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Continuous Modulated Shearlet Transform

Transformation de Shearlet modulée continue

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ABSTRACT. We generalize the well-known transforms such as short-time Fourier transform, wavelet transform and shearlet transform and refer it as *Continuous Modulated Shearlet Transform*. Important properties like Plancherel formula and inversion formula have been investigated. Uncertainty inequalities associated with this transform are presented.
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1. Introduction

The signal processing technology focuses on analyzingl various signals such as speech, image, music and medical signals. For a long period, the classical Fourier transform has been a standard and fundamental tool for analyzing the frequency properties of a given signal. This transform captures all the frequency content of the signal, but fails to capture the moment in time when various frequencies were actually exhibited. For a music signal with high notes in the beginning and low notes during the middle, the Fourier transform captures this information, but there is no indication of when the high or low notes actually occur in time. In such cases, overall frequency obtained from the Fourier transform is inadequate to describe these signals. Therefore a time-frequency analysis is required for characterizing the signals.

The short-time Fourier transform or Gabor transform includes a window function that localizes the given signal over a specific window of time and the frequency content in that window is extracted. The time-filtering window is then slided down the entire signal to pick the frequency information at each instant of time. This transform captures the entire time-frequency content of the signal. According to uncertainty principle, there is a trade-off between the resolution of time and frequency, i.e., higher accuracy in one of these parameters comes at the cost of lower accuracy in the other parameter.

The Gabor transform method can be modified by allowing the scaling window to vary in order to successively extract improvements in the time resolution. In other words, a broad scaling window is used to extract out lower frequencies and poor time resolution. The scaling window is subsequently shortened in order to extract out higher frequencies and better time resolution. The process can be conducted to obtain excellent resolutions in both time and frequency of a given signal. This is the fundamental principle of wavelet theory. Instead of dealing with the time-frequency plane, we decompose the signal into the time-scale plane that is free from the restriction of the uncertainty principle.

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The Shearlet transform is advantageous over the classical wavelet transform as the former precisely resolves the wavefront set of a distribution and provides information about the directionality within the image, for details see [11].

The basic elements of these transforms are

(i) Short-time Fourier transform: Modulation and translation

(ii) Wavelet transform: Scaling and translation

(iii) Shearlet transform: Scaling, shear and translation

We treat all these elements, viz., translation, scaling, shear and modulation simultaneously, to create a new transform referred as *Continuous Modulated Shearlet Transform*.

In quantum physics, the uncertainty principle tells us that the position and momentum of a particle cannot be accurately measured simultaneously. Mathematically, uncertainty principles give limitations on the simultaneous concentration of a non zero function and its Fourier transform. In signal analysis, they tell us that if we observe a signal for a finite period of time, we will lose information about the frequencies of the signal. A precise qualitative formulation of the uncertainty principles is the Heisenberg uncertainty inequality, see [1].

THEOREM 1.1. For any $f \in L^2(\mathbb{R}^n)$ and $a, b \geq 1$, we have

$$\frac{n\|f\|_2^{\left(\frac{1}{a}+\frac{1}{b}\right)}}{4\pi} \leq \left(\int_{\mathbb{R}^n} \|x\|^{2a} |f(x)|^2 \, dx\right)^{\frac{1}{2a}} \left(\int_{\mathbb{R}^n} \|y\|^{2b} |\widehat{f}(y)|^2 \, dy\right)^{\frac{1}{2b}},$$

where $\|\cdot\|_2$ being the L^2-norm and $\|\cdot\|$ the Euclidean norm.

In [16], the Heisenberg uncertainty inequality has been proved for the Fourier transform on the Heisenberg group and further generalizations on the Heisenberg group have been established in [15] and [17]. For more details, see [1] and [7]. In [2], the Heisenberg uncertainty inequality has been proved for Gabor transform on several classes of locally compact groups. The Heisenberg uncertainty inequality has been obtained for wavelet transform in [14] and for shearlet transform in [13].

In this paper, we define continuous modulated shearlet transform that generalizes all the three transforms mentioned previously. Several properties including the Plancherel formula and inversion formula are also proved. As special cases, we obtain Gabor transform, wavelet transform and shearlet transform, followed by some examples. Uncertainty inequalities are showcased for continuous modulated shearlet transform.

2. Continuous Modulated Shearlet Transform

For a separable Hilbert space $\mathscr{H}$, let $\mathcal{B}(\mathscr{H})$ be the set of all bounded operators on $\mathscr{H}$ equipped with the operator norm. An element $T \in \mathcal{B}(\mathscr{H})$ satisfying

$$\sum_k \|Te_k\|^2 < \infty$$

for some orthonormal basis $\{e_k\}$ of $\mathscr{H}$ is known as Hilbert-Schmidt operator. The set $\mathrm{HS}(\mathscr{H})$ of all Hilbert-Schmidt operators on $\mathscr{H}$ is a Hilbert space with the inner product given by

$$\langle T, S\rangle_{\mathrm{HS}(\mathscr{H})} = \mathrm{tr}\,(S^*T) = \sum_k \langle S^*Te_k, e_k\rangle.$$

For each $T \in \mathrm{HS}(\mathscr{H})$, let $\|T\|_{\mathrm{HS}}$ be the Hilbert-Schmidt norm induced by $\langle\cdot,\cdot\rangle_{\mathrm{HS}(\mathscr{H})}$. For more details, refer to [6].

Let G be a second countable, type I and unimodular locally compact group with left Haar measure μ_G and the Plancherel measure on the dual space $\widehat{G}$ as $\mu_{\widehat{G}}$. Let H be a second countable, locally compact Abelian group with Haar measure μ_H. Denote by $\mathrm{Aut}(H)$ the group of automorphisms of H. For a locally compact group L with left Haar measure μ_L, let $\lambda : L \to \mathrm{Aut}(H)$ be a homomorphism $l \mapsto \lambda_l$ such that the mapping $(l,h) \mapsto \lambda_l(h)$ from $L \times H$ onto H is continuous. The semi-direct product $L \times_\lambda H$ is a locally compact group with the operation given by

$$(l,h)(l',h') = (l\,l', h\,\lambda_l(h')).$$

By [8, (15.29)], the left Haar measure μ_D of $D = L \times_\lambda H$ is

$$d\mu_D(l,h) = \delta_\lambda(l)\; d\mu_L(l)\; d\mu_H(h),$$

where δ_λ is a positive continuous homomorphism on L given by

$$d\mu_H(h) = \delta_\lambda(l)\; d\mu_H(\lambda_l(h)).$$

Also the set $\mathcal{S} = D \times G = (L \times_\lambda H) \times G$ is a locally compact group with identity element $1_{\mathcal{S}} = (1_L, 1_H, 1_G)$, where 1_L, 1_H and 1_G denote the identity elements of the groups L, H and G respectively. The left Haar measure of $\mathcal{S}$ is given by

$$d\mu_{\mathcal{S}}(l,h,x) = \delta_\lambda(l)\; d\mu_L(l)\; d\mu_H(h)\; d\mu_G(x).$$

For each $(l,h,x,\pi) \in \mathcal{S} \times \widehat{G}$, let

$$\mathscr{H}_{(l,h,x,\pi)} = \pi(x)\mathrm{HS}(\mathscr{H}_\pi),$$

where $\pi(x)\mathrm{HS}(\mathscr{H}_\pi) = \{\pi(x)T : T \in \mathrm{HS}(\mathscr{H}_\pi)\}$. Then, $\mathscr{H}_{(l,h,x,\pi)}$ is a Hilbert space with the inner product given by

$$\langle \pi(x)T, \pi(x)S\rangle_{\mathscr{H}_{(l,h,x,\pi)}} = \mathrm{tr}\,(S^*T) = \langle T, S\rangle_{\mathrm{HS}(\mathscr{H}_\pi)}.$$

Let $\|\cdot\|_{(l,h,x,\pi)}$ be the norm induced by the inner product on $\mathscr{H}_{(l,h,x,\pi)}$.
One can easily verify that $\mathscr{H}_{(l,h,x,\pi)} = \mathrm{HS}(\mathscr{H}_\pi)$ for all $(l,h,x,\pi) \in \mathcal{S} \times \widehat{G}$. The family $\{\mathscr{H}_{(l,h,x,\pi)} : (l,h,x,\pi) \in \mathcal{S} \times \widehat{G}\}$ of Hilbert spaces indexed by $\mathcal{S} \times \widehat{G}$ is a field of Hilbert spaces over $\mathcal{S} \times \widehat{G}$.
Let $\mathscr{H}^2(\mathcal{S} \times \widehat{G})$ denote the direct integral of $\{\mathscr{H}_{(l,h,x,\pi)} : (l,h,x,\pi) \in \mathcal{S} \times \widehat{G}\}$ with respect to the measure

$$d\sigma(l,h,x,\pi) = d\mu_{\mathcal{S}}(l,h,x)\; d\mu_{\widehat{G}}(\pi),$$

i.e., the space of all measurable vector fields F on $\mathcal{S} \times \widehat{G}$ such that

$$\|F\|^2_{\mathscr{H}^2(\mathcal{S}\times\widehat{G})} = \int_{\mathcal{S}\times\widehat{G}} \|F(l,h,x,\pi)\|^2_{(l,h,x,\pi)}\; d\sigma(l,h,x,\pi) < \infty.$$

$\mathscr{H}^2(\mathcal{S} \times \widehat{G})$ is a Hilbert space with the inner product given by

$$\langle F, K\rangle_{\mathscr{H}^2(\mathcal{S}\times\widehat{G})} = \int\limits_{\mathcal{S}\times\widehat{G}} \operatorname{tr}\left[K(l,h,x,\pi)^* \, F(l,h,x,\pi)\right] d\sigma(l,h,x,\pi).$$

For each $(l,h,x) \in \mathcal{S}$ and $\psi \in L^2(H \times G)$, we define

$$\mathcal{W}_{(l,h,x)} : L^2(H \times G) \to L^2(H \times G)$$

by

$$\mathcal{W}_{(l,h,x)}(\psi)(k,y) = \delta_\lambda^{1/2}(l)\; \psi(\lambda_{l^{-1}}(h^{-1}k), x^{-1}y)$$

for all $(k,y) \in H \times G$.

PROPOSITION 2.1. (i) For each $(l_1,h_1,x_1), (l_2,h_2,x_2) \in \mathcal{S}$, we have

$$\mathcal{W}_{(l_1,h_1,x_1)} \circ \mathcal{W}_{(l_2,h_2,x_2)} = \mathcal{W}_{(l_1l_2,h_1\lambda_{l_1}(h_2),x_1x_2)} = \mathcal{W}_{(l_1,h_1,x_1)(l_2,h_2,x_2)}.$$

(ii) For every $\psi \in L^2(H \times G)$, we have $\mathcal{W}_{(1_L,1_H,1_G)}(\psi) = \psi$.
In other words, $\mathcal{W}_{(1_L,1_H,1_G)}$ is the identity operator on $L^2(H \times G)$.

(iii) $\mathcal{W}_{(l,h,x)}$ is a unitary operator on $L^2(H \times G)$ having inverse $\mathcal{W}_{(l,h,x)^{-1}}$.

Proof. (i) Since δ_λ is a homomorphism, we have for all $\psi \in L^2(H \times G)$, $(k,y) \in H \times G$ and $(l_1,h_1,x_1), (l_2,h_2,x_2) \in \mathcal{S}$,

$$\begin{aligned}
&\left(\mathcal{W}_{(l_1,h_1,x_1)} \circ \mathcal{W}_{(l_2,h_2,x_2)}\right)(\psi)(k,y) \\
&= \delta_\lambda^{1/2}(l_1)\left(\mathcal{W}_{(l_2,h_2,x_2)}(\psi)\right)(\lambda_{l_1^{-1}}(h_1^{-1}k), x_1^{-1}y) \\
&= \delta_\lambda^{1/2}(l_1)\delta_\lambda^{1/2}(l_2)\; \psi\left(\lambda_{l_2^{-1}}\left(h_2^{-1}\,\lambda_{l_1^{-1}}(h_1^{-1}k)\right), x_2^{-1}x_1^{-1}y\right) \\
&= \delta_\lambda^{1/2}(l_1l_2)\; \psi\left(\lambda_{l_2^{-1}}\left(\lambda_{l_1^{-1}}\left(\lambda_{l_1}(h_2^{-1})\; h_1^{-1}k\right)\right), x_2^{-1}x_1^{-1}y\right) \\
&= \delta_\lambda^{1/2}(l_1l_2)\; \psi\left(\lambda_{l_2^{-1}l_1^{-1}}\left((\lambda_{l_1}(h_2))^{-1}\; h_1^{-1}k\right), x_2^{-1}x_1^{-1}y\right) \\
&= \delta_\lambda^{1/2}(l_1l_2)\; \psi\left(\lambda_{(l_1l_2)^{-1}}\left((h_1\lambda_{l_1}(h_2))^{-1}\; k\right), (x_1x_2)^{-1}y\right) \\
&= \mathcal{W}_{(l_1l_2,h_1\lambda_{l_1}(h_2),x_1x_2)}(\psi)(k,y).
\end{aligned}$$

(ii) It follows from the definition of $\mathcal{W}_{(l,h,x)}$ since $\delta_\lambda^{1/2}(1_L) = 1$.

(iii) From (i), it follows that $\mathcal{W}_{(l,h,x)}$ is a linear operator on $L^2(H \times G)$ whose inverse is $\mathcal{W}_{(l,h,x)^{-1}}$.
For each $f, \psi \in L^2(H \times G)$ and $(l,h,x) \in \mathcal{S}$, using transformations $k \mapsto hk$, $y \mapsto xy$ and $k \mapsto \lambda_l(k)$ at appropriate places, we have

$$\begin{aligned}
&\langle \mathcal{W}^*_{(l,h,x)}(f), \psi\rangle \\
&= \int\limits_H \int\limits_G f(k,y)\; \overline{\mathcal{W}_{(l,h,x)}(\psi)(k,y)}\; d\mu_H(k)\; d\mu_G(y) \\
&= \int\limits_H \int\limits_G f(k,y)\; \delta_\lambda^{1/2}(l)\; \overline{\psi(\lambda_{l^{-1}}(h^{-1}k), x^{-1}y)}\; d\mu_H(k)\; d\mu_G(y)
\end{aligned}$$

$$
\begin{aligned}
&= \delta_\lambda^{1/2}(l) \int_H \int_G f(hk, xy)\, \overline{\psi(\lambda_{l^{-1}}(k), y)}\, d\mu_H(k)\, d\mu_G(y) \\
&= \delta_\lambda^{1/2}(l) \int_H \int_G f(h\, \lambda_l(k), xy)\, \overline{\psi(k, y)}\, d\mu_H(\lambda_l(k))\, d\mu_G(y) \\
&= \delta_\lambda^{1/2}(l^{-1}) \int_H \int_G f(\lambda_l(\lambda_{l^{-1}}(h)\, k), xy)\, \overline{\psi(k, y)}\, d\mu_H(k)\, d\mu_G(y) \\
&= \int_H \int_G \delta_\lambda^{1/2}(l^{-1})\, f\left(\lambda_{(l^{-1})^{-1}}((\lambda_{l^{-1}}(h^{-1}))^{-1}\, k), (x^{-1})^{-1} y\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \overline{\psi(k, y)}\, d\mu_H(k)\, d\mu_G(y) \\
&= \int_H \int_G \mathcal{W}_{(l^{-1}, \lambda_{l^{-1}}(h^{-1}), x^{-1})}(f)(k, y)\, \overline{\psi(k, y)}\, d\mu_H(k)\, d\mu_G(y) \\
&= \int_H \int_G \mathcal{W}_{(l,h,x)^{-1}}(f)(k, y)\, \overline{\psi(k, y)}\, d\mu_H(k)\, d\mu_G(y) \\
&= \langle \mathcal{W}_{(l,h,x)^{-1}}(f), \psi \rangle.
\end{aligned}
$$

Thus $\mathcal{W}^*_{(l,h,x)} = \mathcal{W}_{(l,h,x)^{-1}}$, i.e., $\mathcal{W}_{(l,h,x)}$ is a unitary operator.

