spherically symmetric, $\mathring{A}$, $\mathring{\alpha}$, $\mathring{K}$, $\mathring{\beta}$, $\mathring{e} \in C^\infty(\mathbb{R}^3)$ be a regular solution of the constraints equations such that*

$$\mathring{A}(0) \geq 1 + 2\int_0^r \frac{\left(\mathring{A}^{1/2}\mathring{m}\right)(s)}{s^2}\, ds, \tag{28}$$

then there exists a unique regular solution $(A, \alpha, K, \beta, e, f)$ of the asymptotically flat spherically symmetric EVM system with initial data $(\mathring{A}, \mathring{\alpha}, \mathring{K}, \mathring{\beta}, \mathring{e}, \mathring{f})$ on a maximal interval of existence $[0, T[$. If $A(t, 0)$ and $u(t)$ are bounded then $T = +\infty$. Where $u(t)$ is the largest momentum of any particle at time t. In the other words

$$u(t) = \sup\left\{|v|;\ \ f(t, x, v) \neq 0\ for\ some\ x\right\}. \tag{29}$$

Proof of Theorem 3.1 This is obtained by adapting to our new context the corresponding proof in the zero shift vector case [15].

For $r_0 > 0$ and $u_0 > 0$ define the following set of initial data

$D := \Big\{(\mathring{f},\ \mathring{A},\ \mathring{K},\ \mathring{e}) \in\ C^1(\mathbb{R}^6) \times C^2([0, +\infty[) \times (C^1([0, +\infty[))^2;$ $\quad \mathring{f}$ is a non-negative, spherically symmetric with a support contains on $B(r_0) \times B(u_0)$ and $(\mathring{A},\ \mathring{K},\ \mathring{e})$ is a regular solution of the constraints satisfying (28) $\Big\}$

Proposition 3.2. *Consider initial data to the EVM system $(\mathring{f},\ \mathring{A}_f,\ \mathring{K}_f,\ \mathring{e}_f)$, $(\mathring{g},\ \mathring{A}_g,\ \mathring{K}_g,\ \mathring{e}_g) \in D$ such that there exists a constant $\Lambda > 0$ with*

$$|\mathring{A}_f(0) - \mathring{A}_g(0)| \leq \Lambda\|\mathring{f} - \mathring{g}\|_\infty, \tag{30}$$

given a sufficiently small real number $\varepsilon > 0$, If $d := \|\mathring{f} - \mathring{g}\|_\infty < \varepsilon$ then

$$\|\mathring{K}_f\|_\infty, \qquad \|\mathring{e}_f\|_\infty \leq C, \tag{31}$$

$$\begin{aligned} &\|\mathring{A}_f - \mathring{A}_g\|_\infty, \qquad \|\mathring{e}_f - \mathring{e}_g\|_\infty, \qquad \|\mathring{K}_f - \mathring{K}_g\|_\infty, \\ &\|\mathring{A}_f^3\mathring{e}_f - \mathring{A}_g^3\mathring{e}_g\|_\infty, \qquad \|\mathring{A}_f^3\mathring{K}_f - \mathring{A}_g^3\mathring{K}_g\|_\infty \leq C\, d \end{aligned} \tag{32}$$

where the constant C depends on r_0, u_0, $\mathring{g}$, Λ and not on $\mathring{f}$.

Proof: Take $(\mathring{f},\ \mathring{A}_f,\ \mathring{K}_f,\ \mathring{e}_f)$ and $(\mathring{g},\ \mathring{A}_f,\ \mathring{K}_f,\ \mathring{e}_f)$ such that $\|\mathring{f} - \mathring{g}\|_\infty \leq \varepsilon$. Setting $t = 0$ in (12) and replacing f by $\mathring{f}$, we obtain by integration

$$\mathring{e}_f(r) = \frac{q}{r^2}\mathring{A}_f^{-3}\int_0^r s^2\left(\mathring{A}_f^3\mathring{M}_f\right)(s)\, ds,$$

where $\mathring{M}_f$ is defined as in (19) by replacing f by $\mathring{f}$. Distinguishing the cases $r \leq r_0$ and $r \geq r_0$, we obtain, since $\mathring{f}$ is with compact support, $\left|\mathring{e}_f(r)\right| \leq C$. Remarking that the equation (8) can be rearranged to give $\left(r^3A^3K\right)' = 8\pi r^3A^4j$, we also obtain in the same way $\left|\mathring{K}_f(r)\right| \leq C$, where $C = C(r_0, u_0, \mathring{g}, \Lambda)$

is a constant, and (31) holds. Note that using (6), A can be expressed in term of ρ via an integration. To prove inequalities (32), one writes

$$\left|\mathring{A}_f - \mathring{A}_g\right|(r) \leq \left|\mathring{A}_f(0) - \mathring{A}_g(0)\right| \;+\; C\int_0^r \frac{\mathring{m}_f(s)}{s^2}\left|\mathring{A}_f - \mathring{A}_g\right|(s)\,ds$$

$$+\; C\int_0^r \frac{\left|\mathring{m}_f - \mathring{m}_g\right|(s)}{s^2}\,ds, \tag{33}$$

where $\mathring{m}_f = m_f(0,r)$, $m(t,r)$ is the mass function given by (27). We obtain using (30), with the Gronwall lemma

$$\left|\mathring{A}_f - \mathring{A}_g\right|(r) \leq C\|\mathring{f} - \mathring{g}\|_\infty \exp\left(C\int_0^r \left[\frac{\mathring{m}_f}{s^2} \;+\; (r+1)^2\Big(\left|\mathring{j}_f\right| \;+\; \left|\mathring{M}_f\right|\Big)\right](s)\,ds\right) \leq C\,d,$$

and the rest of inequalities in (32) follow immediately from those above. Thus, the proof of Proposition 3.2 is complete.

We now state the essential result of this section.

Theorem 3.2. *There exist a constant $\varepsilon > 0$, a positive increasing function $C \in C\big([0, T_g[\big)$, and a positive decreasing function $\eta \in C\big(]0, \varepsilon[\big)$ such that*

$$\lim_{s\to 0} \eta(s) = T_g, \tag{34}$$

and for any solution $(\alpha_f, \beta_f, A_f, K_f, e_f, f)$ with initial datum satisfying the condition (30) of proposition 3.2 and $d = \left\|\mathring{f} - \mathring{g}\right\|_\infty \leq \varepsilon$, we have the following estimates

$$T_f > \eta(d), \tag{35}$$

$$\|f(t) - g(t)\|_\infty \;+\; \|A_f(t) - A_g(t)\|_\infty \;+\; \|A'_f(t) - A'_g(t)\|_\infty$$

$$+\; \|A''_f(t) - A''_g(t)\|_\infty \;+\; \|\alpha'_f(t) - \alpha'_g(t)\|_\infty \;+\; \|\beta_f(t) - \beta_g(t)\|_\infty \;+\; \|\beta'_f(t) - \beta'_g(t)\|_\infty$$

$$+\; \|K_f(t) - K_g(t)\|_\infty \;+\; \|e_f(t) - e_g(t)\|_\infty \leq C(t)d, \tag{36}$$

for $t \in [0, \eta(d)]$. The analogous assertion holds for $t \leq 0$.