□

For each $\psi \in L^2(H \times G)$ and $(l, h, x) \in \mathcal{S}$, we define $\mathcal{U}^\psi_{(l,h,x)} : H \times G \to \mathbb{C}$ by

$$
\mathcal{U}^\psi_{(l,h,x)}(k, y) := \mathcal{W}_{(l,h,x)}(\psi)(k, y) = \delta_\lambda^{1/2}(l)\, \psi(\lambda_{l^{-1}}(h^{-1}k), x^{-1}y) \tag{2.1}
$$

and for each $f \in L^2(H \times G)$, $\mathcal{L}^\psi_{(l,h,x)} f : H \times G \to \mathbb{C}$ by

$$
\mathcal{L}^\psi_{(l,h,x)} f(k, y) = f(k, y)\, \overline{\mathcal{U}^\psi_{(l,h,x)}(k, y)}
$$

for all $(k, y) \in H \times G$.

PROPOSITION 2.2. For each $\psi \in L^2(H \times G)$ and $(l, h, x) \in \mathcal{S}$, we have $\mathcal{U}^\psi_{(l,h,x)} \in L^2(H \times G)$ and it satisfies

$$
\|\mathcal{U}^\psi_{(l,h,x)}\|_{L^2(H\times G)} = \|\psi\|_{L^2(H\times G)}.
$$

Also, the operator $\mathcal{W}_{(l,h,x)}$ is an isometry and hence an isomorphism from $L^2(H \times G)$ onto itself.

Proof. For every $(l, h, x) \in \mathcal{S}$ and $\psi \in L^2(H \times G)$, applying transformations $k \mapsto hk$, $y \mapsto xy$ and $k \mapsto \lambda_l(k)$ at appropriate places, we have

$$
\begin{aligned}
&\int_H \int_G \left| \mathcal{U}^\psi_{(l,h,x)}(k, y) \right|^2 d\mu_H(k)\, d\mu_G(y) \\
&= \int_H \int_G \left| \delta_\lambda^{1/2}(l)\, \psi(\lambda_{l^{-1}}(h^{-1}k), x^{-1}y) \right|^2 d\mu_H(k)\, d\mu_G(y)
\end{aligned}
$$

$$= \delta_\lambda(l) \int_H \int_G |\psi(\lambda_{l^{-1}}(k), y)|^2 \, d\mu_H(k) \, d\mu_G(y)$$

$$= \delta_\lambda(l) \int_H \int_G |\psi(k, y)|^2 \, d\mu_H(\lambda_l(k)) \, d\mu_G(y)$$

$$= \int_H \int_G |\psi(k, y)|^2 \, d\mu_H(k) \, d\mu_G(y) = \|\psi\|^2_{L^2(H\times G)} < \infty.$$

So, $\mathcal{U}^\psi_{(l,h,x)} \in L^2(H \times G)$ and $\|\mathcal{U}^\psi_{(l,h,x)}\|_{L^2(H\times G)} = \|\psi\|_{L^2(H\times G)}$.
Also, we have

$$\|\mathcal{W}_{(l,h,x)}(\psi)\|_{L^2(H\times G)} = \|\mathcal{U}^\psi_{(l,h,x)}\|_{L^2(H\times G)} = \|\psi\|_{L^2(H\times G)}.$$

Hence, $\mathcal{W}_{(l,h,x)}$ is an isometry and so is an isomorphism from $L^2(H \times G)$ onto itself. □

REMARK 2.3. Although we will not be using $\mathcal{U}^\psi_{(l,h,x)}$ for $\psi \in L^p(H \times G)$ (where $p \geq 1$), even then one may note the following:
For each $\psi \in L^p(H \times G)$ (where $p \geq 1$) and $(l, h, x) \in \mathcal{S}, \mathcal{U}^\psi_{(l,h,x)} \in L^p(H \times G)$ and it satisfies

$$\|\mathcal{U}^\psi_{(l,h,x)}\|_{L^p(H\times G)} = \delta_\lambda^{(p-2)/2p}(l) \, \|\psi\|_{L^p(H\times G)}.$$

DEFINITION 2.4. A pair of functions $\psi, \varphi \in L^2(H \times G)$ is said to be an *admissible pair* if

$$C_{\psi,\varphi} := \int_L \int_G \mathscr{F}_H \, \widetilde{\psi}(\eta \circ \lambda_l, x) \, \overline{\mathscr{F}_H \, \widetilde{\varphi}(\eta \circ \lambda_l, x)} \, d\mu_L(l) \, d\mu_G(x) < \infty,$$

which is independent of almost every $\eta \in \widehat{H}$, $\mathscr{F}_H$ being the Fourier transform on H and $\widetilde{\psi}$, $\widetilde{\varphi}$ are the involutions of ψ, φ respectively.
In particular, a function $\psi \in L^2(H \times G)$ is said to be *admissible* if ψ, ψ is an admissible pair. We denote $C_{\psi,\psi}$ by C_ψ and we have

$$C_\psi := \int_L \int_G \left| \mathscr{F}_H \, \widetilde{\psi}(\eta \circ \lambda_l, x) \right|^2 \, d\mu_L(l) \, d\mu_G(x) < \infty,$$

which is independent of almost every $\eta \in \widehat{H}$.

DEFINITION 2.5. Let $f \in C_c(H \times G)$, the set of all complex-valued continuous functions on $H \times G$ with compact supports and let $\psi \in L^2(H \times G)$ be an admissible function. We define *continuous modulated shearlet transform* of f with respect to ψ as a measurable field of operators on $(L \times_\lambda H) \times G \times \widehat{G}$ by

$$\mathcal{MS}_\psi f(l, h, x, \pi) = \int_H \int_G f(k, y) \, \overline{\mathcal{U}^\psi_{(l,h,x)}(k, y)} \, \pi(y)^* \, d\mu_H(k) \, d\mu_G(y). \tag{2.2}$$

We consider the operator-valued integral (2.2) in the weak sense. In other words, for each $(l, h, x, \pi) \in (L \times_\lambda H) \times G \times \widehat{G}$ and $\zeta, \xi \in \mathscr{H}_\pi$, we have

$$\langle \mathcal{MS}_\psi f(l, h, x, \pi)\zeta, \xi\rangle = \int_H \int_G f(k, y) \overline{\mathcal{U}^\psi_{(l,h,x)}(k, y)} \, \langle \pi(y)^* \zeta, \xi\rangle \, d\mu_H(k) \, d\mu_G(y).$$

Since the map $y \mapsto \langle \pi(y)^* \zeta, \xi \rangle$ is a bounded continuous function on G, the right hand side integral is the ordinary integral of a function in $L^1(H \times G)$.
Using transformations $k \mapsto hk$, $y \mapsto xy$, $k \mapsto \lambda_l(k)$ and Cauchy-Schwarz inequality at appropriate places, we have

$$
\begin{aligned}
&|\langle \mathcal{MS}_\psi f(l,h,x,\pi)\zeta, \xi\rangle| \\
&\leq \int_H \int_G |f(k,y)| \, |\mathcal{U}^{\psi}_{(l,h,x)}(k,y)| \, |\langle \pi(y)^*\zeta, \xi\rangle| \, d\mu_H(k) \, d\mu_G(y) \\
&\leq \|\zeta\| \, \|\xi\| \int_H \int_G |f(k,y)| \, |\mathcal{U}^{\psi}_{(l,h,x)}(k,y)| \, d\mu_H(k) \, d\mu_G(y) \\
&\leq \|\zeta\| \, \|\xi\| \, \|f\|_{L^2(H\times G)} \\
&\qquad \left(\int_H \int_G |\psi(\lambda_{l^{-1}}(h^{-1}k), x^{-1}y)|^2 \, \delta_\lambda(l) \, d\mu_H(k) \, d\mu_G(y) \right)^{1/2} \\
&= \|\zeta\| \, \|\xi\| \, \|f\|_{L^2(H\times G)} \left(\int_H \int_G |\psi(\lambda_{l^{-1}}(k), y)|^2 \, \delta_\lambda(l) \, d\mu_H(k) \, d\mu_G(y) \right)^{1/2} \\
&= \|\zeta\| \, \|\xi\| \, \|f\|_{L^2(H\times G)} \left(\int_H \int_G |\psi(k, y)|^2 \, \delta_\lambda(l) \, d\mu_H(\lambda_l(k)) \, d\mu_G(y) \right)^{1/2} \\
&= \|\zeta\| \, \|\xi\| \, \|f\|_{L^2(H\times G)} \left(\int_H \int_G |\psi(k, y)|^2 \, d\mu_H(k) \, d\mu_G(y) \right)^{1/2} \\
&= \|\zeta\| \, \|\xi\| \, \|f\|_{L^2(H\times G)} \, \|\psi\|_{L^2(H\times G)}.
\end{aligned}
$$

So for each $(l,h,x,\pi) \in (L \times_\lambda H) \times G \times \widehat{G}$, $\mathcal{MS}_\psi f(l,h,x,\pi)$ is a bounded linear operator on $\mathscr{H}_\pi$ and its operator norm satisfies

$$
\|\mathcal{MS}_\psi f(l,h,x,\pi)\| \leq \|f\|_{L^2(H\times G)} \, \|\psi\|_{L^2(H\times G)}. \tag{2.3}
$$

Note that $\mathcal{L}^{\psi}_{(l,h,x)} f \in L^1(H \times G) \cap L^2(H \times G)$. So

$$
\mathscr{F}_H \left(\mathcal{L}^{\psi}_{(l,h,x)} f \right) (I, \cdot) \in L^1(G) \cap L^2(G).
$$

Here $I \in \widehat{H}$ is the identity element. Let $\mathscr{F}_G$ represent the Fourier transform on G. By [6, Page 253], it follows that $\mathscr{F}_G \mathscr{F}_H \left(\mathcal{L}^{\psi}_{(l,h,x)} f \right) (I, \pi)$ is a Hilbert-Schmidt operator for every $(l,h,x) \in \mathcal{S}$ and for almost every $\pi \in \widehat{G}$.
It can be easily observed that for all $(l,h,x,\pi) \in (L \times_\lambda H) \times G \times \widehat{G}$,

$$
\mathcal{MS}_\psi f(l,h,x,\pi) = \mathscr{F}_G \mathscr{F}_H \left(\mathcal{L}^{\psi}_{(l,h,x)} f \right) (I, \pi) = \widehat{\mathcal{L}^{\psi}_{(l,h,x)}} f(I, \pi). \tag{2.4}
$$

Hence, $\mathcal{MS}_\psi f(l,h,x,\pi)$ is a Hilbert-Schmidt operator for all $(l,h,x) \in \mathcal{S}$ and for almost every $\pi \in \widehat{G}$.

PROPOSITION 2.6. For every $f \in C_c(H \times G)$ and every admissible function $\psi \in L^2(H \times G)$, we have

$$
\mathcal{MS}_\psi f(l,h,x,\pi)^* = \widehat{\left(\widetilde{\mathcal{L}^{\psi}_{(l,h,x)}} f \right)} (I, \pi).
$$

Proof. Using transformations $k \mapsto k^{-1}$ and $y \mapsto y^{-1}$ at appropriate places, we have for each $\zeta, \xi \in \mathscr{H}_\pi$,

$$
\begin{aligned}
&\langle \mathcal{MS}_\psi f(l,h,x,\pi)^* \zeta, \xi \rangle \\
&= \langle \zeta, \mathcal{MS}_\psi f(l,h,x,\pi) \xi \rangle \\
&= \int_H \int_G \langle \zeta, \mathcal{L}^{\psi}_{(l,h,x)} f(k,y)\, \pi(y)^* \xi \rangle \, d\mu_H(k)\, d\mu_G(y) \\
&= \int_H \int_G \langle \overline{\mathcal{L}^{\psi}_{(l,h,x)} f(k,y)}\, \pi(y) \zeta, \xi \rangle \, d\mu_H(k)\, d\mu_G(y) \\
&= \int_H \int_G \langle \overline{\mathcal{L}^{\psi}_{(l,h,x)} f(k^{-1},y^{-1})}\, \pi(y)^* \zeta, \xi \rangle \, d\mu_H(k)\, d\mu_G(y) \\
&= \int_H \int_G \langle \widetilde{\mathcal{L}^{\psi}_{(l,h,x)}} f(k,y)\, \pi(y)^* \zeta, \xi \rangle \, d\mu_H(k)\, d\mu_G(y) = \left\langle \widehat{\left(\widetilde{\mathcal{L}^{\psi}_{(l,h,x)}} f \right)}(I,\pi) \zeta, \xi \right\rangle.
\end{aligned}
$$

Hence, $\mathcal{MS}_\psi f(l,h,x,\pi)^* = \widehat{\left(\widetilde{\mathcal{L}^{\psi}_{(l,h,x)}} f \right)}(I,\pi)$. □

Simple computations will lead to the following lemmas.

LEMMA 2.7. Let $f, \psi \in L^2(H \times G)$, then for every $(l,h,x) \in \mathcal{S}$ and $y \in G$,

$$\mathscr{F}_H \left(\mathcal{L}^{\psi}_{(l,h,x)} f \right)(I,y) = \left(f *_H \mathcal{U}^{\widetilde{\psi}}_{(l,1_H,yx^{-1}y)} \right)(h,y),$$

where $*_H$ denotes convolution on H.

LEMMA 2.8. For every $\psi \in L^2(H \times G)$, $l \in L$, $x \in G$ and $(\eta, y) \in \widehat{H} \times G$,

$$\mathscr{F}_H \widetilde{\psi}(\eta \circ \lambda_l, y^{-1}x) = \delta_\lambda^{1/2}(l)\, \mathscr{F}_H \left(\mathcal{U}^{\widetilde{\psi}}_{(l,1_H,yx^{-1}y)} \right)(\eta, y).$$

LEMMA 2.9. For every admissible pair of functions $\psi, \varphi \in L^2(H \times G)$, $l \in L$, $x \in G$ and $(\eta, y) \in \widehat{H} \times G$,

$$C_{\psi,\varphi} = \int_L \int_G \mathscr{F}_H \left(\mathcal{U}^{\widetilde{\psi}}_{(l,1_H,yx^{-1}y)} \right)(\eta,y)\, \overline{\mathscr{F}_H \left(\mathcal{U}^{\widetilde{\varphi}}_{(l,1_H,yx^{-1}y)} \right)(\eta,y)}\, \delta_\lambda(l)\, d\mu_L(l)\, d\mu_G(x)$$

is satisfied. In particular, $C_{\psi,\varphi} = \overline{C_{\varphi,\psi}}$.

COROLLARY 2.10. Let $\psi \in L^2(H \times G)$ be an admissible function. For every $l \in L$, $x \in G$ and $(\eta, y) \in \widehat{H} \times G$,

$$C_\psi = \int_L \int_G \left| \mathscr{F}_H \left(\mathcal{U}^{\widetilde{\psi}}_{(l,1_H,yx^{-1}y)} \right)(\eta,y) \right|^2 \delta_\lambda(l)\, d\mu_L(l)\, d\mu_G(x)$$

is satisfied.

PROPOSITION 2.11. For every admissible function $\psi \in L^2(H \times G)$, the linear operator $\mathcal{MS}_\psi : C_c(H \times G) \to \mathscr{H}^2(\mathcal{S} \times \widehat{G})$ given by $f \mapsto \mathcal{MS}_\psi f$ satisfies

$$\|\mathcal{MS}_\psi f\|_{\mathscr{H}^2(\mathcal{S} \times \widehat{G})} = C_\psi^{1/2}\, \|f\|_{L^2(H \times G)}.$$

Proof. By Plancherel formula, Lemma 2.7 and Corollary 2.10, we have

$$\begin{aligned}
&\|\mathcal{MS}_\psi f\|^2_{\mathscr{H}^2(\mathcal{S}\times\widehat{G})} \\
&= \int\limits_{\mathcal{S}\times\widehat{G}} \mathrm{tr}[\mathcal{MS}_\psi f(l,h,x,\pi)^* \, \mathcal{MS}_\psi f(l,h,x,\pi)] \, d\sigma(l,h,x,\pi) \\
&= \int\limits_{\mathcal{S}} \int\limits_{\widehat{G}} \mathrm{tr}\left[\left(\mathscr{F}_G\mathscr{F}_H\left(\mathcal{L}^\psi_{(l,h,x)}f\right)(I,\pi)\right)^* \left(\mathscr{F}_G\mathscr{F}_H\left(\mathcal{L}^\psi_{(l,h,x)}f\right)(I,\pi)\right)\right] \\
&\qquad\qquad d\mu_{\mathcal{S}}(l,h,x)\, d\mu_{\widehat{G}}(\pi) \\
&= \int\limits_{\mathcal{S}} \int\limits_{G} \overline{\mathscr{F}_H\left(\mathcal{L}^\psi_{(l,h,x)}f\right)(I,y)}\, \mathscr{F}_H\left(\mathcal{L}^\psi_{(l,h,x)}f\right)(I,y)\, d\mu_{\mathcal{S}}(l,h,x)\, d\mu_G(y) \\
&= \int\limits_{L} \int\limits_{H} \int\limits_{G} \int\limits_{G} \left|\mathscr{F}_H\left(\mathcal{L}^\psi_{(l,h,x)}f\right)(I,y)\right|^2 \delta_\lambda(l)\, d\mu_L(l)\, d\mu_H(h)\, d\mu_G(x)\, d\mu_G(y) \\
&= \int\limits_{L} \int\limits_{G} \int\limits_{G} \left[\int\limits_{H} \left|\left(f *_H \mathcal{U}^{\widetilde{\psi}}_{(l,1_H,yx^{-1}y)}\right)(h,y)\right|^2 d\mu_H(h)\right] \\
&\qquad\qquad \delta_\lambda(l)\, d\mu_L(l)\, d\mu_G(x)\, d\mu_G(y) \\
&= \int\limits_{L} \int\limits_{G} \int\limits_{G} \left[\int\limits_{\widehat{H}} |\mathscr{F}_H f(\eta,y)|^2 \left|\mathscr{F}_H\left(\mathcal{U}^{\widetilde{\psi}}_{(l,1_H,yx^{-1}y)}\right)(\eta,y)\right|^2 d\mu_{\widehat{H}}(\eta)\right] \\
&\qquad\qquad \delta_\lambda(l)\, d\mu_L(l)\, d\mu_G(x)\, d\mu_G(y) \\
&= \int\limits_{G} \int\limits_{\widehat{H}} |\mathscr{F}_H f(\eta,y)|^2 \left[\int\limits_{L} \int\limits_{G} \left|\mathscr{F}_H\left(\mathcal{U}^{\widetilde{\psi}}_{(l,1_H,yx^{-1}y)}\right)(\eta,y)\right|^2 \delta_\lambda(l)\, d\mu_L(l)\, d\mu_G(x)\right] \\
&\qquad\qquad d\mu_{\widehat{H}}(\eta)\, d\mu_G(y) \\
&= C_\psi \int\limits_{G} \int\limits_{\widehat{H}} |\mathscr{F}_H f(\eta,y)|^2 \, d\mu_{\widehat{H}}(\eta)\, d\mu_G(y) \\
&= C_\psi \int\limits_{G} \int\limits_{H} |f(k,y)|^2 \, d\mu_H(k)\, d\mu_G(y) \\
&= C_\psi \, \|f\|^2_{L^2(H\times G)}.
\end{aligned}$$

Hence, $\|\mathcal{MS}_\psi f\|_{\mathscr{H}^2(\mathcal{S}\times\widehat{G})} = C_\psi^{1/2}\, \|f\|_{L^2(H\times G)}$. □

The above equality shows that $\mathcal{MS}_\psi : C_c(H\times G) \to \mathscr{H}^2(\mathcal{S}\times\widehat{G})$ defined by $f \mapsto \mathcal{MS}_\psi f$ is a multiple of an isometry. So, we can extend $\mathcal{MS}_\psi$ uniquely to a bounded linear operator from $L^2(H\times G)$ into a closed subspace N of $\mathscr{H}^2(\mathcal{S}\times\widehat{G})$ which we still denote by $\mathcal{MS}_\psi$ and this extension satisfies

$$\|\mathcal{MS}_\psi f\|_{\mathscr{H}^2(\mathcal{S}\times\widehat{G})} = C_\psi^{1/2}\, \|f\|_{L^2(H\times G)}, \tag{2.5}$$

for each $f \in L^2(H\times G)$.

THEOREM 2.12. For every function $f \in L^2(H \times G)$ and admissible function $\psi \in L^2(H \times G)$, we have

$$\mathcal{MS}_\psi f(l,h,x,\pi) = \widehat{\mathcal{L}^{\psi}_{(l,h,x)} f}(I,\pi)$$

for all $(l,h,x,\pi) \in \mathcal{S} \times \widehat{G}$.

Proof. Let $f \in L^2(H \times G)$. Since $C_c(H \times G)$ is dense in $L^2(H \times G)$, there exists a sequence $\{\phi_n\}$ in $C_c(H \times G)$ such that $f = \lim_{n\to\infty} \phi_n$ in the $L^2(H \times G)$-norm. Then using (2.4), we have for all $n \in \mathbb{N}$

$$\mathcal{MS}_\psi \phi_n(l,h,x,\pi) = \widehat{\mathcal{L}^{\psi}_{(l,h,x)} \phi_n}(I,\pi).$$

By (2.5), the operator $\mathcal{MS}_\psi : L^2(H \times G) \to N \subseteq \mathscr{H}^2(\mathcal{S} \times \widehat{G})$ satisfies

$$\begin{aligned} \|\mathcal{MS}_\psi f - \mathcal{MS}_\psi \phi_n\|^2_{\mathscr{H}^2(\mathcal{S}\times\widehat{G})} &= \|\mathcal{MS}_\psi (f - \phi_n)\|^2_{\mathscr{H}^2(\mathcal{S}\times\widehat{G})} \\ &= C_\psi \, \|f - \phi_n\|^2_{L^2(H\times G)}. \end{aligned}$$

Therefore $\mathcal{MS}_\psi f = \lim_{n\to\infty} \mathcal{MS}_\psi \phi_n$ in the $\mathscr{H}^2(\mathcal{S} \times \widehat{G})$-norm.
Using Plancherel formula, Lemma 2.7 and equation (2.4), we can write

$$\begin{aligned} &\|\mathcal{MS}_\psi f - \mathcal{MS}_\psi \phi_n\|^2_{\mathscr{H}^2(\mathcal{S}\times\widehat{G})} \\ &= C_\psi \, \|f - \phi_n\|^2_{L^2(H\times G)} \\ &= C_\psi \int_G \int_H |(f-\phi_n)(k,y)|^2 \, d\mu_H(k)\, d\mu_G(y) \\ &= C_\psi \int_G \int_{\widehat{H}} |\mathscr{F}_H(f-\phi_n)(\eta,y)|^2 \, d\mu_{\widehat{H}}(\eta)\, d\mu_G(y) \\ &= \int_G \int_{\widehat{H}} |\mathscr{F}_H(f-\phi_n)(\eta,y)|^2 \left[\int_L \int_G \left| \mathscr{F}_H \left(\mathcal{U}^{\widetilde{\psi}}_{(l,1_H,yx^{-1}y)} \right)(\eta,y) \right|^2 \delta_\lambda(l)\, d\mu_L(l)\, d\mu_G(x) \right] \\ &\qquad d\mu_{\widehat{H}}(\eta)\, d\mu_G(y) \\ &= \int_L \int_G \int_G \left[\int_{\widehat{H}} |\mathscr{F}_H(f-\phi_n)(\eta,y)|^2 \left| \mathscr{F}_H \left(\mathcal{U}^{\widetilde{\psi}}_{(l,1_H,yx^{-1}y)} \right)(\eta,y) \right|^2 d\mu_{\widehat{H}}(\eta) \right] \\ &\qquad \delta_\lambda(l)\, d\mu_L(l)\, d\mu_G(x)\, d\mu_G(y) \\ &= \int_L \int_G \int_G \left[\int_H \left| \left((f-\phi_n) *_H \mathcal{U}^{\widetilde{\psi}}_{(l,1_H,yx^{-1}y)} \right)(h,y) \right|^2 d\mu_H(h) \right] \\ &\qquad \delta_\lambda(l)\, d\mu_L(l)\, d\mu_G(x)\, d\mu_G(y) \\ &= \int_L \int_H \int_G \int_G \left| \mathscr{F}_H \left(\mathcal{L}^{\psi}_{(l,h,x)} (f-\phi_n) \right)(I,y) \right|^2 \delta_\lambda(l)\, d\mu_L(l)\, d\mu_H(h)\, d\mu_G(x)\, d\mu_G(y) \\ &= \int_{\mathcal{S}} \int_G \left| \mathscr{F}_H \left(\mathcal{L}^{\psi}_{(l,h,x)} f \right)(I,y) - \mathscr{F}_H \left(\mathcal{L}^{\psi}_{(l,h,x)} \phi_n \right)(I,y) \right|^2 d\mu_{\mathcal{S}}(l,h,x)\, d\mu_G(y) \end{aligned}$$

$$= \int_{\mathcal{S}\times\widehat{G}} \|\widehat{\mathcal{L}^{\psi}_{(l,h,x)}f}(I,\pi) - \widehat{\mathcal{L}^{\psi}_{(l,h,x)}\phi_n}(I,\pi)\|^2_{\mathrm{HS}}\, d\sigma(l,h,x,\pi)$$

$$= \int_{\mathcal{S}\times\widehat{G}} \|\Phi(l,h,x,\pi) - \mathcal{MS}_{\psi}\phi_n(l,h,x,\pi)\|^2_{\mathrm{HS}}\, d\sigma(l,h,x,\pi)$$

$$= \|\Phi - \mathcal{MS}_{\psi}\phi_n\|^2_{\mathscr{H}^2(\mathcal{S}\times\widehat{G})},$$

where $\Phi : \mathcal{S}\times\widehat{G} \to \mathscr{H}^2(\mathcal{S}\times\widehat{G})$ defined by $\Phi(l,h,x,\pi) = \widehat{\mathcal{L}^{\psi}_{(l,h,x)}f}(I,\pi)$.

Thus $\quad \|\mathcal{MS}_{\psi}f - \mathcal{MS}_{\psi}\phi_n\|_{\mathscr{H}^2(\mathcal{S}\times\widehat{G})} = \|\Phi - \mathcal{MS}_{\psi}\phi_n\|_{\mathscr{H}^2(\mathcal{S}\times\widehat{G})}$

$\Rightarrow \quad \lim_{n\to\infty}\|\mathcal{MS}_{\psi}f - \mathcal{MS}_{\psi}\phi_n\|_{\mathscr{H}^2(\mathcal{S}\times\widehat{G})} = \lim_{n\to\infty}\|\Phi - \mathcal{MS}_{\psi}\phi_n\|_{\mathscr{H}^2(\mathcal{S}\times\widehat{G})}$

So $\quad \|\mathcal{MS}_{\psi}f - \lim_{n\to\infty}\mathcal{MS}_{\psi}\phi_n\|_{\mathscr{H}^2(\mathcal{S}\times\widehat{G})} = \|\Phi - \lim_{n\to\infty}\mathcal{MS}_{\psi}\phi_n\|_{\mathscr{H}^2(\mathcal{S}\times\widehat{G})} = 0$

$\Rightarrow \quad \mathcal{MS}_{\psi}f = \lim_{n\to\infty}\mathcal{MS}_{\psi}\phi_n = \Phi.$

Hence, $\mathcal{MS}_{\psi}f(l,h,x,\pi) = \Phi(l,h,x,\pi) = \widehat{\mathcal{L}^{\psi}_{(l,h,x)}f}(I,\pi)$. □

As a corollary to the above theorem, the inequality (2.3) for $f \in C_c(H\times G)$ can be extended for functions in $L^2(H\times G)$.

COROLLARY 2.13. For every function $f \in L^2(H\times G)$ and admissible function $\psi \in L^2(H\times G)$, we have

$$\|\mathcal{MS}_{\psi}f(l,h,x,\pi)\| \leq \|f\|_{L^2(H\times G)}\,\|\psi\|_{L^2(H\times G)}.$$

We now prove the following orthogonality relation.