Proof: Consider the solution $(\alpha_f, \beta_f, A_f, K_f, e_f, f)$ with initial data close enough to $\mathring{g}$, define on the support of f

$$T_0(f) := \sup\Big\{\; t \in \big]0, \min(T_f, T_g)\big[;\; \textit{such that}\; \forall s \in [0,t],\; \|A_f(s) - A_g(s)\|_\infty \;+\; \|K_f(s)$$

$$-\; K_g(s)\|_\infty + \|e_f(s) - e_g(s)\|_\infty \;+\; \|A'_f(s) - A'_g(s)\|_\infty \;+\; \|\alpha'_f(s) - \alpha'_g(s)\|_\infty \leq 1\Big\}$$

Notice that for d small enough, say $d < \varepsilon_1$, for a suitable defined $\varepsilon_1 > 0$, the estimate defining $T_0(f)$ holds at $t = 0$ so that by continuity, $T_0(f) > 0$. Let $(\alpha_f, \beta_f, A_f, K_f, e_f, f)$ be the solution with d small enough, the characteristics can be bounded on $\big[0, T_0(f)\big[$ by

$$|\dot{x}(s)| \leq 4,$$

$$|\dot{v}(s)| \leq C\Big(\|\alpha'_f(s)\|_\infty \ + \ \|K_f(s)\|_\infty \ + \ \|A'_f(s)\|_\infty \ + \ \|A_f(s)\|_\infty \|e_f(s)\|_\infty\Big)(1+|v(s)|),$$

with $C = C(q)$ being a constant. Via the Gronwall inequality this implies that

$$\mathtt{Supp} f(t) \subseteq \Big\{(x,v) \in \mathbb{R}^6; \ \ |x| \leq r_0 + 4t \; and \; |v| \leq U_g(t)\Big\}, \tag{37}$$

for $t \in \big[0, T_0(f)\big[$, where

$$U_g(t) = (1+u_0)\exp\Bigg(\int_0^t C\Big(1 + \|\alpha'_g\|_\infty + \|K_g\|_\infty + \|A'_g\|_\infty$$

$$+ \|A_g\|_\infty + \|e_g\|_\infty + \|A_g\|_\infty \|e_g\|_\infty\Big)(s)ds\Bigg).$$

Denote by C a continuous, increasing function on $[0, T_g[$ which depends only on $(\alpha_g, A_g, K_g, e_g, g)$, for example $C(t) = U_g(t)$. We have the following estimates on $\big[0, T_0(f)\big[$

$$w(t) \leq C(t)\Big(\|f(t) - g(t)\|_\infty + \|A_f(t) - A_g(t)\|_\infty + \|e_f(t) - e_g(t)\|_\infty\Big), \tag{38}$$

where we setting

$$w(t) := \|\rho_f(t) - \rho_g(t)\|_\infty + \|trS_f(t) - trS_g(t)\|_\infty + \|j_f(t) - j_g(t)\|_\infty.$$

We now, find an estimate for $\frac{d}{ds}(f-g)$ along the characteristics $Z_f = (X_f, V_f)$ corresponding to f. From the fact that f is constant along these characteristics and from the Vlasov equation for g, using the mean value theorem it follows that

$$\left|\frac{d}{ds}(f-g)\big(s, Z_f(s,t,z)\big)\right| \leq C(s)\Big(\|\alpha_f - \alpha_g\|_\infty + \|\beta_f - \beta_g\|_\infty + \|A_f - A_g\|_\infty + \|K_f - K_g\|_\infty$$

$$+ \|e_f - e_g\|_\infty + \|\alpha'_f - \alpha'_g\|_\infty + \|A'_f - A'_g\|_\infty\Big)(s),$$

and by integration with respect to s from 0 to t, one has

$$\|f(t) - g(t)\|_\infty \leq \|\mathring{f} - \mathring{g}\|_\infty \ + \ \int_0^t C(s)\Big(\|\alpha_f - \alpha_g\|_\infty \ + \ \|\beta_f - \beta_g\|_\infty \ + \ \|A_f - A_g\|_\infty$$

$$+ \|K_f - K_g\|_\infty \ + \ \|e_f - e_g\|_\infty \ + \ \|\alpha'_f - \alpha'_g\|_\infty \ + \ \|A'_f - A'_g\|_\infty\Big)(s)ds, \tag{39}$$

using (38) and the fact that $\|f\|_\infty = \|\mathring{f}\|_\infty \leq \varepsilon_1 + \|\mathring{g}\|_\infty$, we obtain the estimate, for any $t \in \ \big[0, T_0(f)\big[$

$$\|\rho_f(t)\|_\infty + \|trS_f(t)\|_\infty + \|j_f(t)\|_\infty + \|M_f(t)\|_\infty + \|N_f(t)\|_\infty \leq C(t), \tag{40}$$

on the other hand, the equation (8) in K can be rewritten as $(r^3A^3K)' = 8\pi r^3 A^4 j$. The integration on $[0, r]$ gives, using (38) and (40)

$$\big\|A_f^3 K_f - A_g^3 K_g\big\|_\infty(t) \leq C(t)\Big(\|f - g\|_\infty + \|A_f - A_g\|_\infty + \|e_f - e_g\|_\infty\Big)(t), \tag{41}$$

using the fact $A_g \geq 1$, the mean value theorem implies the following inequality for K

$$\|K_f - K_g\|_\infty(t) \leq C(t)\big(\|f-g\|_\infty + \|A_f - A_g\|_\infty + \|e_f - e_g\|_\infty\big)(t). \tag{42}$$

Equation (7) can be rewritten as

$$(r^2 A\alpha')' = \alpha A^3 r^2\Big(\frac{3}{2}K^2 + 4\pi(\rho + trS)\Big).$$

So, in the same way as above, we also obtain after integrating on $[0, r]$

$$\|A_f\alpha'_f - A_g\alpha'_g\|_\infty(t) \leq C(t)\big(\|f-g\|_\infty + \|A_f - A_g\|_\infty + \|e_f - e_g\|_\infty\big)(t), \tag{43}$$

hence we can deduce that

$$\|\alpha'_f - \alpha'_g\|_\infty(t) \leq C(t)\big(\|f-g\|_\infty + \|A_f - A_g\|_\infty + \|e_f - e_g\|_\infty\big)(t). \tag{44}$$

From equation (6), we obtain an analogous estimate for A'

$$\|A'_f - A'_g\|_\infty(t) \leq C(t)\big(\|f-g\|_\infty + \|A_f - A_g\|_\infty + \|e_f - e_g\|_\infty\big)(t). \tag{45}$$

Equation (9) can be rearranged as $(r^{-1}\beta)' = \frac{3}{2}r^{-1}\alpha K$, integrating on $[r, +\infty[$ and using the fact that $\lim_{r\to\infty}\beta(r) = 0$, we also deduce that

$$\|\beta_f - \beta_g\|_\infty(t) \leq C(t)\big(\|f-g\|_\infty + \|A_f - A_g\|_\infty + \|e_f - e_g\|_\infty\big)(t). \tag{46}$$

We deduce from (39), using the estimates have been established above, that

$$\|f-g\|_\infty \leq \|\mathring{f} - \mathring{g}\|_\infty + \int_0^t C(s)\Big(\|f-g\|_\infty + \|A_f - A_g\|_\infty + \|e_f - e_g\|_\infty\Big)(s)ds \tag{47}$$

Evolution's equations (10) and (13) of A and e respectively also yield by virtue of above estimates

$$\big|\partial_s A_f - \partial_s A_g\big|(s,r) \leq C(s)\Big(\|f-g\|_\infty \ + \ \|A_f - A_g\|_\infty \ + \ \|e_f - e_g\|_\infty\Big)(s),$$

$$\big|\partial_s(A_f^3 e_f) - \partial_s(A_g^3 e_g)\big|(s,r) \leq C(s)\Big(\|f-g\|_\infty \ + \ \|A_f - A_g\|_\infty \ + \ \|e_f - e_g\|_\infty\Big)(s)$$

and

$$\big|\partial_s e_f - \partial_s e_g\big|(s,r) \leq C(s)\Big(\|f-g\|_\infty \ + \ \|A_f - A_g\|_\infty \ + \ \|e_f - e_g\|_\infty\Big)(s).$$

Integrating with respect to s from 0 to t, we get

$$\begin{aligned}\|A_f(t) - A_g(t)\|_\infty \leq \|\mathring{A}_f - \mathring{A}_g\|_\infty \ + \ &\int_0^t C(s)\Big(\|f-g\|_\infty \\ &+ \ \|A_f - A_g\|_\infty \ + \ \|e_f - e_g\|_\infty\Big)(s)ds,\end{aligned} \tag{48}$$

$$\|A_f^3 e_f(t) - A_g^3 e_g(t)\|_\infty \leq \|\mathring{A}_f^3 \mathring{e}_f - \mathring{A}_g^3 \mathring{e}_g\|_\infty \; + \; \int_0^t C(s)\Big(\|f-g\|_\infty$$

$$+ \; \|A_f - A_g\|_\infty \; + \; \|e_f - e_g\|_\infty\Big)(s)ds \tag{49}$$

and

$$\|e_f(t) - e_g(t)\|_\infty \leq \|\mathring{e}_f - \mathring{e}_g\|_\infty \; + \; \int_0^t C(s)\Big(\|f-g\|_\infty$$

$$+ \; \|A_f - A_g\|_\infty \; + \; \|e_f - e_g\|_\infty\Big)(s)ds, \tag{50}$$

setting

$$z(t) := \|f(t) - g(t)\|_\infty + \|A_f(t) - A_g(t)\|_\infty + \|e_f(t) - e_g(t)\|_\infty + \|A_f^3 e_f(t) - A_g^3 e_g(t)\|_\infty.$$

It follows that on the interval $[0, T_0(f)[$, using (32), the inequality

$$z(t) \leq C\|\mathring{f} - \mathring{g}\|_\infty \; + \; \int_0^t C(s)\Big(\|f-g\|_\infty \; + \; \|A_f - A_g\|_\infty$$

$$+ \; \|e_f - e_g\|_\infty + \|A_f^3 e_f - A_g^3 e_g\|_\infty\Big)(s)ds,$$

holds, where the function $C \in C\big([0, T_0(f)[\big)$ depends only on g and can be taken strictly increasing with $\lim_{t\to T_g} C(t) = +\infty$. By the Gronwall inequality, we have

$$z(t) \leq C\|\mathring{f} - \mathring{g}\|_\infty \exp\Big(\int_0^t C(s)ds\Big) \leq C(t)d. \tag{51}$$

The rest terms of inequality (36) of theorem 3.2 follow from (41), (42), (44), (45) and (46) using (51). Since $C(t)$ can be taken as $C(0) > 0$, define

$$\varepsilon = \min\Big\{\varepsilon_1; \frac{1}{2C(0)}\Big\}; \qquad \eta(s) = C^{-1}\Big(\frac{1}{2s}\Big) \qquad s \in]0, \varepsilon[,$$

then $\eta \in C\big(]0,\varepsilon[\big)$ is positve decreasing and $\lim_{s\to 0}\eta(s) = T_g$. Then on interval $\Big[0;\, \min\big(\eta(d), T_0(f)\big)\Big]$, it follows that since C is strictly increasing

$$B(t) := \|A_f - A_g\|_\infty + \|K_f - K_g\|_\infty + \|A'_f - A'_g\|_\infty + \|\alpha'_f - \alpha'_g\|_\infty + \|e_f - e_g\|_\infty$$
$$\leq C(t)d \leq C\big(\eta(d)\big)d = \frac{1}{2d}d = \frac{1}{2}$$

thus

$$B(t) \leq \frac{1}{2} \qquad for\; t < \eta(d). \tag{52}$$

Assume that $T_f \leq \eta(d)$. Then by definition of $T_0(f)$ and (52) we obtain the identity $T_0(f) = \min(T_f, T_g) = T_f$, in particular the estimate

$$|v| \leq U_g\big(\eta(d)\big) < +\infty,$$

holds for all $(x,v) \in \mathtt{Supp}\big(f(t)\big)$ and $t \in [0, T_f[$ since $T_f \leq \eta(d) < +\infty$, this is a contracdiction to the local existence and continuation criterion theorem, so we have shown that $\eta(d) < T_f$. Futhermore, (52) implies that $T_0(f) > \eta(d)$ so that the estimates which were established on the interval $[0, T_0(f)[$ hold on $[0, \eta(d)]$, and the proof of theorem 3.2 is complete.

4. Global existence

In this section global existence of solutions of the EVM system will be proved when the spherically symmetric initial data and the charge of particle are sufficiently small. The essential ingredient of the proof is the concept of Fermi derivative along a geodesic that is necessary to control the bound of particle momentum and obtain certain decay properties . We also observe that the obtained spacetime is found to be geodesically complete.

Theorem 4.1. *Let $r_0 > 0$ and $u_0 > 0$ be positive constants. There exists $\varepsilon > 0$ such that if $(f, \alpha, \beta, A, K, e)$ is the maximal solution of the asymptotically flat spherically symmetric Einstein-Vlasov-Maxwell system with initial data satisfying*

$$\textit{Supp}\, \mathring{f} \subseteq B(r_0) \times B(u_0), \qquad \|\mathring{f}\| < \varepsilon,$$

then the solution exists globally in time t. Moreover, for this solution the metric coefficients α^{-1} and A are bounded and the following estimates hold for some constant $C > 0$

$$\|\alpha'(t)\|_\infty + \|A'(t)\|_\infty + \|K(t)\|_\infty \leq C(1+t)^{-2},$$

$$\|e(t)\|_\infty + \|K_{\mu\nu}(t)\|_\infty + \|\gamma^\alpha_{\mu\nu}(t)\|_\infty \leq C(1+t)^{-2},$$

$$\|A''(t)\|_\infty + \|\rho(t)\|_\infty + \|trS(t)\|_\infty + \|j(t)\|_\infty \leq C(1+t)^{-3},$$

$$\|M(t)\|_\infty + \|N(t)\|_\infty + \|S_R(t)\|_\infty + \|R_{\alpha\beta,\mu\nu}(t)\|_\infty \leq C(1+t)^{-3},$$

for $t \in \mathbb{R}$, and the geodesics are defined on $\mathbb{R}$.