THEOREM 2.14. For every $f,g \in L^2(H\times G)$ and for every admissible pair of functions $\psi,\varphi \in L^2(H\times G)$, we have

$$\langle \mathcal{MS}_{\psi}f, \mathcal{MS}_{\varphi}g\rangle_{\mathscr{H}^2(\mathcal{S}\times\widehat{G})} = C_{\psi,\varphi}\,\langle f,g\rangle_{L^2(H\times G)}.$$

Proof. By Parseval formula, Lemma 2.7 and Lemma 2.9, we have

$$\langle \mathcal{MS}_{\psi}f, \mathcal{MS}_{\varphi}g\rangle_{\mathscr{H}^2(\mathcal{S}\times\widehat{G})}$$

$$= \int_{\mathcal{S}\times\widehat{G}} \mathrm{tr}[\mathcal{MS}_{\varphi}g(l,h,x,\pi)^*\, \mathcal{MS}_{\psi}f(l,h,x,\pi)]\, d\sigma(l,h,x,\pi)$$

$$= \int_{\mathcal{S}}\int_{\widehat{G}} \mathrm{tr}\left[\left(\mathscr{F}_G\mathscr{F}_H\left(\mathcal{L}^{\varphi}_{(l,h,x)}g\right)(I,\pi)\right)^*\,\left(\mathscr{F}_G\mathscr{F}_H\left(\mathcal{L}^{\psi}_{(l,h,x)}f\right)(I,\pi)\right)\right] d\mu_{\widehat{G}}(\pi)\, d\mu_{\mathcal{S}}(l,h,x)$$

$$= \int_{\mathcal{S}}\int_{G} \overline{\mathscr{F}_H\left(\mathcal{L}^{\varphi}_{(l,h,x)}g\right)(I,y)}\,\mathscr{F}_H\left(\mathcal{L}^{\psi}_{(l,h,x)}f\right)(I,y)\, d\mu_G(y)\, d\mu_{\mathcal{S}}(l,h,x)$$

$$= \int_L\int_H\int_G\int_G \overline{\mathscr{F}_H\left(\mathcal{L}^{\varphi}_{(l,h,x)}g\right)(I,y)}\,\mathscr{F}_H\left(\mathcal{L}^{\psi}_{(l,h,x)}f\right)(I,y)\,\delta_\lambda(l)\, d\mu_L(l)\, d\mu_H(h)\, d\mu_G(x)\, d\mu_G(y)$$

$$= \int_L \int_G \int_G \left[\int_H \overline{\left(g *_H \mathcal{U}^{\widetilde{\varphi}}_{(l,1_H,yx^{-1}y)}\right)(h,y)} \left(f *_H \mathcal{U}^{\widetilde{\psi}}_{(l,1_H,yx^{-1}y)}\right)(h,y)\, d\mu_H(h) \right] \delta_\lambda(l)\, d\mu_L(l)\, d\mu_G(x)\, d\mu_G(y)$$

$$= \int_L \int_G \int_G \left[\int_{\widehat{H}} \overline{\mathscr{F}_H \left(g *_H \mathcal{U}^{\widetilde{\varphi}}_{(l,1_H,yx^{-1}y)}\right)(\eta,y)}\, \mathscr{F}_H \left(f *_H \mathcal{U}^{\widetilde{\psi}}_{(l,1_H,yx^{-1}y)}\right)(\eta,y)\, d\mu_{\widehat{H}}(\eta) \right] \delta_\lambda(l)\, d\mu_L(l)\, d\mu_G(x)\, d\mu_G(y)$$

$$= \int_L \int_G \int_G \left[\int_{\widehat{H}} \overline{\mathscr{F}_H\, g(\eta,y)}\; \overline{\mathscr{F}_H \left(\mathcal{U}^{\widetilde{\varphi}}_{(l,1_H,yx^{-1}y)}\right)(\eta,y)} \times \mathscr{F}_H f(\eta,y)\, \mathscr{F}_H \left(\mathcal{U}^{\widetilde{\psi}}_{(l,1_H,yx^{-1}y)}\right)(\eta,y)\, d\mu_{\widehat{H}}(\eta) \right] \delta_\lambda(l)\, d\mu_L(l)\, d\mu_G(x)\, d\mu_G(y)$$

$$= \int_G \int_{\widehat{H}} \mathscr{F}_H f(\eta,y)\, \overline{\mathscr{F}_H\, g(\eta,y)} \times \left[\int_L \int_G \mathscr{F}_H \left(\mathcal{U}^{\widetilde{\psi}}_{(l,1_H,yx^{-1}y)}\right)(\eta,y)\, \overline{\mathscr{F}_H \left(\mathcal{U}^{\widetilde{\varphi}}_{(l,1_H,yx^{-1}y)}\right)(\eta,y)}\, \delta_\lambda(l)\, d\mu_L(l)\, d\mu_G(x) \right] d\mu_{\widehat{H}}(\eta)\, d\mu_G(y)$$

$$= C_{\psi,\varphi} \int_G \int_{\widehat{H}} \mathscr{F}_H f(\eta,y)\, \overline{\mathscr{F}_H\, g(\eta,y)}\, d\mu_{\widehat{H}}(\eta)\, d\mu_G(y)$$

$$= C_{\psi,\varphi} \int_G \int_H f(k,y)\, \overline{g(k,y)}\, d\mu_H(k)\, d\mu_G(y)$$

$$= C_{\psi,\varphi} \langle f, g \rangle_{L^2(H\times G)}.$$

Hence, $\langle \mathcal{MS}_\psi f, \mathcal{MS}_\varphi g \rangle_{\mathscr{H}^2(\mathcal{S}\times\widehat{G})} = C_{\psi,\varphi} \langle f, g \rangle_{L^2(H\times G)}$. □

COROLLARY 2.15. Let G be a compact group and $\psi \in L^2(H \times G)$ be an admissible function. Then for each $f \in L^2(H \times G)$, we have

$$\int_{\mathcal{S}} \sum_{\pi\in\widehat{G}} \|\mathcal{MS}_\psi f(l,h,x,\pi)\|^2_{\mathrm{HS}}\, d\mu_{\mathcal{S}}(l,h,x) = C_\psi \|f\|^2_{L^2(H\times G)}.$$

3. Inversion Formula

For $(s,t) \in H \times G$, the left translation operator $T_{s,t}$ on $L^2(H \times G)$ is given by

$$T_{s,t}\, g(h,x) = g(s^{-1}h, t^{-1}x)$$

for all $g \in L^2(H \times G)$ and $(h,x) \in H \times G$. Simple computations show the following:

PROPOSITION 3.1. Let $f, \psi \in L^2(H \times G)$ with ψ an admissible function. Then for each $(l,h,x,\pi) \in \mathcal{S} \times \widehat{G}$ and $(s,t) \in H \times G$ we have

$$\mathcal{MS}_\psi[T_{s,t}\, f](l,h,x,\pi) = \pi(t)^*\, \mathcal{MS}_\psi f(l, s^{-1}h, t^{-1}x, \pi).$$

For every $m \in L$, we define an operator S_m on $L^2(H \times G)$ given by

$$(S_m\psi)(k,y) = \delta_\lambda^{1/2}(m)\, \psi(\lambda_{m^{-1}}(k), y)$$

for all $\psi \in L^2(H \times G)$ and $(k,y) \in H \times G$. Note that $\widetilde{S_m\psi} = S_m\widetilde{\psi}$.

PROPOSITION 3.2. If $\psi \in L^2(H\times G)$ is an admissible function, then $S_m\psi$ is also an admissible function for all $m \in L$.

Proof. Given that ψ is an admissible function. Therefore

$$C_\psi = \int_L \int_G \left| \mathscr{F}_H\, \widetilde{\psi}(\eta \circ \lambda_l, x)\right|^2 d\mu_L(l)\, d\mu_G(x) < \infty.$$

For each $m \in L$, using transformations $k \mapsto \lambda_m(k)$ and $l \mapsto lm^{-1}$ at appropriate places, we have

$$\begin{aligned}
&\int_L \int_G \left| \mathscr{F}_H\, \widetilde{S_m\psi}(\eta \circ \lambda_l, x)\right|^2 d\mu_L(l)\, d\mu_G(x) \\
&= \int_L \int_G \left| \int_H \widetilde{S_m\psi}(k,x)\, (\eta\circ\lambda_l)(k^{-1})\, d\mu_H(k)\right|^2 d\mu_L(l)\, d\mu_G(x) \\
&= \int_L \int_G \left| \int_H \delta_\lambda^{1/2}(m)\, \widetilde{\psi}(\lambda_{m^{-1}}(k),x)\, (\eta\circ\lambda_l)(k^{-1})\, d\mu_H(k)\right|^2 d\mu_L(l)\, d\mu_G(x) \\
&= \int_L \int_G \left| \int_H \delta_\lambda^{1/2}(m)\, \widetilde{\psi}(k,x)\, (\eta\circ\lambda_l)(\lambda_m(k^{-1}))\, d\mu_H(\lambda_m(k))\right|^2 d\mu_L(l)\, d\mu_G(x) \\
&= \int_L \int_G \left| \int_H \delta_\lambda^{1/2}(m)\, \widetilde{\psi}(k,x)\, (\eta\circ\lambda_{lm})(k^{-1})\, d\mu_H(\lambda_m(k))\right|^2 d\mu_L(l)\, d\mu_G(x) \\
&= \delta_\lambda^{-1}(m) \int_L \int_G \left| \int_H \widetilde{\psi}(k,x)\, (\eta\circ\lambda_{lm})(k^{-1})\, d\mu_H(k)\right|^2 d\mu_L(l)\, d\mu_G(x) \\
&= \delta_\lambda^{-1}(m)\, \Delta(m^{-1}) \int_L \int_G \left| \int_H \widetilde{\psi}(k,x)\, (\eta\circ\lambda_{l})(k^{-1})\, d\mu_H(k)\right|^2 d\mu_L(l)\, d\mu_G(x) \\
&= \delta_\lambda^{-1}(m)\, \Delta(m^{-1}) \int_L \int_G \left| \mathscr{F}_H\, \widetilde{\psi}(\eta \circ \lambda_l, x)\right|^2 d\mu_L(l)\, d\mu_G(x) < \infty,
\end{aligned}$$

where Δ being the modular function of L.
Hence, $S_m\psi$ is also an admissible function for all $m \in L$. □

The definition of continuous modulated shearlet transform and the operator S_m will give the following:

PROPOSITION 3.3. Let $f \in L^2(H \times G)$ and $\psi \in L^2(H \times G)$ be an admissible function. Then for each $(l,h,x,\pi) \in \mathcal{S} \times \widehat{G}$ and $m \in L$, we have

$$\mathcal{MS}_{[S_m\psi]} f(l,h,x,\pi) = \mathcal{MS}_\psi f(lm,h,x,\pi).$$

For every $\pi \in \widehat{G}$, the modulation operator

$$M_\pi : L^2(H \times G) \to L^2(H \times G, \mathcal{B}(\mathscr{H}_\pi))$$

is defined by $(M_\pi f)(k,y) = f(k,y)\,\pi(y)$ for all $(k,y) \in H \times G$.
It is easy to check that M_π is an isometry.
Expressing $\mathcal{U}^\psi_{(l,h,x)}$ in terms of the operators S_l and $T_{h,x}$, we have

$$\mathcal{U}^\psi_{(l,h,x)}(k,y)\,\pi(y) = M_\pi(T_{h,x}(S_l\,\psi))(k,y) = M_\pi\,T_{h,x}\,S_l\,\psi(k,y)$$

for all $(k,y) \in H \times G$ and $(l,h,x,\pi) \in \mathcal{S} \times \widehat{G}$.

LEMMA 3.4. For every $K \in \mathscr{H}^2(\mathcal{S} \times \widehat{G})$ and for every admissible function $\varphi \in L^2(H \times G)$, the function $\ell^K_\varphi : L^2(H \times G) \to \mathbb{C}$, defined by

$$\ell^K_\varphi(g) = \int\limits_{\mathcal{S}\times\widehat{G}} \mathrm{tr}\left[K(l,h,x,\pi)^*\,\mathcal{M}\mathcal{S}_\varphi g(l,h,x,\pi)\right] d\sigma(l,h,x,\pi)$$

for all $g \in L^2(H \times G)$, is a bounded linear functional on $L^2(H \times G)$ and it also satisfies $\ell^K_\varphi(g) = \langle g, g_0\rangle_{L^2(H\times G)}$, where g_0 defined on $H \times G$ is given by

$$g_0(k,y) = \int\limits_{\mathcal{S}\times\widehat{G}} \mathrm{tr}\left[K(l,h,x,\pi)\;M_\pi T_{h,x} S_l\,\varphi(k,y)\right] d\sigma(l,h,x,\pi).$$

Proof. For all $g \in L^2(H \times G)$, we use (2.5) and Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
|\ell^K_\varphi(g)| &= \left|\int\limits_{\mathcal{S}\times\widehat{G}} \mathrm{tr}\left[K(l,h,x,\pi)^*\,\mathcal{M}\mathcal{S}_\varphi g(l,h,x,\pi)\right] d\sigma(l,h,x,\pi)\right| \\
&= \left|\langle \mathcal{M}\mathcal{S}_\varphi g, K\rangle_{\mathscr{H}^2(\mathcal{S}\times\widehat{G})}\right| \\
&\le \|\mathcal{M}\mathcal{S}_\varphi g\|_{\mathscr{H}^2(\mathcal{S}\times\widehat{G})}\,\|K\|_{\mathscr{H}^2(\mathcal{S}\times\widehat{G})} \\
&= C_\varphi^{1/2}\,\|g\|_{L^2(H\times G)}\,\|K\|_{\mathscr{H}^2(\mathcal{S}\times\widehat{G})}.
\end{aligned}$$