Before coming to the proof of theorem 4.1, we need to establish some results.

Lemma 4.1. *Let $\delta \in]0, 1]$ and $C_1 > 0$. Then there exists a constant $C_2 > 0$ only depending on r_0, C_1 and δ such that, if $(f, \alpha, \beta, A, K, e)$ is a spherically symmetric asymptotically flat solution of the EVM system on a time interval $[0, T[$, satisfying the estimates*

$$\sup\Big\{|x|;\ \ (x, v) \in \mathtt{Supp}\big(f(t)\big)\Big\} \leq r_0 + 4t,$$

$$A(t, 0) \leq 3 \quad \forall\, t \in [0, T[,$$

$$\|\rho(t)\|_\infty \leq C_1(1+t)^{-2-\delta} \quad \forall\, t \in [0, T[, \tag{53}$$

then we have the following decay conditions for $(t, r) \in [0, T[\times[0, +\infty[$

$$\begin{cases} |r^{-1}A'(t)| + |r^{-1}\alpha'(t)| + |r^{-1}K(t)| + |r^{-1}e(t)| \leq C_2(1+t)^{-2-\delta} \\ \|A''(t)\|_\infty + \|\alpha''(t)\|_\infty + \|trS(t)\|_\infty \leq C_2(1+t)^{-2-\delta} \\ \|M(t)\|_\infty + \|N(t)\|_\infty + \|j(t)\|_\infty \leq C_2(1+t)^{-2-\delta} \\ \|S_R(t)\|_\infty + \|R_{\alpha\beta,\mu\nu}(t)\|_\infty \leq C_2(1+t)^{-2-\delta} \\ \|A'(t)\|_\infty + \|\alpha'(t)\|_\infty + \|K(t)\|_\infty + \|e(t)\|_\infty \leq C_2(1+t)^{-1-\delta} \end{cases} \tag{54}$$

Proof: The main idea of the proof of lemma is to show by virtue of differents equations (6), (7), (8) and (12) that $\|A'(t)\|_\infty$, $\|\alpha'(t)\|_\infty$, $\|K(t)\|_\infty$ and $\|e(t)\|_\infty$ can be bounded by a constant multiple of $\left(\|\rho(t)\|_\infty r_0 + r_0^{-2}\right)$. Remarking that the latter function of r_0 has a critical point at $r_0^* = \left(\frac{1}{2}\|\rho(t)\|_\infty\right)^{-1/3}$. Then the proof of lemma follows.

The following result comes immediately from the Gronwall inequality.

Lemma 4.2. *Let $\delta \in]0,1]$, $r_0 > 0$ and $u_0 > 0$. If C_2 is taken sufficiently small such that the solution of the EVM system satisfies* (54)*, $A(t,0) \leq 3$ on $[0,T[$ and $\mathtt{Supp}\,\mathring{f} \subseteq B(r_0) \times B(u_0)$, then the largest momentum satisfies the estimate*

$$u(t) \leq 1 + u_0\,, \qquad \forall\, t \,\in\, [0,T[. \tag{55}$$

Next, Consider a timelike curve γ described by parametric relation $x^\mu(\tau)$ in which τ is proper time measured along γ, starting at $t=0$. Suppose that γ intersects the initial hypersurface $t=0$ and passing through a point (t,x).

Definition 4.1. *Let $u^\mu = \frac{dx^\mu}{d\tau}$ be the tangent vector to γ and $a^\mu = \frac{Du^\mu}{d\tau} = u^\nu \nabla_\nu u^\mu$ its acceleration vector. A vector field v^μ is said to be Fermi-Walker transported on γ if it is a solution to the differential equation*

$$\frac{Dv^\mu}{d\tau} = (v_\nu a^\nu) u^\mu - (v_\nu u^\nu) a^\mu. \tag{56}$$

Remark 4.1. *The operation of Fermi-Walker transport have two important properties*

1) *$e'_0 = u^\mu = \frac{dx^\mu}{d\tau}$ is automatically Fermi-Walker transported along γ. Indeed $a_\mu u^\mu = 0$ and $u_\mu u^\mu = -1$.*

2) *If v^μ and w^μ is Fermi-Walker transported along γ, then their inner product is constant along γ i.e. $\frac{D}{d\tau}(v_\mu w^\mu) = 0$.*

Let $e'_0 = \frac{dx^\mu}{d\tau}$ be the curve tangent vector, we can complete this vector to obtain an orthonormal frame $\{e'_\sigma\}$ which is Fermi-Walker transported along γ. Let $\{\theta'^\sigma\}$ be the dual coframe which is also Fermi-Walker transported along γ. Define by Z^s the separation of two infinitesimally neighbouring particles, Z^s satisfies the following equation ($see\ for\ more\ details,\ [10],\ p.82$), known as the worldlines deviation or Jacobi equation which gives the relative acceleration of two particles. Note that here the families of worldlines of particles have the same charge q

$$\frac{d^2 Z^s}{d\tau^2} = \Big(- R^s_{\beta\gamma\delta}\, e'^\beta_0 e'^\gamma_t e'^\delta_0 + e'^\gamma_t \nabla_\gamma \dot{u}^s + e'^\gamma_t \dot{u}^s \dot{u}_\gamma \Big) Z^t, \tag{57}$$

where

$$\dot{u}^\alpha = u^\beta \nabla_\beta\, u^\alpha = e'^\beta_0 \nabla_\beta\, e'^\alpha_0. \tag{58}$$

Now the crucial step to prove the global existence theorem is to estimate the jacobian determinant of the mapping

$$v \to X(0,t,x,v), \tag{59}$$

for fixed values t and x, $X(s,t,x,v)$ is part of the solution of the characteristics system of the Vlasov equation written in terms of the frame components. Remark that, the derivative of the mapping (59) can conveniently be written as a composition $L_3 L_2 L_1$ of three linear mappings from $\mathbb{R}^3$ to itself. The first mapping L_1 is the one sending the components of a vector tangent to the mass shell in the basis $\{\partial/\partial v^i\}$ to the components Y^s. Let $L_2(Y^s)$ be the value at $t=0$ of the solution of (57) with the initial data $Z^s=0$ and $dZ^s/d\tau = Y^s$ at the point (t,x). Let $L_3(Z^s)$ be the cartesian components of the vector obtained by projecting the vector $Z^s e'_s$ onto the hypersurface $t=0$ along the vector e'_0.