Hence, ℓ^K_φ is a bounded linear functional on $L^2(H \times G)$.
By Riesz representation theorem [9, Corollary 5.5.1], there exists a unique element $g_0 \in L^2(H \times G)$ such that $\ell^K_\varphi(g) = \langle g, g_0\rangle_{L^2(H\times G)}$. We have

$$\begin{aligned}
&\ell^K_\varphi(g) \\
&= \int\limits_{\mathcal{S}\times\widehat{G}} \mathrm{tr}\left[K(l,h,x,\pi)^*\,\mathcal{M}\mathcal{S}_\varphi g(l,h,x,\pi)\right] d\sigma(l,h,x,\pi) \\
&= \int\limits_{\mathcal{S}\times\widehat{G}} \mathrm{tr}\left[K(l,h,x,\pi)^*\left(\int\limits_H\int\limits_G g(k,y)\,\overline{\mathcal{U}^\varphi_{(l,h,x)}(k,y)}\,\pi(y)^*\,d\mu_H(k)\,d\mu_G(y)\right)\right] d\sigma(l,h,x,\pi) \\
&= \int\limits_H\int\limits_G g(k,y)\left(\int\limits_{\mathcal{S}\times\widehat{G}} \mathrm{tr}\left[K(l,h,x,\pi)^*\,\overline{\mathcal{U}^\varphi_{(l,h,x)}(k,y)}\,\pi(y)^*\right] d\sigma(l,h,x,\pi)\right) d\mu_H(k)\,d\mu_G(y)
\end{aligned}$$

$$= \int\limits_{H}\int\limits_{G} g(k,y)\,\overline{g_0(k,y)}\,d\mu_H(k)\,d\mu_G(y),$$

$$\text{where}\ \ g_0(k,y) = \int\limits_{\mathcal{S}\times\widehat{G}} \overline{\operatorname{tr}\left[K(l,h,x,\pi)^*\,\overline{\mathcal{U}^{\varphi}_{(l,h,x)}(k,y)}\,\pi(y)^*\right]}\,d\sigma(l,h,x,\pi)$$
$$= \int\limits_{\mathcal{S}\times\widehat{G}} \operatorname{tr}\left[K(l,h,x,\pi)\,(M_\pi T_{h,x} S_l\,\varphi)(k,y)\right]d\sigma(l,h,x,\pi).$$

□

THEOREM 3.5 (Inversion Formula). For every $f \in L^2(H\times G)$ and for every admissible pair of functions $\psi, \varphi \in L^2(H\times G)$ such that $C_{\psi,\varphi} \neq 0$, we have

$$f = C^{-1}_{\psi,\varphi} \int\limits_{\mathcal{S}\times\widehat{G}} \operatorname{tr}\left[\mathcal{MS}_\psi f(l,h,x,\pi)\,M_\pi T_{h,x} S_l\,\varphi\right] d\sigma(l,h,x,\pi).$$

Proof. Using (2.5), it is clear that $\mathcal{MS}_\psi f \in \mathscr{H}^2(\mathcal{S}\times\widehat{G})$. On applying Lemma 3.4 for $K = \mathcal{MS}_\psi f$ and then using Theorem 2.14, we obtain

$$\ell^K_\varphi(g) = \int\limits_{\mathcal{S}\times\widehat{G}} \operatorname{tr}\left[\mathcal{MS}_\psi f(l,h,x,\pi)^*\,\mathcal{MS}_\varphi g(l,h,x,\pi)\right] d\sigma(l,h,x,\pi)$$
$$= \langle \mathcal{MS}_\varphi g, \mathcal{MS}_\psi f\rangle_{\mathscr{H}^2(\mathcal{S}\times\widehat{G})}$$
$$= C_{\varphi,\psi}\,\langle g, f\rangle_{L^2(H\times G)}$$
$$= \langle g, \overline{C_{\varphi,\psi}}\,f\rangle_{L^2(H\times G)}$$
$$= \langle g, C_{\psi,\varphi}\,f\rangle_{L^2(H\times G)}.$$

Again applying Lemma 3.4 for $K = \mathcal{MS}_\psi f$, we have $\ell^K_\varphi(g) = \langle g, g_0\rangle_{L^2(H\times G)}$, where for all $(k,y) \in H\times G$,

$$g_0(k,y) = \int\limits_{\mathcal{S}\times\widehat{G}} \operatorname{tr}\left[\mathcal{MS}_\psi f(l,h,x,\pi)\,M_\pi T_{h,x} S_l\,\varphi(k,y)\right] d\sigma(l,h,x,\pi).$$

$$\text{So}\qquad C_{\psi,\varphi}\,f = \int\limits_{\mathcal{S}\times\widehat{G}} \operatorname{tr}\left[\mathcal{MS}_\psi f(l,h,x,\pi)\,M_\pi T_{h,x} S_l\,\varphi\right] d\sigma(l,h,x,\pi)$$

$$\text{i.e.,}\qquad f = C^{-1}_{\psi,\varphi} \int\limits_{\mathcal{S}\times\widehat{G}} \operatorname{tr}\left[\mathcal{MS}_\psi f(l,h,x,\pi)\,M_\pi T_{h,x} S_l\,\varphi\right] d\sigma(l,h,x,\pi).$$

□

Following is an immediate consequence of Theorem 3.5.

COROLLARY 3.6. For every $f \in L^2(H\times G)$ and for every admissible function $\psi \in L^2(H\times G)$ such that $C_\psi \neq 0$, we have

$$f = C^{-1}_{\psi} \int\limits_{\mathcal{S}\times\widehat{G}} \operatorname{tr}\left[\mathcal{MS}_\psi f(l,h,x,\pi)\,M_\pi T_{h,x} S_l\,\psi\right] d\sigma(l,h,x,\pi).$$

LEMMA 3.7. For every admissible function $\varphi \in L^2(H \times G)$, the adjoint operator of $\mathcal{MS}_\varphi$ is an operator $\mathcal{MS}^*_\varphi : \mathscr{H}^2(\mathcal{S} \times \widehat{G}) \to L^2(H \times G)$ given by

$$\mathcal{MS}^*_\varphi(K) = \int\limits_{\mathcal{S}\times\widehat{G}} \operatorname{tr}\left[K(l,h,x,\pi)\; M_\pi T_{h,x} S_l\,\varphi\right] d\sigma(l,h,x,\pi).$$

Proof. Let $J_\varphi : \mathscr{H}^2(\mathcal{S} \times \widehat{G}) \to L^2(H \times G)$ be defined by

$$J_\varphi(K) = \int\limits_{\mathcal{S}\times\widehat{G}} \operatorname{tr}\left[K(l,h,x,\pi)\; M_\pi T_{h,x} S_l\,\varphi\right] d\sigma(l,h,x,\pi)$$

for all $K \in \mathscr{H}^2(\mathcal{S} \times \widehat{G})$. Then J_φ is a bounded linear operator.

By Lemma 3.4, we have for all $g \in L^2(H \times G)$ and $K \in \mathscr{H}^2(\mathcal{S} \times \widehat{G})$,

$$\begin{aligned}\langle g, J_\varphi(K)\rangle_{L^2(H\times G)} &= \ell^K_\varphi(g)\\ &= \int\limits_{\mathcal{S}\times\widehat{G}} \operatorname{tr}\left[K(l,h,x,\pi)^*\; \mathcal{MS}_\varphi g(l,h,x,\pi)\right] d\sigma(l,h,x,\pi)\\ &= \langle \mathcal{MS}_\varphi\, g, K\rangle_{\mathscr{H}^2(\mathcal{S}\times\widehat{G})} = \langle g, \mathcal{MS}^*_\varphi(K)\rangle_{L^2(H\times G)}.\end{aligned}$$

So $J_\varphi(K) = \mathcal{MS}^*_\varphi(K)$ for all $K \in \mathscr{H}^2(\mathcal{S} \times \widehat{G})$. Hence, we have

$$\mathcal{MS}^*_\varphi(K) = \int\limits_{\mathcal{S}\times\widehat{G}} \operatorname{tr}\left[K(l,h,x,\pi)\; M_\pi T_{h,x} S_l\,\varphi\right] d\sigma(l,h,x,\pi).$$

□

COROLLARY 3.8. For every admissible pair of functions $\psi, \varphi \in L^2(H \times G)$ such that $C_{\psi,\varphi} \neq 0$, we have

$$\mathcal{MS}^*_\varphi \mathcal{MS}_\psi = C_{\psi,\varphi}\, I_{L^2(H\times G)},$$

where $I_{L^2(H\times G)}$ denotes the identity operator on $L^2(H \times G)$.

Proof. On applying Lemma 3.7 for $K = \mathcal{MS}_\psi f$ and then using Theorem 3.5, we have for all $f \in L^2(H \times G)$,

$$\begin{aligned}(\mathcal{MS}^*_\varphi \mathcal{MS}_\psi)(f) &= \int\limits_{\mathcal{S}\times\widehat{G}} \operatorname{tr}\left[\mathcal{MS}_\psi f(l,h,x,\pi)\; M_\pi T_{h,x} S_l\,\varphi\right] d\sigma(l,h,x,\pi)\\ &= C_{\psi,\varphi}\, f\\ &= C_{\psi,\varphi}\, I_{L^2(H\times G)}(f).\end{aligned}$$

Hence, $\mathcal{MS}^*_\varphi \mathcal{MS}_\psi = C_{\psi,\varphi}\, I_{L^2(H\times G)}$. □

For $f \in L^2(H \times G)$ and $\Psi \in L^2(H \times G, \mathcal{B}(\mathscr{H}_\pi))$, we consider an operator $\langle f, \Psi\rangle_\pi$ on $\mathscr{H}_\pi$ as

$$\langle f, \Psi\rangle_\pi = \int\limits_H \int\limits_G f(k,y)\, \Psi(k,y)^*\, d\mu_H(k)\, d\mu_G(y).$$

Using Cauchy-Schwarz inequality, we have for all $\zeta, \xi \in \mathscr{H}_\pi$,

$$\begin{aligned}|\langle\langle f, \Psi\rangle_\pi \zeta, \xi\rangle| &= \left|\int_H \int_G f(k,y) \langle \Psi(k,y)^* \zeta, \xi\rangle \, d\mu_H(k)\, d\mu_G(y)\right| \\ &\leq \int_H \int_G |f(k,y) \langle \Psi(k,y)^* \zeta, \xi\rangle| \, d\mu_H(k)\, d\mu_G(y) \\ &\leq \|\zeta\| \, \|\xi\| \int_H \int_G |f(k,y)| \, \|\Psi(k,y)\| \, d\mu_H(k)\, d\mu_G(y) \\ &\leq \|\zeta\| \, \|\xi\| \, \|f\|_{L^2(H\times G)} \, \|\Psi\|_{L^2(H\times G, \mathcal{B}(\mathscr{H}_\pi))}.\end{aligned}$$

So for each $\pi \in \widehat{G}$, $\langle f, \Psi\rangle_\pi$ defines a bounded linear operator on $\mathscr{H}_\pi$ and its operator norm satisfies

$$\|\langle f, \Psi\rangle_\pi\| \leq \|f\|_{L^2(H\times G)} \, \|\Psi\|_{L^2(H\times G, \mathcal{B}(\mathscr{H}_\pi))}.$$

Thus

$$\langle \cdot, \cdot\rangle_\pi : L^2(H\times G) \times L^2(H\times G, \mathcal{B}(\mathscr{H}_\pi)) \to \mathcal{B}(\mathscr{H}_\pi)$$

is a separately continuous sesquilinear map with values in $\mathcal{B}(\mathscr{H}_\pi)$.

REMARK 3.9. We can rewrite the expression of continuous modulated shearlet transform as follows:

$$\begin{aligned}\mathcal{MS}_\psi f(l,h,x,\pi) &= \int_H \int_G f(k,y) \, \overline{\mathcal{U}^\psi_{(l,h,x)}(k,y)} \, \pi(y)^* \, d\mu_H(k)\, d\mu_G(y) \\ &= \int_H \int_G f(k,y) \, \delta_\lambda^{1/2}(l) \, \overline{\psi(\lambda_{l^{-1}}(h^{-1}k), x^{-1}y)} \, \pi(y)^* \, d\mu_H(k)\, d\mu_G(y) \\ &= \int_H \int_G f(k,y) \, \overline{(T_{h,x}(S_l\psi))(k,y)} \, \pi(y)^* \, d\mu_H(k)\, d\mu_G(y) \\ &= \int_H \int_G f(k,y) \, [M_\pi T_{h,x} S_l \, \psi(k,y)]^* \, d\mu_H(k)\, d\mu_G(y) \\ &= \langle f, M_\pi T_{h,x} S_l \, \psi\rangle_\pi.\end{aligned}$$

Let $\phi_1, \phi_2 \in L^2(H\times G, \mathcal{B}(\mathscr{H}_\pi))$. Consider an operator

$$\phi_1 \otimes_\pi \phi_2 : L^2(H\times G) \to L^2(H\times G, \mathcal{B}(\mathscr{H}_\pi))$$

defined by $(\phi_1 \otimes_\pi \phi_2)(f) = \langle f, \phi_2\rangle_\pi \, \phi_1$ for all $f \in L^2(H\times G)$.
The inversion formula given as Theorem 3.5 can be restated as follows:

COROLLARY 3.10. For every admissible pair of functions $\psi, \varphi \in L^2(H\times G)$ such that $C_{\psi,\varphi} \neq 0$, we have

$$f = C^{-1}_{\psi,\varphi} \int_{\mathcal{S}\times\widehat{G}} \operatorname{tr}\left[((M_\pi T_{h,x} S_l \, \varphi) \otimes_\pi (M_\pi T_{h,x} S_l \, \psi))(f)\right] d\sigma(l,h,x,\pi).$$

4. Special Cases

4.1. *Gabor Transform*

Let us consider the particular case when $L = \{1_L\}$ and $H = \{1_H\}$. Then f and ψ can be considered as functions in $L^2(G)$. The condition for ψ to be admissible becomes

$$C_\psi := \int\limits_G |\widetilde{\psi}(x)|^2 \, d\mu_G(x) = \|\psi\|^2_{L^2(G)} < \infty,$$

which is true. So every $\psi \in L^2(G)$ is an admissible function.
Also the group $\mathcal{S} \times \widehat{G} = (\{1_L\} \times_\lambda \{1_H\}) \times G \times \widehat{G}$ is isomorphic to $G \times \widehat{G}$, so the continuous modulated shearlet transform of $f \in L^2(G)$ with respect to $\psi \in L^2(G)$ can be written as a measurable field of operators on $G \times \widehat{G}$ by

$$\mathcal{MS}_\psi f(x, \pi) = \int\limits_G f(y) \, \overline{\psi(x^{-1}y)} \, \pi(y)^* \, d\mu_G(y),$$

which is just the Gabor transform. For details, see [5].
The equation (2.5) becomes

$$\|\mathcal{MS}_\psi f\|_{\mathscr{H}^2(G\times\widehat{G})} = \|\psi\|_{L^2(G)} \, \|f\|_{L^2(G)},$$

which is same as [2, (3.2)]. Other properties of Gabor transform can also be established in similar manner.