Lemma 4.3. *Let ε, r_0, $u_0 > 0$. Then there exists a constant $C_3 > 0$ depending only on ε, r_0, u_0 and C_2 such that if (f,α,β,A,K,e) is the spherically symmetric asymptotically flat solution of the EVM system satisfying (54) for some constant $C_2 > 0$ on a time interval $[0,T[$, with*

$$\mathtt{Supp}\,\mathring{f} \subseteq B(r_0)\times B(u_0), \qquad \|\mathring{f}\|_\infty \leq \varepsilon \quad \textit{and} \quad A(t,0)\leq 3,$$

then

$$\alpha(t,r) \geq 1/2 \quad \textit{and} \quad \|\rho(t)\|_\infty \leq C_3 t^{-3}, \qquad (t,r)\in\ ]0,T[\times[0,+\infty[.$$

Proof: Firstly, we prove that $\|K(t)\|_\infty^2$, $\|e(t)\|_\infty^2$, $\|\rho(t)\|_\infty < C_3\varepsilon$. In fact, by virtue of equation (12) and distinguishing the cases $r\leq r_0$ and $r\geq r_0$, one get the following inequality

$$|e(t,r)| \leq C_3\left(r_0\|\mathring{f}\|_\infty + r_0^{-2}\right), \qquad \forall\, r\geq 0,\ t\in[0,T[.$$

The function of r_0 occuring on the right hand side in this last inequality has a critical point to $r_0^* = \left(\frac{1}{2}\|\mathring{f}\|_\infty\right)^{-1/3}$, it follows that $|e(t,r)| \leq C_3\|\mathring{f}\|_\infty^{2/3}$, hence $\|e(t)\|_\infty \leq C_3\|\mathring{f}\|_\infty^{2/3}$ or $\|e(t)\|_\infty^2 \leq C_3\|\mathring{f}\|_\infty$ for $t\in[0,T[$ with ε sufficiently small. In the same way we also get that $\|K(t)\|_\infty \leq C_3\|\mathring{f}\|_\infty^{2/3}$ or $\|K(t)\|_\infty^2 \leq C_3\|\mathring{f}\|_\infty$ for $t\in[0,T[$. By defintion (15) of ρ and using lemma 4.2 we also have $\|\rho(t)\|_\infty \leq C_3\,\|\mathring{f}\|_\infty$ for $t\in[0,T[$. Since $\|\mathring{f}\|_\infty\leq\varepsilon$, we conclude that $\|K(t)\|_\infty^2$, $\|e(t)\|_\infty^2$ and $\|\rho(t)\|_\infty \leq C_3\,\varepsilon$. Next, from (7), one has

$$(r^2 A\alpha')' = \alpha A^3 r^2\left(\frac{3}{2}K^2 + 4\pi\big(\rho + trS\big)\right).$$

It follows that $\alpha'(r) \leq C_3\,\varepsilon\,\min(r, r^{-2})$, integrating on $[r,+\infty[$ and use the fact that $\lim_{r\to+\infty}\alpha(r) = 1$, one has

$$\alpha(r) \geq 1 - \frac{3}{2}C_3\,\varepsilon.$$

If we choose ε so that $C_3\,\varepsilon < 1/3$, then the last inequality will give $\alpha(t,r)\geq 1/2$.

The inequality (55) implies, since $e'_0 = \frac{dx^\mu}{d\tau}$ is the momentum of the particle, a uniform bound for the cartesian components of e'_0 if the tangent vector to the curve γ is contained in the support of the distribution function (only the curves of this kind are interested here). In order to estimate the cartesian components of the remaining frame vectors e'_s, note that it suffices, under the assumptions of the lemma, to estimate their components in the frame e_μ. Let these components be denoted by U_s^μ so that $e'_s = U_s^\mu e_\mu$. Since e'_s are Fermi-Walker transported along γ, we get using the definition of Fermi derivative

$$\frac{DU_s^\mu}{d\tau} = \Big(a_\nu U_s^\nu\Big)U_0^\mu - \underbrace{\Big(U_{0,\nu}U_s^\nu\Big)a^\mu}_{=0},$$

$$e_0'^\nu\partial_\nu U_s^\mu + \gamma^\mu_{\nu\lambda}U_s^\lambda U_0^\nu = qF_{\nu\lambda}U_s^\nu U_0^\lambda U_0^\mu,$$

where equation of motion of the particles and a simple computation give $a^{\nu} = qF^{\nu}{}_{\lambda}U_0^{\lambda}$ and $a_{\nu} = g_{\nu\alpha}a^{\alpha} = qF_{\nu\lambda}U_0^{\lambda}$. Hence the components U_s^{μ} satisfies

$$\frac{dU_s^{\mu}}{d\tau} + \gamma^{\mu}_{\nu\lambda}U_s^{\lambda}U_0^{\nu} = qF_{\nu\lambda}U_s^{\nu}U_0^{\lambda}U_0^{\mu}, \tag{60}$$

and the spatial components of (60) can be written as

$$\frac{dU_s^{i}}{d\tau} + \gamma^{i}_{\nu\lambda}U_s^{\lambda}U_0^{\nu} = qF_{\nu\lambda}U_s^{\nu}U_0^{\lambda}U_0^{i}. \tag{61}$$

Since the electromagnetic field $F^{\alpha\beta}$ is reduced only its electric field part $F^{0i} = \alpha^{-1}E^i = \alpha^{-1}e\frac{x^i}{r}$, it is easy to see by virtue of (54) the following inequality

$$|F^{0i}| \leq C_3(1+t)^{-1-\delta} \qquad \forall\, t \in [0, T[, \tag{62}$$

and the same estimate holds for the Ricci rotation coefficients $\gamma^i_{\nu\lambda}$ since $1/2 \leq \alpha \leq 1$. The relation between the proper time and the coordinate time is given by

$$\frac{d\tau}{dt} = \alpha\big(1+|v|^2\big)^{-1/2}. \tag{63}$$

Since $\alpha \leq 1$, equations $(61) - (63)$ imply that

$$\left|\frac{dU_s^i}{dt}\right| \leq C_3\Big(|U_s^i| + |U_s^0|\Big)(1+t)^{-1-\delta}, \tag{64}$$

since e'_s is an unit vector, $|U_s^0| \leq C_3|U_s^i|$, (64) becomes

$$\left|\frac{dU_s^i}{dt}\right| \leq C_3|U_s^i|(1+t)^{-1-\delta},$$

Gronwall's lemma implies, after integrating the last inequality on $[0,t]$

$$|U_s^i| \leq |U_s^i(0)|e^{C_3/\delta} \qquad \forall t \in [0, T[. \tag{65}$$

Considering now that the vectors e'_s along all trajectories are contained in the support of f; we have a uniform bound for $|U_s^i(0)|$ and so (65) imply the bound of $|U_s^i|$ and hence of $|U_s^0|$. As indicated above, this implies a bound for the cartesian coordinate components of the frame vectors e'_s. (63) implies that there are positive constants K_1 and K_2 such that

$$K_1 t \leq \tau \leq K_2 t, \tag{66}$$

along the trajectory γ. We can assume without loss of generality that $K_2 \geq 1$ and then

$$(1+t)^{-2-\delta} \leq K_2(1+\tau)^{-2-\delta}. \tag{67}$$

Thus if $K^s_t := -R^s_{\beta\gamma\delta}\, e'^{\beta}_0 e'^{\gamma}_t e'^{\delta}_0 + e'^{\gamma}_t \nabla_{\gamma}\dot{u}^s + e'^{\gamma}_t \dot{u}^s \dot{u}_{\gamma}$ define in equation (57), an estimate of the form