4.2. *Wavelet Transform*

Let us consider the case when $G = \{1_G\}$. Then the dual group $\widehat{G}$ becomes equal to $\{I\}$. The functions f and ψ can be regarded as functions in $L^2(H)$. In this case, the function ψ is said to be admissibile if

$$C_\psi := \int\limits_L |\mathscr{F}_H \widetilde{\psi}(\eta \circ \lambda_l)|^2 \, d\mu_L(l) = \int\limits_L |\widehat{\psi}(\eta \circ \lambda_l)|^2 \, d\mu_L(l) < \infty,$$

which is independent of almost every $\eta \in \widehat{H}$. It is the required admissibility condition for ψ in the case of abstract wavelet transform. For details, refer to [12, Lemma 10]. Also the group $\mathcal{S} \times \widehat{G} = (L \times_\lambda H) \times \{1_G\} \times \{I\}$ is isomorphic to $L \times_\lambda H$, so the continuous modulated shearlet transform of $f \in L^2(H)$ with respect to admissible function $\psi \in L^2(H)$ can be written as

$$\mathcal{MS}_\psi f(l, h) = \int\limits_H f(k) \, \delta_\lambda^{1/2}(l) \, \overline{\psi(\lambda_{l^{-1}}(h^{-1}k))} \, d\mu_H(k),$$

which is just the wavelet transform. For details, see [12, Definition 13].
The equation (2.5) becomes

$$\|\mathcal{MS}_\psi f\|_{\mathscr{H}^2(L\times_\lambda H)} = C_\psi^{1/2} \, \|f\|_{L^2(H)},$$

which is same as [12, (12)]. Other properties of wavelet transform can also be deduced in similar manner.

4.3. *Shearlet Transform*

Consider the particular case when $G = \{1_G\}$ and $L = A \times_\tau B$, where A and B are locally compact groups with left Haar measures $d\mu_A(a)$ and $d\mu_B(b)$ respectively. Let $\mathrm{Aut}(B)$ be the group of automorphisms of B. Assume that $\tau : A \to \mathrm{Aut}(B)$ be the homomorphism $a \mapsto \tau_a$ such that the mapping $(a, b) \mapsto \tau_a(b)$ from $A \times B$ onto B is continuous. The functions f and ψ can be regarded as functions in $L^2(H)$. In this case, the function ψ is said to be admissibile if

$$C_\psi := \int\limits_L |\mathscr{F}_H \widetilde{\psi}(\eta \circ \lambda_{(a,b)})|^2 \, d\mu_L(a, b)$$
$$= \int\limits_{A \times_\tau B} |\widehat{\psi}(\eta \circ \lambda_{(a,b)})|^2 \, \delta_\tau(a) \, d\mu_A(a) \, d\mu_B(b) < \infty,$$

which is independent of almost every $\eta \in \widehat{H}$. It is the required admissibility condition for ψ in the case of abstract shearlet transform. For details, refer to [10, Theorem 3.9].
Also the group $\mathcal{S} \times \widehat{G} = ((A \times_\tau B) \times_\lambda H) \times \{1_G\} \times \{I\}$ is isomorphic to $(A \times_\tau B) \times_\lambda H$. So the continuous modulated shearlet transform of $f \in L^2(H)$ with respect to admissible function $\psi \in L^2(H)$ can be written as

$$\mathcal{MS}_\psi f(a, b, h) = \int\limits_H f(k) \, \delta_\lambda^{1/2}(a, b) \, \overline{\psi(\lambda_{(a,b)^{-1}}(h^{-1}k))} \, d\mu_H(k),$$

which is just the shearlet transform. For details, see [10, Definition 3.3].

The equation (2.5) becomes

$$\|\mathcal{MS}_\psi f\|_{\mathscr{H}^2((A \times_\tau B) \times_\lambda H)} = C_\psi^{1/2} \, \|f\|_{L^2(H)},$$

which is same as [10, Theorem 3.9]. Other properties of shearlet transform can be investigated in similar manner.

5. Examples

In this section, we shall discuss some examples of the continuous modulated shearlet transform.

EXAMPLE 5.1. In this example, we shall extend the definition of shearlet group as discussed in [3]. Consider *anisotropic (parabolic) scaling* matrix $A_a = \begin{bmatrix} a & 0 \\ 0 & \sqrt{a} \end{bmatrix}$ and *shear* matrix $S_s = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$ acting on the Euclidean plane, where $a \in \mathbb{R}^+$ and $s \in \mathbb{R}$.
We consider $L = \mathbb{R}^+ \times_\tau \mathbb{R}$, $H = \mathbb{R}^2$ and $G = \mathbb{R}$. Thus consider the group

$$\mathbb{MS} = (\mathbb{R}^+ \times_\tau \mathbb{R}) \times_\lambda \mathbb{R}^2 \times \mathbb{R} \times \widehat{\mathbb{R}} = \mathcal{S} \times \widehat{\mathbb{R}}.$$

Here $\tau : \mathbb{R}^+ \to \mathrm{Aut}(\mathbb{R})$ is a homomorphism $a \mapsto \tau_a$ with

$$\tau_a(s) = \sqrt{a} \, s$$

and $\lambda : \mathbb{R}^+ \times_\tau \mathbb{R} \to \mathrm{Aut}(\mathbb{R}^2)$ is a homomorphism $(a, s) \mapsto \lambda_{(a,s)}$ with

$$\lambda_{(a,s)}(t) = S_s \, A_a \, t$$

such that $(a,s) \mapsto \tau_a(s)$ from $\mathbb{R}^+ \times \mathbb{R}$ onto $\mathbb{R}$ and $((a,s),t) \mapsto \lambda_{(a,s)}(t)$ from $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$ onto $\mathbb{R}^2$ are continuous, where $t \in \mathbb{R}^2$.
The multiplication operation on $\mathbb{MS}$ is

$$(a,s,t,x,\xi)(a',s',t',x',\xi') = (a\,a', s + s'\sqrt{a}, t + S_s\,A_a\,t', x + x', \xi + \xi')$$

with left Haar measure given by

$$d\mu_{\text{left}}(a,s,t,x,\xi) = \frac{da}{a^3}\,ds\,dt\,dx\,d\xi,$$

where $a \in \mathbb{R}^+$, $s \in \mathbb{R}$, $t \in \mathbb{R}^2$, $x \in \mathbb{R}$ and $\xi \in \widehat{\mathbb{R}}$.
A function $\psi \in L^2(\mathbb{R}^3)$ is admissible if

$$\int_{\mathbb{R}^3} |\widehat{\psi}(v_1,v_2,\zeta)|^2\,\frac{dv_1}{v_1^2}\,dv_2\,d\zeta < \infty. \tag{5.1}$$

The continuous modulated shearlet transform of $f \in L^2(\mathbb{R}^3)$ with respect to admissible function $\psi \in L^2(\mathbb{R}^3)$ is given by

$$\mathcal{MS}_\psi f(a,s,t,x,\xi) = \int_{\mathbb{R}^3} f(k,y)\,\overline{\mathcal{U}^{\psi}_{(a,s,t,x)}(k,y)}\,e^{-2\pi i\xi y}\,dk\,dy,$$

where $\mathcal{U}^{\psi}_{(a,s,t,x)}(k,y) = a^{-3/4}\,\psi(A_a^{-1}S_s^{-1}(k-t), y-x)$, for all $k \in \mathbb{R}^2$, $y \in \mathbb{R}$.

Also, we have $\mathcal{L}^{\psi}_{(a,s,t,x)}f(k,y) = f(k,y)\,\overline{\mathcal{U}^{\psi}_{(a,s,t,x)}(k,y)}$, for all $k \in \mathbb{R}^2$, $y \in \mathbb{R}$.
We now verify the admissibility condition (5.1). We have

$$\begin{aligned}
&\|\mathcal{MS}_\psi f\|^2_{\mathscr{H}^2(\mathcal{S}\times\widehat{\mathbb{R}})} \\
&= \int_0^\infty \int_{\mathbb{R}^5} |\mathcal{MS}_\psi f(a,s,t_1,t_2,x,\xi)|^2\,ds\,dt_1\,dt_2\,dx\,d\xi\,\frac{da}{a^3} \\
&= \int_0^\infty \int_{\mathbb{R}^5} \left|\mathscr{F}_{123}\left(\mathcal{L}^{\psi}_{(a,s,t_1,t_2,x)}f\right)(0,0,\xi)\right|^2 ds\,dt_1\,dt_2\,dx\,d\xi\,\frac{da}{a^3} \\
&= \int_0^\infty \int_{\mathbb{R}^5} \left|\mathscr{F}_{12}\left(\mathcal{L}^{\psi}_{(a,s,t_1,t_2,x)}f\right)(0,0,y)\right|^2 ds\,dt_1\,dt_2\,dx\,dy\,\frac{da}{a^3} \\
&= \int_0^\infty \int_{\mathbb{R}^5} \left|\left(f *_{12} \mathcal{U}^{\widetilde{\psi}}_{(a,s,0,0,yx^{-1}y)}\right)(t_1,t_2,y)\right|^2 ds\,dt_1\,dt_2\,dx\,dy\,\frac{da}{a^3} \\
&= \int_0^\infty \int_{\mathbb{R}^5} |\mathscr{F}_{12}f(\omega_1,\omega_2,y)|^2\,\left|\mathscr{F}_{12}\left(\mathcal{U}^{\widetilde{\psi}}_{(a,s,0,0,yx^{-1}y)}\right)(\omega_1,\omega_2,y)\right|^2 ds\,d\omega_1\,d\omega_2\,dx\,dy\,\frac{da}{a^3} \\
&= \int_0^\infty \int_{\mathbb{R}^5} |\mathscr{F}_{12}f(\omega_1,\omega_2,y)|^2\,\left|\mathscr{F}_{12}\widetilde{\psi}(a\omega_1,\sqrt{a}(s\omega_1+\omega_2),x)\right|^2 ds\,d\omega_1\,d\omega_2\,dx\,dy\,\frac{da}{a^{3/2}}.
\end{aligned}$$

We substitute $\sqrt{a}(s\omega_1+\omega_2)=v_2$. If $\omega_1>0$, then v_2 varies from $-\infty$ to ∞ and if $\omega_1<0$, then v_2 varies from ∞ to $-\infty$.

$$
\begin{aligned}
&\|\mathcal{MS}_\psi f\|^2_{\mathscr{H}^2(\mathcal{S}\times\widehat{\mathbb{R}})}\\
&=\int_0^\infty\int_{-\infty}^\infty\int_0^\infty\int_{\mathbb{R}^3}|\mathscr{F}_{12}f(\omega_1,\omega_2,y)|^2\left|\mathscr{F}_{12}\widetilde{\psi}(a\omega_1,v_2,x)\right|^2\frac{1}{\omega_1}\,d\omega_2\,dx\,dy\,d\omega_1\,dv_2\,\frac{da}{a^2}\\
&-\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^0\int_{\mathbb{R}^3}|\mathscr{F}_{12}f(\omega_1,\omega_2,y)|^2\left|\mathscr{F}_{12}\widetilde{\psi}(a\omega_1,v_2,x)\right|^2\frac{1}{\omega_1}\,d\omega_2\,dx\,dy\,d\omega_1\,dv_2\,\frac{da}{a^2}.
\end{aligned}
$$

On substituting $a\omega_1=v_1$, we have

$$
\begin{aligned}
&\|\mathcal{MS}_\psi f\|^2_{\mathscr{H}^2(\mathcal{S}\times\widehat{\mathbb{R}})}\\
&=\int_0^\infty\int_{-\infty}^\infty\int_0^\infty\int_{\mathbb{R}^3}|\mathscr{F}_{12}f(\omega_1,\omega_2,y)|^2\left|\mathscr{F}_{12}\widetilde{\psi}(v_1,v_2,x)\right|^2d\omega_2\,dx\,dy\,d\omega_1\,dv_2\,\frac{dv_1}{v_1^2}\\
&+\int_{-\infty}^0\int_{-\infty}^\infty\int_{-\infty}^0\int_{\mathbb{R}^3}|\mathscr{F}_{12}f(\omega_1,\omega_2,y)|^2\left|\mathscr{F}_{12}\widetilde{\psi}(v_1,v_2,x)\right|^2d\omega_2\,dx\,dy\,d\omega_1\,dv_2\,\frac{dv_1}{v_1^2}\\
&=\int_0^\infty\int_{\mathbb{R}^2}|\widehat{f}(\omega_1,\omega_2,\xi)|^2\,d\omega_2\,d\xi\,d\omega_1\int_0^\infty\int_{\mathbb{R}^2}|\widehat{\psi}(v_1,v_2,\zeta)|^2\,dv_2\,d\zeta\,\frac{dv_1}{v_1^2}\\
&+\int_{-\infty}^0\int_{\mathbb{R}^2}|\widehat{f}(\omega_1,\omega_2,\xi)|^2\,d\omega_2\,d\xi\,d\omega_1\int_{-\infty}^0\int_{\mathbb{R}^2}|\widehat{\psi}(v_1,v_2,\zeta)|^2\,dv_2\,d\zeta\,\frac{dv_1}{v_1^2}\\
&=C_\psi^+\int_0^\infty\int_{\mathbb{R}^2}|\widehat{f}(\omega_1,\omega_2,\xi)|^2\,d\omega_2\,d\xi\,d\omega_1+C_\psi^-\int_{-\infty}^0\int_{\mathbb{R}^2}|\widehat{f}(\omega_1,\omega_2,\xi)|^2\,d\omega_2\,d\xi\,d\omega_1,
\end{aligned}
$$

where

$$
C_\psi^+=\int_0^\infty\int_{\mathbb{R}^2}|\widehat{\psi}(v_1,v_2,\zeta)|^2\,dv_2\,d\zeta\,\frac{dv_1}{v_1^2}\text{ and }C_\psi^-=\int_{-\infty}^0\int_{\mathbb{R}^2}|\widehat{\psi}(v_1,v_2,\zeta)|^2\,dv_2\,d\zeta\,\frac{dv_1}{v_1^2}.
$$

It is clear that

$$
\int_{\mathbb{R}^3}|\widehat{\psi}(v_1,v_2,\zeta)|^2\,\frac{dv_1}{v_1^2}\,dv_2\,d\zeta=C_\psi^++C_\psi^-,
$$

which implies

$$
\int_{\mathbb{R}^3}|\widehat{\psi}(v_1,v_2,\zeta)|^2\,\frac{dv_1}{v_1^2}\,dv_2\,d\zeta<\infty\text{ if and only if }C_\psi^+<\infty\text{ and }C_\psi^-<\infty.
$$

But ψ is admissible if $\|\mathcal{MS}_\psi f\|_{\mathscr{H}^2(\mathcal{S}\times\widehat{\mathbb{R}})}<\infty$, i.e., if $C_\psi^+<\infty$ and $C_\psi^-<\infty$, i.e., if

$$
\int_{\mathbb{R}^3}|\widehat{\psi}(v_1,v_2,\zeta)|^2\,\frac{dv_1}{v_1^2}\,dv_2\,d\zeta<\infty.
$$

EXAMPLE 5.2. In this example, we extend the definition of three dimensional shearlets as discussed in [4]. Consider *anisotropic (parabolic) scaling* matrix $A_a = \begin{bmatrix} a & 0 & 0 \\ 0 & \sqrt[3]{a} & 0 \\ 0 & 0 & \sqrt[3]{a} \end{bmatrix}$ and *shear* matrix $S_s = \begin{bmatrix} 1 & s_1 & s_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ acting on the Euclidean plane, where $a \in \mathbb{R}^*$ and $s = (s_1, s_2) \in \mathbb{R}^2$.