$$|K^s_t(\tau)| \leq C_3(1+\tau)^{-2-\delta} \tag{68}$$

holds along γ. In fact by (54), the curvature tensor is bounded by $C_3(1+\tau)^{-2-\delta}$, so $|R^s_{\beta\gamma\delta}\, e'^{\beta}_0 e'^{\gamma}_t e'^{\delta}_0| \leq C_3(1+\tau)^{-2-\delta}$. Since $\dot{u}^{\mu} = qF^{\mu}{}_{\lambda}e'^{\lambda}_0$ and using (62), it is bounded by a constant time $C_3(1+\tau)^{-1-\delta}$, hence

$$|e'^{\gamma}_t \dot{u}^s \dot{u}_{\gamma}| \leq C_3(1+\tau)^{-2-\delta}.$$

To obtain (68) it suffices to bound $e_t^{\prime\gamma}\nabla_\gamma \dot{u}^s$ by $C_3(1+\tau)^{-2-\delta}$. Without loss of generality we can write $\nabla_\gamma \dot{u}^s = \partial_\gamma \dot{u}^s + \gamma^s_{\gamma\beta}\dot{u}^\beta$, the second term of the right hand side is already bounded by $C_3(1+\tau)^{-2-\delta}$; it rests to prove that $|\partial_\gamma \dot{u}^s| \leq C_3(1+\tau)^{-2-\delta}$. The equation of motion of the particles permit us to have

$$\Big|\frac{d^2x^\mu}{d\tau^2}\Big| = |-\gamma^\mu_{\lambda\beta}e_0^{\prime\lambda}e_0^{\prime\beta} + qF^\mu{}_\lambda e_0^{\prime\lambda}| \leq C_3(1+\tau)^{-1-\delta}, \tag{69}$$

for $\mu = 0$ it implies that $|d^2t/d\tau^2| \leq C_3(1+\tau)^{-1-\delta}$. Since $dt/d\tau = e_0^{\prime 0} = \alpha^{-1}\sqrt{1+|v|^2}$ and $e_0^{\prime\mu} = dx^\mu/d\tau$, we deduce from (69) that

$$|\partial_t\big(\alpha^{-1}\big)(\tau)| \leq C_3(1+\tau)^{-1-\delta}, \tag{70}$$

and

$$|\partial_t\, e_0^{\prime\mu}(\tau)| \leq C_3(1+\tau)^{-1-\delta}. \tag{71}$$

Maxwell's evolution equation (13) of the electric part e allows us to obtain a similary bound for the time derivative of e

$$|\partial_t\, e(\tau)| \leq C_3(1+\tau)^{-2-\delta}, \tag{72}$$

using (70), (72) and the fact that $F^{0s} = \alpha^{-1}\, e\frac{x^s}{r}$, we also have

$$|\partial_t F^{0s}(\tau)| \leq C_3(1+\tau)^{-2-\delta}, \tag{73}$$

(71), (73) and $\dot{u}^\mu = qF^\mu{}_\lambda e_0^{\prime\lambda}$ imply that $|\partial_t\, \dot{u}^s(\tau)| \leq C_3(1+\tau)^{-2-\delta}$. In the similar manner as above we also get $|\partial_i\, \dot{u}^s(\tau)| \leq C_3(1+\tau)^{-2-\delta}$. Hence, we finally get $|\partial_\gamma\, \dot{u}^s(\tau)| \leq C_3(1+\tau)^{-2-\delta}$. It completes the proof of the inequality (68).

Let τ_0 be the value of τ at the point (t,x) and let $Z^s(\tau)$ be the solution of (57) with $Z^s(\tau_0) = 0$ and $\frac{dZ^s(\tau_0)}{d\tau} = Y^s$. Define

$$E^s(\tau) = Z^s(\tau) - (\tau-\tau_0)Y^s. \tag{74}$$

Then by Taylor's theorem with integral remainder on interval $[\tau,\tau_0]$, one has

$$\begin{aligned}
E^s(\tau) &= E^s(\tau_0) + (\tau-\tau_0)\frac{dE^s(\tau_0)}{d\tau} + \int_{\tau_0}^{\tau}(\tau-\sigma)\frac{d^2E^s(\sigma)}{d\tau^2}d\sigma \\
&= \int_{\tau_0}^{\tau}(\tau-\sigma)\frac{d^2Z^s(\sigma)}{d\tau^2}d\sigma \\
&= \int_{\tau_0}^{\tau}(\tau-\sigma)K^s_t(\sigma)Z^t(\sigma)d\sigma \\
&= \int_{\tau_0}^{\tau}(\tau-\sigma)K^s_t(\sigma)E^t(\sigma)d\sigma + \int_{\tau_0}^{\tau}(\tau-\sigma)(\sigma-\tau_0)K^s_t(\sigma)Y^t d\sigma
\end{aligned}$$

hence

$$E^s(\tau) = \int_{\tau_0}^{\tau}(\tau-\sigma)(\sigma-\tau_0)K^s_t(\sigma)Y^t d\sigma + \int_{\tau_0}^{\tau}(\tau-\sigma)K^s_t(\sigma)E^t(\sigma)d\sigma, \tag{75}$$

the first integral can be estimate as follow using (68)

$$\begin{aligned}
\Big|\int_{\tau_0}^{\tau}(\tau-\sigma)(\sigma-\tau_0)K_t^s(\sigma)Y^t d\sigma\Big| &\leq \Big|\int_{\tau}^{\tau_0}(\tau-\sigma)(\sigma-\tau_0)K_t^s(\sigma)Y^t d\sigma\Big| \\
&\leq C_3\,|Y^t|\int_{\tau}^{\tau_0}|\tau-\sigma||\sigma-\tau_0|(1+\sigma)^{-2-\delta}d\sigma \\
&\leq C_3\,(\tau_0-\tau)|Y^t|\int_{\tau}^{\tau_0}\sigma(1+\sigma)^{-2-\delta}d\sigma \\
&\leq C_3\,(\tau_0-\tau)|Y^t|\underbrace{\int_{0}^{+\infty}\sigma(1+\sigma)^{-2-\delta}d\sigma}_{=\frac{1}{\delta(1+\delta)}} \\
&\leq C_3\,(\tau_0-\tau)|Y^t|
\end{aligned}$$

thus we obtain from (75) the following inequality

$$|E^s(\tau)| \leq C_3\,(\tau_0-\tau)|Y^t| + C_3\int_{\tau}^{\tau_0}\sigma(1+\sigma)^{-2-\delta}|E^s(\sigma)|d\sigma,$$

Gronwall's inequality gives

$$\begin{aligned}
|E^s(\tau)| &\leq C_3\,(\tau_0-\tau)|Y^t| exp\Big(C_3\int_{\tau}^{\tau_0}\sigma(1+\sigma)^{-2-\delta}d\sigma\Big) \\
&\leq C_3\,(\tau_0-\tau)|Y^t|. \qquad (76)
\end{aligned}$$