We assume $L = \mathbb{R}^* \times_\tau \mathbb{R}^2$, $H = \mathbb{R}^3$ and $G = \mathbb{R}$. Consider the group

$$\mathbb{MS} = (\mathbb{R}^* \times_\tau \mathbb{R}^2) \times_\lambda \mathbb{R}^3 \times \mathbb{R} \times \widehat{\mathbb{R}} = \mathcal{S} \times \widehat{\mathbb{R}}.$$

Here $\tau : \mathbb{R}^* \to \mathrm{Aut}(\mathbb{R}^2)$ is a homomorphism $a \mapsto \tau_a$ with

$$\tau_a(s) = a^{2/3}\, s$$

and $\lambda : \mathbb{R}^* \times_\tau \mathbb{R}^2 \to \mathrm{Aut}(\mathbb{R}^3)$ is a homomorphism $(a, s) \mapsto \lambda_{(a,s)}$ with

$$\lambda_{(a,s)}(t) = S_s\, A_a\, t$$

such that $(a, s) \mapsto \tau_a(s)$ from $\mathbb{R}^* \times \mathbb{R}^2$ onto $\mathbb{R}^2$ and $((a, s), t) \mapsto \lambda_{(a,s)}(t)$ from $\mathbb{R}^* \times \mathbb{R}^2 \times \mathbb{R}^3$ onto $\mathbb{R}^3$ are continuous, where $t \in \mathbb{R}^3$.

The multiplication operation on $\mathbb{MS}$ is

$$(a, s, t, x, \xi)(a', s', t', x', \xi') = (a\, a', s + a^{2/3}\, s', t + S_s\, A_a\, t', x + x', \xi + \xi')$$

with left Haar measure given by

$$d\mu_{\text{left}}(a, s, t, x, \xi) = \frac{da}{a^4}\, ds\; dt\; dx\; d\xi,$$

where $a \in \mathbb{R}^*$, $s \in \mathbb{R}^2$, $t \in \mathbb{R}^3$, $x \in \mathbb{R}$ and $\xi \in \widehat{\mathbb{R}}$.

A function $\psi \in L^2(\mathbb{R}^4)$ is admissible if

$$\int_{\mathbb{R}^4} |\widehat{\psi}(v_1, v_2, v_3, \zeta)|^2\, \frac{dv_1}{|v_1|^3}\, dv_2\; dv_3\; d\zeta < \infty. \tag{5.2}$$

The continuous modulated shearlet transform of $f \in L^2(\mathbb{R}^4)$ with respect to admissible function $\psi \in L^2(\mathbb{R}^4)$ is given by

$$\mathcal{MS}_\psi f(a, s, t, x, \xi) = \int_{\mathbb{R}^4} f(k, y)\, \overline{\mathcal{U}^\psi_{(a,s,t,x)}(k, y)}\, e^{-2\pi i \xi y}\, dk\; dy,$$

where $\mathcal{U}^\psi_{(a,s,t,x)}(k, y) = a^{-5/6}\, \psi(A_a^{-1} S_s^{-1}(k - t), y - x)$, for all $k \in \mathbb{R}^3$, $y \in \mathbb{R}$.

Also, we have $\mathcal{L}^\psi_{(a,s,t,x)} f(k, y) = f(k, y)\, \overline{\mathcal{U}^\psi_{(a,s,t,x)}(k, y)}$, for all $k \in \mathbb{R}^3$, $y \in \mathbb{R}$.

We now verify the admissibility condition (5.2). Proceeding as in previous example, we have

$$\begin{aligned} &\|\mathcal{MS}_\psi f\|^2_{\mathscr{H}^2(\mathcal{S} \times \widehat{\mathbb{R}})} \\ &= \int_{\mathbb{R}^8} |\mathcal{MS}_\psi f(a, s_1, s_2, t_1, t_2, t_3, x, \xi)|^2\, \frac{da}{a^4}\, ds_1\; ds_2\; dt_1\; dt_2\; dt_3\; dx\; d\xi \end{aligned}$$

$$= \int_{\mathbb{R}^8} |\mathscr{F}_{123} f(\omega_1, \omega_2, \omega_3, y)|^2 \left| \mathscr{F}_{123} \widetilde{\psi}(a\omega_1, a^{1/3}(s_1\omega_1 + \omega_2), a^{1/3}(s_2\omega_1 + \omega_3), x) \right|^2 \frac{da}{|a|^{7/3}} \, ds_1 \, ds_2 \, d\omega_1 \, d\omega_2 \, d\omega_3 \, dx \, dy.$$

On substituting $a^{1/3}(s_1\omega_1 + \omega_2) = v_2$ and $a^{1/3}(s_2\omega_1 + \omega_3) = v_3$, we have

$$\|\mathcal{MS}_\psi f\|^2_{\mathscr{H}^2(\mathcal{S}\times\widehat{\mathbb{R}})} = \int_{\mathbb{R}^8} |\mathscr{F}_{123} f(\omega_1, \omega_2, \omega_3, y)|^2 \left| \mathscr{F}_{123} \widetilde{\psi}(a\omega_1, v_2, v_3, x) \right|^2 \frac{da}{|a|^3} \frac{dv_2}{\omega_1} \frac{dv_3}{\omega_1} \, d\omega_1 \, d\omega_2 \, d\omega_3 \, dx \, dy.$$

We now substitute $a\omega_1 = v_1$. If $\omega_1 > 0$, then v_1 varies from $-\infty$ to ∞ and if $\omega_1 < 0$, then v_1 varies from ∞ to $-\infty$. We obtain

$$\|\mathcal{MS}_\psi f\|^2_{\mathscr{H}^2(\mathcal{S}\times\widehat{\mathbb{R}})} = C_\psi \, \|f\|_2^2,$$

where

$$C_\psi = \int_{\mathbb{R}^4} |\widehat{\psi}(v_1, v_2, v_3, \zeta)|^2 \frac{dv_1}{|v_1|^3} \, dv_2 \, dv_3 \, d\zeta.$$

Thus, the admissibility condition for ψ can be stated as $C_\psi < \infty$.

6. Uncertainty Principles

We shall now prove certain versions of Heisenberg uncertainty principle. The weak uncertainty principle given below follows immediately by using Corollary 2.13.

THEOREM 6.1. Let $f \in L^2(H\times G)$ and $\psi \in L^2(H\times G)$ be an admissible function such that $\|f\|_{L^2(H\times G)} = \|\psi\|_{L^2(H\times G)} = 1$. Suppose that $U \subseteq \mathcal{S} \times \widehat{G}$ and $\epsilon > 0$ satisfy

$$\int_U \|\mathcal{MS}_\psi f(l, h, x, \pi)\|^2 \, d\sigma(l, h, x, \pi) \geq 1 - \epsilon.$$

Then $\sigma(U) \geq 1 - \epsilon$.

THEOREM 6.2 (Heisenberg Uncertainty Inequality). Let H and L as above and $G = \mathbb{R}^n$. For any $f \in L^2(H \times G)$, admissible function $\psi \in L^2(H \times G)$ and $a, b \geq 1$, we have

$$\frac{n}{4\pi} C_\psi^{\frac{1}{2b}} \, \|f\|^{\frac{1}{a}+\frac{1}{b}}_{L^2(H\times G)} \leq \left(\int_H \int_G |y|^{2a} \, |f(k, y)|^2 \, d\mu_H(k) \, d\mu_G(y) \right)^{\frac{1}{2a}} \times \left(\int_{\mathcal{S}\times\widehat{G}} |\xi|^{2b} \, |\mathcal{MS}_\psi f(l, h, x, \xi)|^2 \, d\sigma(l, h, x, \xi) \right)^{\frac{1}{2b}}.$$

Proof. Using Plancherel theorem and Lemma 2.7, we have

$$\begin{aligned}
& C_\psi \, \|f\|^2_{L^2(H\times G)} \\
&= C_\psi \int\limits_G \int\limits_H |f(k,y)|^2 \, d\mu_H(k) \, d\mu_G(y) \\
&= C_\psi \int\limits_G \int\limits_{\widehat{H}} |\mathscr{F}_H f(\eta,y)|^2 \, d\mu_{\widehat{H}}(\eta) \, d\mu_G(y) \\
&= \int\limits_G \int\limits_{\widehat{H}} |\mathscr{F}_H f(\eta,y)|^2 \left[\int\limits_L \int\limits_G \left| \mathscr{F}_H \left(\mathcal{U}^{\widetilde{\psi}}_{(l,1_H,yx^{-1}y)} \right)(\eta,y) \right|^2 \delta_\lambda(l) \, d\mu_L(l) \, d\mu_G(x) \right] d\mu_{\widehat{H}}(\eta) \, d\mu_G(y) \\
&= \int\limits_L \int\limits_G \int\limits_G \left[\int\limits_{\widehat{H}} |\mathscr{F}_H f(\eta,y)|^2 \, \left| \mathscr{F}_H \left(\mathcal{U}^{\widetilde{\psi}}_{(l,1_H,yx^{-1}y)} \right)(\eta,y) \right|^2 d\mu_{\widehat{H}}(\eta) \right] \delta_\lambda(l) \, d\mu_L(l) \, d\mu_G(x) \, d\mu_G(y) \\
&= \int\limits_L \int\limits_G \int\limits_G \left[\int\limits_H \left| \left(f *_H \mathcal{U}^{\widetilde{\psi}}_{(l,1_H,yx^{-1}y)} \right)(h,y) \right|^2 d\mu_H(h) \right] \delta_\lambda(l) \, d\mu_L(l) \, d\mu_G(x) \, d\mu_G(y) \\
&= \int\limits_L \int\limits_H \int\limits_G \int\limits_G \left| \mathscr{F}_H \left(\mathcal{L}^{\psi}_{(l,h,x)} f \right)(I,y) \right|^2 \delta_\lambda(l) \, d\mu_L(l) \, d\mu_H(h) \, d\mu_G(x) \, d\mu_G(y). \qquad (6.1)
\end{aligned}$$

So $\mathscr{F}_H \left(\mathcal{L}^{\psi}_{(l,h,x)} f \right)(I,\cdot) \in L^2(G)$ for almost every $(l,h,x) \in \mathcal{S}$.
Using Theorem 1.1 for $a = b = 1$, we have

$$\begin{aligned}
& \frac{n}{4\pi} \int\limits_G \left| \mathscr{F}_H \left(\mathcal{L}^{\psi}_{(l,h,x)} f \right)(I,y) \right|^2 d\mu_G(y) \\
&\le \left(\int\limits_G |y|^2 \, \left| \mathscr{F}_H \left(\mathcal{L}^{\psi}_{(l,h,x)} f \right)(I,y) \right|^2 d\mu_G(y) \right)^{1/2} \\
&\quad \times \left(\int\limits_{\widehat{G}} |\xi|^2 \, \left| \mathscr{F}_G \mathscr{F}_H \left(\mathcal{L}^{\psi}_{(l,h,x)} f \right)(I,\xi) \right|^2 d\mu_{\widehat{G}}(\xi) \right)^{1/2}.
\end{aligned}$$

Applying Cauchy-Schwarz inequality, equation (6.1) can be written as

$$\begin{aligned}
& \frac{n}{4\pi} \, C_\psi \, \|f\|^2_{L^2(H\times G)} \\
&\le \int\limits_L \int\limits_H \int\limits_G \left(\int\limits_G |y|^2 \, \left| \mathscr{F}_H \left(\mathcal{L}^{\psi}_{(l,h,x)} f \right)(I,y) \right|^2 d\mu_G(y) \right)^{1/2} \\
&\quad \times \left(\int\limits_{\widehat{G}} |\xi|^2 \, \left| \mathscr{F}_G \mathscr{F}_H \left(\mathcal{L}^{\psi}_{(l,h,x)} f \right)(I,\xi) \right|^2 d\mu_{\widehat{G}}(\xi) \right)^{1/2} \delta_\lambda(l) \, d\mu_L(l) \, d\mu_H(h) \, d\mu_G(x)
\end{aligned}$$

$$\leq \left(\int\limits_L \int\limits_H \int\limits_G \int\limits_G |y|^2 \, \left| \mathscr{F}_H \left(\mathcal{L}^{\psi}_{(l,h,x)} f \right) (I, y) \right|^2 \delta_\lambda(l) \, d\mu_L(l) \, d\mu_H(h) \, d\mu_G(x) \, d\mu_G(y) \right)^{1/2}$$

$$\times \left(\int\limits_L \int\limits_H \int\limits_G \int\limits_{\widehat{G}} |\xi|^2 \, \left| \mathscr{F}_G \mathscr{F}_H \left(\mathcal{L}^{\psi}_{(l,h,x)} f \right) (I, \xi) \right|^2 \delta_\lambda(l) \, d\mu_L(l) \, d\mu_H(h) \, d\mu_G(x) \, d\mu_{\widehat{G}}(\xi) \right)^{1/2} .$$

Using Lemma 2.7, equation (2.4), Plancherel formula and Corollary 2.10, we obtain

$$\frac{n}{4\pi} C_\psi \, \|f\|^2_{L^2(H\times G)}$$

$$\leq \left(\int\limits_L \int\limits_G \int\limits_G |y|^2 \left[\int\limits_H \left| \left(f *_H \mathcal{U}^{\widetilde{\psi}}_{(l, 1_H, yx^{-1}y)} \right) (h, y) \right|^2 d\mu_H(h) \right] \delta_\lambda(l) \, d\mu_L(l) \, d\mu_G(x) \, d\mu_G(y) \right)^{1/2}$$

$$\times \left(\int\limits_{\mathcal{S}\times\widehat{G}} |\xi|^2 \, |\mathcal{M}\mathcal{S}_\psi f(l,h,x,\xi)|^2 \, d\sigma(l,h,x,\xi) \right)^{1/2}$$

$$= \left(\int\limits_L \int\limits_G \int\limits_G |y|^2 \left[\int\limits_{\widehat{H}} |\mathscr{F}_H f(\eta, y)|^2 \left| \mathscr{F}_H \left(\mathcal{U}^{\widetilde{\psi}}_{(l, 1_H, yx^{-1}y)} \right) (\eta, y) \right|^2 d\mu_{\widehat{H}}(\eta) \right] \delta_\lambda(l) \, d\mu_L(l) \, d\mu_G(x) \, d\mu_G(y) \right)^{1/2}$$

$$\times \left(\int\limits_{\mathcal{S}\times\widehat{G}} |\xi|^2 \, |\mathcal{M}\mathcal{S}_\psi f(l,h,x,\xi)|^2 \, d\sigma(l,h,x,\xi) \right)^{1/2}$$

$$= \left(\int\limits_G \int\limits_{\widehat{H}} |y|^2 \, |\mathscr{F}_H f(\eta, y)|^2 \left[\int\limits_L \int\limits_G \left| \mathscr{F}_H \left(\mathcal{U}^{\widetilde{\psi}}_{(l, 1_H, yx^{-1}y)} \right) (\eta, y) \right|^2 \delta_\lambda(l) \, d\mu_L(l) \, d\mu_G(x) \right] d\mu_{\widehat{H}}(\eta) \, d\mu_G(y) \right)^{1/2}$$

$$\times \left(\int\limits_{\mathcal{S}\times\widehat{G}} |\xi|^2 \, |\mathcal{M}\mathcal{S}_\psi f(l,h,x,\xi)|^2 \, d\sigma(l,h,x,\xi) \right)^{1/2}$$

$$= C_\psi^{1/2} \left(\int\limits_G \int\limits_{\widehat{H}} |y|^2 \, |\mathscr{F}_H f(\eta, y)|^2 \, d\mu_{\widehat{H}}(\eta) \, d\mu_G(y) \right)^{1/2}$$

$$\times \left(\int\limits_{\mathcal{S}\times\widehat{G}} |\xi|^2 \, |\mathcal{M}\mathcal{S}_\psi f(l,h,x,\xi)|^2 \, d\sigma(l,h,x,\xi) \right)^{1/2}$$

$$= C_\psi^{1/2} \left(\int\limits_G \int\limits_H |y|^2 \, |f(k, y)|^2 \, d\mu_H(k) \, d\mu_G(y) \right)^{1/2}$$

$$\times \left(\int\limits_{\mathcal{S}\times\widehat{G}} |\xi|^2 \, |\mathcal{M}\mathcal{S}_\psi f(l,h,x,\xi)|^2 \, d\sigma(l,h,x,\xi) \right)^{1/2} .$$

Thus

$$\frac{n}{4\pi}\, C_\psi^{1/2}\, \|f\|^2_{L^2(H\times G)} \leq \left(\int_H\int_G |y|^2\, |f(k,y)|^2\, d\mu_H(k)\, d\mu_G(y)\right)^{1/2} \times \left(\int_{\mathcal{S}\times\widehat{G}} |\xi|^2\, |\mathcal{MS}_\psi f(l,h,x,\xi)|^2\, d\sigma(l,h,x,\xi)\right)^{1/2}. \tag{6.2}$$

Now using Hölder's inequality, we have

$$\begin{aligned}
&\int_H\int_G |y|^2\, |f(k,y)|^2\, d\mu_H(k)\, d\mu_G(y)\\
&= \int_H\int_G |y|^2\, |f(k,y)|^{\frac{2}{a}}\, |f(k,y)|^{2\left(1-\frac{1}{a}\right)}\, d\mu_H(k)\, d\mu_G(y)\\
&\leq \left(\int_H\int_G |y|^{2a}\, |f(k,y)|^2\, d\mu_H(k)\, d\mu_G(y)\right)^{\frac{1}{a}} \left(\int_H\int_G |f(k,y)|^2\, d\mu_H(k)\, d\mu_G(y)\right)^{1-\frac{1}{a}}\\
&= \left(\int_H\int_G |y|^{2a}\, |f(k,y)|^2\, d\mu_H(k)\, d\mu_G(y)\right)^{\frac{1}{a}} \left(\|f\|^2_{L^2(H\times G)}\right)^{1-\frac{1}{a}}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathcal{S}\times\widehat{G}} |\xi|^2\, |\mathcal{MS}_\psi f(l,h,x,\xi)|^2\, d\sigma(l,h,x,\xi)\\
&= \int_{\mathcal{S}\times\widehat{G}} |\xi|^2\, |\mathcal{MS}_\psi f(l,h,x,\xi)|^{\frac{2}{b}}\, |\mathcal{MS}_\psi f(l,h,x,\xi)|^{2\left(1-\frac{1}{b}\right)}\, d\sigma(l,h,x,\xi)\\
&\leq \left(\int_{\mathcal{S}\times\widehat{G}} |\xi|^{2b}\, |\mathcal{MS}_\psi f(l,h,x,\xi)|^2\, d\sigma(l,h,x,\xi)\right)^{\frac{1}{b}}\\
&\quad\times \left(\int_{\mathcal{S}\times\widehat{G}} |\mathcal{MS}_\psi f(l,h,x,\xi)|^2\, d\sigma(l,h,x,\xi)\right)^{1-\frac{1}{b}}\\
&= \left(\int_{\mathcal{S}\times\widehat{G}} |\xi|^{2b}\, |\mathcal{MS}_\psi f(l,h,x,\xi)|^2\, d\sigma(l,h,x,\xi)\right)^{\frac{1}{b}} \left(C_\psi\, \|f\|^2_{L^2(H\times G)}\right)^{1-\frac{1}{b}}.
\end{aligned}$$

Thus equation (6.2) can be written as

$$\begin{aligned}
&\frac{n}{4\pi}\, C_\psi^{1/2}\, \|f\|^2_{L^2(H\times G)}\\
&\leq \left(\int_H\int_G |y|^{2a}\, |f(k,y)|^2\, d\mu_H(k)\, d\mu_G(y)\right)^{\frac{1}{2a}} \left(\|f\|^2_{L^2(H\times G)}\right)^{\frac{1}{2}-\frac{1}{2a}}
\end{aligned}$$

$$\times \left(\int_{\mathcal{S}\times\widehat{G}} |\xi|^{2b} \, |\mathcal{MS}_\psi f(l,h,x,\xi)|^2 \, d\sigma(l,h,x,\xi) \right)^{\frac{1}{2b}} \left(C_\psi \, \|f\|^2_{L^2(H\times G)} \right)^{\frac{1}{2}-\frac{1}{2b}},$$

which implies

$$\frac{n}{4\pi} \, C_\psi^{\frac{1}{2b}} \, \|f\|^{\frac{1}{a}+\frac{1}{b}}_{L^2(H\times G)} \leq \left(\int_H \int_G |y|^{2a} \, |f(k,y)|^2 \, d\mu_H(k) \, d\mu_G(y) \right)^{\frac{1}{2a}}$$
$$\times \left(\int_{\mathcal{S}\times\widehat{G}} |\xi|^{2b} \, |\mathcal{MS}_\psi f(l,h,x,\xi)|^2 \, d\sigma(l,h,x,\xi) \right)^{\frac{1}{2b}}.$$

□

THEOREM 6.3 (Another form). Let $H = \mathbb{R}^n$. For any $f \in L^2(H\times G)$, admissible function $\psi \in L^2(H\times G)$ and $a, b \geq 1$, we have

$$\frac{n}{4\pi} \, C_\psi^{\frac{1}{2a}} \, \|f\|^{\frac{1}{a}+\frac{1}{b}}_{L^2(H\times G)} \leq \left(\int_{\mathcal{S}\times\widehat{G}} |h|^{2a} \, |\mathcal{MS}_\psi f(l,h,x,\pi)|^2 \, d\sigma(l,h,x,\pi) \right)^{\frac{1}{2a}}$$
$$\times \left(\int_G \int_{\widehat{H}} |\eta|^{2b} \, \left|\mathscr{F}_H f(\eta,y)\right|^2 \, d\mu_G(y) \, d\mu_{\widehat{H}}(\eta) \right)^{\frac{1}{2b}}.$$

Proof. Using (2.5), (2.4), Plancherel formula and Lemma 2.7, we have

$$\begin{aligned}
& C_\psi \, \|f\|^2_{L^2(H\times G)} \\
& = \|\mathcal{MS}_\psi f\|^2_{\mathscr{H}^2(\mathcal{S}\times\widehat{G})} \\
& = \int_{\mathcal{S}\times\widehat{G}} \|\mathcal{MS}_\psi f(l,h,x,\pi)\|^2_{\text{HS}} \, d\sigma(l,h,x,\pi) \\
& = \int_L \int_H \int_G \int_{\widehat{G}} \left\| \mathscr{F}_G \mathscr{F}_H \left(\mathcal{L}^\psi_{(l,h,x)} f \right)(I,\pi) \right\|^2_{\text{HS}} \delta_\lambda(l) \, d\mu_L(l) \, d\mu_H(h) \, d\mu_G(x) \, d\mu_{\widehat{G}}(\pi) \\
& = \int_L \int_H \int_G \int_G \left| \mathscr{F}_H \left(\mathcal{L}^\psi_{(l,h,x)} f \right)(I,y) \right|^2 \delta_\lambda(l) \, d\mu_L(l) \, d\mu_H(h) \, d\mu_G(x) \, d\mu_G(y) \\
& = \int_L \int_H \int_G \int_G \left| \left(f *_H \mathcal{U}^{\widetilde{\psi}}_{(l,1_H,yx^{-1}y)} \right)(h,y) \right|^2 \delta_\lambda(l) \, d\mu_L(l) \, d\mu_H(h) \, d\mu_G(x) \, d\mu_G(y).
\end{aligned} \tag{6.3}$$

So $\left(f *_H \mathcal{U}^{\widetilde{\psi}}_{(l,1_H,yx^{-1}y)} \right)(\cdot,y) \in L^2(H)$ for almost every $l \in L$ and $x, y \in G$.
Using Theorem 1.1 for $a = b = 1$ and then using Lemma 2.7 and Lemma 2.8, we obtain

$$\frac{n}{4\pi} \int_H \left| \left(f *_H \mathcal{U}^{\widetilde{\psi}}_{(l,1_H,yx^{-1}y)} \right)(h,y) \right|^2 d\mu_H(h)$$

$$\leq \left(\int_H |h|^2 \left| \left(f *_H \mathcal{U}^{\widetilde{\psi}}_{(l,1_H,yx^{-1}y)} \right)(h,y) \right|^2 d\mu_H(h) \right)^{1/2}$$

$$\times \left(\int_{\widehat{H}} |\eta|^2 \left| \mathscr{F}_H \left(f *_H \mathcal{U}^{\widetilde{\psi}}_{(l,1_H,yx^{-1}y)} \right)(\eta,y) \right|^2 d\mu_{\widehat{H}}(\eta) \right)^{1/2}$$

$$= \left(\int_H |h|^2 \left| \mathscr{F}_H \left(\mathcal{L}^{\psi}_{(l,h,x)} f \right)(I,y) \right|^2 d\mu_H(h) \right)^{1/2}$$

$$\times \left(\int_{\widehat{H}} |\eta|^2 \, |\mathscr{F}_H f(\eta,y)|^2 \left| \mathscr{F}_H \widetilde{\psi}(\eta \circ \lambda_l, y^{-1}x) \right|^2 \delta_\lambda^{-1}(l) \, d\mu_{\widehat{H}}(\eta) \right)^{1/2}.$$

Applying Cauchy-Schwarz inequality, (6.3) can be written as

$$\frac{n}{4\pi} C_\psi \, \|f\|^2_{L^2(H\times G)}$$

$$\leq \int_L \int_G \int_G \left(\int_H |h|^2 \left| \mathscr{F}_H \left(\mathcal{L}^{\psi}_{(l,h,x)} f \right)(I,y) \right|^2 d\mu_H(h) \right)^{1/2}$$

$$\times \left(\int_{\widehat{H}} |\eta|^2 \, |\mathscr{F}_H f(\eta,y)|^2 \left| \mathscr{F}_H \widetilde{\psi}(\eta \circ \lambda_l, y^{-1}x) \right|^2 \delta_\lambda^{-1}(l) \, d\mu_{\widehat{H}}(\eta) \right)^{1/2} \delta_\lambda(l) \, d\mu_L(l) \, d\mu_G(x) \, d\mu_G(y)$$

$$\leq \left(\int_L \int_G \int_G \int_H |h|^2 \left| \mathscr{F}_H \left(\mathcal{L}^{\psi}_{(l,h,x)} f \right)(I,y) \right|^2 \delta_\lambda(l) \, d\mu_L(l) \, d\mu_G(x) \, d\mu_G(y) \, d\mu_H(h) \right)^{1/2}$$

$$\times \left(\int_L \int_G \int_G \int_{\widehat{H}} |\eta|^2 \, |\mathscr{F}_H f(\eta,y)|^2 \left| \mathscr{F}_H \widetilde{\psi}(\eta \circ \lambda_l, y^{-1}x) \right|^2 d\mu_L(l) \, d\mu_G(x) \, d\mu_G(y) \, d\mu_{\widehat{H}}(\eta) \right)^{1/2}$$

$$= \left(\int_L \int_G \int_{\widehat{G}} \int_H |h|^2 \left| \mathscr{F}_G \mathscr{F}_H \left(\mathcal{L}^{\psi}_{(l,h,x)} f \right)(I,\pi) \right|^2 \delta_\lambda(l) \, d\mu_L(l) \, d\mu_G(x) \, d\mu_{\widehat{G}}(\pi) \, d\mu_H(h) \right)^{1/2}$$

$$\times \left(\int_G \int_{\widehat{H}} |\eta|^2 \, |\mathscr{F}_H f(\eta,y)|^2 \left\{ \int_L \int_G \left| \mathscr{F}_H \widetilde{\psi}(\eta \circ \lambda_l, x) \right|^2 d\mu_L(l) \, d\mu_G(x) \right\} d\mu_G(y) \, d\mu_{\widehat{H}}(\eta) \right)^{1/2}$$

$$= C_\psi^{1/2} \left(\int_{\mathcal{S}\times\widehat{G}} |h|^2 \, |\mathcal{MS}_\psi f(l,h,x,\pi)|^2 \, d\sigma(l,h,x,\pi) \right)^{1/2}$$

$$\times \left(\int_G \int_{\widehat{H}} |\eta|^2 \, |\mathscr{F}_H f(\eta, y)|^2 \, d\mu_G(y) \, d\mu_{\widehat{H}}(\eta) \right)^{1/2}.$$

Thus

$$\frac{n}{4\pi} C_\psi^{1/2} \|f\|_{L^2(H\times G)}^2 \leq \left(\int_{\mathcal{S}\times\widehat{G}} |h|^2 \, |\mathcal{MS}_\psi f(l,h,x,\pi)|^2 \, d\sigma(l,h,x,\pi) \right)^{1/2}$$
$$\times \left(\int_G \int_{\widehat{H}} |\eta|^2 \, |\mathscr{F}_H f(\eta, y)|^2 \, d\mu_G(y) \, d\mu_{\widehat{H}}(\eta) \right)^{1/2}.$$

As in Theorem 6.2, applying Hölder's inequality repeatedly, we obtain the required inequality. □

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