Using this last inequality and the definition (74) of $E^s(\tau)$ at $\tau=0$, we obtain $|\frac{1}{\tau_0}L_2(Y^s)+Y^s| \leq C_3\,|Y^t|$. Since the tangent vectors are contained in the support of the distribution function, the determinant of L_1 can be bounded uniformly so we can have without loss of generality that det$(L_1) \geq C_3$, it implies taking C_3 small enough and using the properties of determinant that det$(L_2) \geq C_3\tau_0^3$. The inequality (66) implies that τ_0 can be remplaced by t, so det$(L_2) \geq C_3\,t^3$. Since the explicite form of components of the vector $L_3(Z^s)$ are given by $Z^s\Big(e_s'^i - (e_s'^0/e_0'^0)e_0'^i\Big)$, it follows that the determinant of L_3 can be uniformly bounded and so $\det(L_3) \geq C_3$. Since the Jacobian determinant of the mapping (59) is equal to det$(L_3L_2L_1)$, we obtain

$$\Big|\det\Big(\partial_v X(0,t,x,.)\Big)\Big| \geq C_3\,t^3. \qquad (77)$$

We now able to change the variable on $\mathbb{R}^3$ on the definition of matter term ρ to obtain

$$\begin{aligned}
\int_{\mathbb{R}^3}\sqrt{1+|v|^2}f(t,x,v)dv &\leq \int_{Supp\,f(t,x,.)}\sqrt{1+|v|^2}f(t,x,v)dv \\
&\leq \sqrt{1+(1+u_0)^2}\int_{Supp\,f(t,x,.)}\mathring{f}\Big(\big(X,V\big)(0,t,x,v)\Big)dv
\end{aligned}$$

$$\leq C_3 \int\limits_{\chi} \left| \det\Big(\partial_v X(0,t,x,.)\Big)\right|^{-1} dX,$$

where $\chi = \Big\{X(0,t,x,v);\ v \in Supp\, f(t,x,.)\Big\}$, note that for all $(x,v) \in\ Suppf(t)$, $|X(0,t,x,v)| \leq r_0$. Using the estimate on the determinant (77), one has

$$\int\limits_{\mathbb{R}^3} \sqrt{1+|v|^2} f(t,x,v)dv \leq C_3\, t^{-3}, \qquad \forall t \in [0,T[. \tag{78}$$

By virtue of (54), we have for C_2 small enough $|e^2(t,r)| \leq {C_2}^2\,(1+t)^{-2-2\delta} \leq t^{-3}$. And since $A(t,0) \leq 3$, we finally get that $\|\rho(t)\|_\infty \leq (C_3 + \frac{9}{2})\, t^{-3}$. The proof of lemma 4.3 is complete.

Remark 4.2. We are not surprised by the fact that ρ decays like t^{-3}, since even in the Minkowski spacetime, free particles starting in a compact set spread out linearly with time and the associated density decays like t^{-3} as $t \to \infty$.

The following result comes immediately from the Gronwall lemma.

Lemma 4.4. *Let r_0, $u_0 > 0$. There exists $\varepsilon > 0$ such that if the initial data of the EVM system satisfy*

$$\mathtt{Supp}\mathring{f} \subseteq B(r_0) \times B(u_0), \qquad \|\mathring{f}\|_\infty < \varepsilon,$$

then there exists $T_1 > 0$ such that the solution to this initial value problem exists on $[0,T_1]$. Moreover $A(t,0) \leq 3$, $u(t) \leq 3u(0)$ and $\|\rho(t)\|_\infty$ is as small as desired on this time interval.

***Proof of theorem 4.1*:** Let r_0, $u_0 > 0$ be fixed constants. By virtue of lemma 4.4, define T to be the supremum of those times T_1 such that the solution (f,α,β,A,K,e) of the EVM system satisfies (28) and

$$\mathtt{Supp}\,\mathring{f} \subseteq B(r_0) \times B(u_0), \qquad \|\mathring{f}\|_\infty \leq \varepsilon$$

$$A(t,0) \leq 3\,, \quad \forall\, t \in [0,T[.$$

Here $T = \infty$, in fact we will show that it is the only possibility. Take $g = \beta_g = K_g = e_g = 0$, $A = A_g$, $\alpha = \alpha_g$ and $T_g = 1$, applying theorem 3.2, there exists $\varepsilon > 0$ small enough, a positive increasing function $C \in C\big([0,1[\big)$ and a positive decreasing function $\eta \in C\big([0,\varepsilon[\big)$ such that $\lim_{s\to 0} \eta(s) = 1$ and since $d := \|\mathring{f}\|_\infty \leq \varepsilon$, we have the estimate below

$$\|f(t)\|_\infty + \|e(t)\|_\infty \leq C(t)\,\varepsilon, \quad \forall\, t \in \big[0,\eta(\varepsilon)\big].$$

Then by definition of ρ, one gets

$$\|\rho(t)\|_\infty \leq \lambda\, C(t)\,\varepsilon, \quad \forall\, t \in \big[0,\eta(\varepsilon)\big],$$

where the constant λ depends only on u_0. So for $t \in\ [0,1]$

$$\|\rho(t)\|_\infty \leq \lambda\,\varepsilon \sup_{t\in[0,1]} C(t) \leq 1, \qquad with\ \varepsilon\ sufficiently\ small.$$

Next, take $\delta = 1/2$ and note by ${C_3}^*$ the constant C_3 corresponding to C_2 in lemma 4.3 and define

$$C^* := 8({C_3}^* + 1)$$

applying lemma 4.1, note ${C_2}^*$ the corresponding constant to $C_1 = C^*$ and we take for instance $T_1 = 4\Big(\frac{{C_2}^*}{C_2}\Big)^2 + 1$ to have

$$ {C_2}^*(1+t)^{-1} \leq C_2(1+t)^{-1/2}, \quad for\ t \geq T_1. $$

Using theorem 3.2 with $g = 0$, we can choose ε such that the solution $(f, \alpha, \beta, A, K, e)$ exists on $[0, T_1]$ and on this interval (54) is satisfying with parameters $\delta = 1/2$ and C_2, provided $\|\overset{\circ}{f}\| < \varepsilon$. Define

$$ T_2 := \sup\Big\{t \in [0, T[;\ \ (f, \alpha, \beta, A, K, e)\ satisfies\ (54)\ on\ [0, t]\Big\} $$

Then by definition $T_2 > T_1$ and using lemma 4.3

$$ \|\rho(t)\|_\infty \leq {C_3}^* t^{-3}, \quad t \in]0, T_2[, $$

the fact that $\|\rho(t)\|_\infty \leq 1$ for $t \in [0, 1]$ allows us to establish the following inequality

$$ \|\rho(t)\|_\infty \leq C^*(1+t)^{-3}, \quad t \in [0, T_2[. $$

Note that the inequality above is obtained, distinguishing the cases $0 \leq t \leq 1$ and $1 < t < T_2$. Now, using lemma 4.1, (54) holds with the parameters $\delta = 1$ and ${C_2}^*$, and with the choice of T_1, (54) holds again on $[T_1, T_2[$ with parameters $\delta = 1/2$ and C_2. By the construction of T_2 we obtain $T_2 = T$. We deduce from lemma 4.2

$$ \mathtt{Supp}\, f(t) \subseteq \mathbb{R}^3 \times B(1 + u_0), \quad t \in [0, T[, $$

and using the theorem 3.1, we conclude that $T = +\infty$.

On the other hand, the decay conditions with respect to coordinate time of the matter terms, Riemannian tensor $R_{\alpha\beta,\mu\nu}(t)$, the Ricci rotation coefficients $\gamma^\alpha_{\mu\nu}(t)$, the second fundamental form $K_{\mu\nu}(t)$, $K(t)$ and electric field $e(t)$ come with the proof. To prove that the solution $(f, \alpha, \beta, A, K, e)$ is asymptotically flat in time coordinate, we have just to show that $\alpha(t, r)$ and $A(t, r)$ tend to unity as $t \to +\infty$ for each fixed r while $\beta(t, r)$ and $e(t, r) \to 0$. Since $\lim_{r\to+\infty} A(t, r) = \lim_{r\to+\infty} \alpha(t, r) = 1$ and $\lim_{r\to+\infty} \beta(t, r) = 0$; the equations (6), (7) and (9) by integration on $[r, +\infty[$, and recalling the defintions (26) and (27) of ADM mass and mass function respectively, give for $r_0 \geq 0$, $t \geq 0$, the fact that A is bounded and with estimates obtained in theorem 4.1

$$ \begin{aligned} |A(t, r)| &\leq 1 + \int_r^{+\infty} |A'(t, s)|\, ds \\ &\leq 1 + \int_0^{+\infty} \frac{2}{s^2} A^{1/2} m(t, s)\, ds \\ &\leq 1 + 2\sqrt{3} \int_0^{r_0+4t} \frac{m(t, s)}{s^2} ds + 2\sqrt{3} \int_{r_0+4t}^{+\infty} \frac{m_{ADM}}{s^2}\, ds \\ &\leq 1 + \frac{\sqrt{3}}{4} \int_0^{r_0+4t} \frac{1}{s^2} \int_0^s s'^2 A^{5/2}\Big(\frac{3}{2}K^2 + 16\pi\rho\Big)(t, s')\, ds' ds + 2\, m_{ADM}\sqrt{3}\big(r_0 + 4t\big)^{-1} \\ &\leq 1 + C_2(1+t)^{-3}\big(r_0 + 4t\big)^2 + 2\, m_{ADM}\sqrt{3}\big(r_0 + 4t\big)^{-1}. \end{aligned} $$

In the same way we also get the similary estimates for α and β, in fact

$$\alpha(t,r) = 1 - \int_r^{+\infty} \alpha'(t,s)\,ds$$
$$= 1 - \int_r^{+\infty} \frac{1}{s^2 A} \int_0^s \alpha A^3 s'^2 \Big(\frac{3}{2}K^2 + 4\pi(\rho + trS)\Big)(t,s')\,ds'ds,$$

since $\alpha' \geq 0$

$$\int_r^{+\infty} \alpha'(t,s)\,ds = \int_r^{+\infty} |\alpha'(t,s)|\,ds$$
$$= \int_r^{+\infty} \frac{1}{s^2 A} \int_0^s \alpha A^3 s'^2 \Big(\frac{3}{2}K^2 + 4\pi(\rho + trS)\Big)(t,s')\,ds'ds$$
$$\leq C_2 \int_0^{+\infty} \frac{1}{s^2} \int_0^s s'^2 A^{5/2} \Big(\frac{3}{2}K^2 + 16\pi\rho\Big)(t,s')\,ds'ds,$$

hence

$$|\alpha(t,r)| \geq 1 - \int_r^{+\infty} |\alpha'(t,s)|\,ds$$
$$\geq 1 - \int_r^{+\infty} \frac{1}{s^2 A} \int_0^s \alpha A^3 s'^2 \Big(\frac{3}{2}K^2 + 4\pi(\rho + trS)\Big)(t,s')\,ds'ds$$
$$\geq 1 - C_2 \int_0^{+\infty} \frac{1}{s^2} \int_0^s s'^2 A^{5/2} \Big(\frac{3}{2}K^2 + 16\pi\rho\Big)(t,s')\,ds'ds$$
$$\geq 1 - C_2 \int_0^{r_0+4t} \frac{1}{s^2} \int_0^s s'^2 A^{5/2} \Big(\frac{3}{2}K^2 + 16\pi\rho\Big)(t,s')\,ds'ds - C_2 \int_{r_0+4t}^{+\infty} \frac{m_{ADM}}{s^2}\,ds$$
$$\geq 1 - C_2(1+t)^{-3}(r_0+4t)^2 + C_2(r_0+4t)^{-1},$$

next, using $\big(r^3A^3K\big)' = 8\pi r^3 A^4 j$

$$|\beta(t,r)| \leq \frac{3}{2} r \int_r^{+\infty} \frac{\alpha|K|}{s}(t,s)\,ds$$
$$\leq 12\pi r \int_0^{+\infty} \frac{1}{s^4} \int_0^s s'^3 A^4 |j|(t,s')\,ds'ds$$
$$\leq 12\pi r \int_0^{r_0+4t} \frac{1}{s^4} \int_0^s s'^3 A^4 |j|(t,s')\,ds'ds + 12\pi r \int_{r_0+4t}^{+\infty} \frac{1}{s^4} \int_0^s s'^3 A^4 |j|(t,s')\,ds'ds$$

$$\leq C_2\, r(1+t)^{-3}\big(r_0+4t\big) + C_2\, r \int_{r_0+4t}^{+\infty} \frac{m_{ADM}}{s^3}\, ds$$
$$\leq C_2\, r(1+t)^{-3}\big(r_0+4t\big) + C_2\, r\big(r_0+4t\big)^{-2}.$$

From above inequalities, using the fact that $1 \leq A(t,r)$, $\alpha(t,r) \leq 1$ and since we get $\|e(t)\|_\infty \leq \sqrt{2}\|\rho(t)\|_\infty$, one concludes that

$$\lim_{t\to+\infty} A(t,r) = \lim_{t\to+\infty} \alpha(t,r) = 1$$

and

$$\lim_{t\to+\infty} \beta(t,r) = \lim_{t\to+\infty} e(t,r) = 0.$$

So, the solution $(f, \alpha, \beta, A, K, e)$ is asymptotically flat in time coordinate. To end the proof of theorem 4.1, we notice that all geodesics are complete. To see the latter, we have shown that the proper time and coordinate time are equivalent along a timelike geodesic, combining this with the global existence in coordinate time shows that geodesics are complete.

Remark 4.3. One can prove as in the zero shift vector case that if singularity exists, the first one has to occur at the centre with arbitrary initial data.

5. Conclusion

In this work, we proved a global existence theorem for the EVM system in the spherically symmetric setting. To achieve this goal, the concept of Fermi derivative was used to control the particle momentum on the support of f. We hope to investigate in the next paper the global results to the EVM system without any symmetry assumptions.

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