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Existence Results for Singular $p(x)$-Laplacian Equation

Résultats d'existence pour l'equation du p(x)-laplacien singulier

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ABSTRACT. This paper is concerned with the existence of solutions for the following class of singular fourth order elliptic equations

$$\begin{cases} \Delta\Big(|x|^{p(x)}|\Delta u|^{p(x)-2}\Delta u\Big) = a(x)u^{-\gamma(x)} + \lambda f(x,u), & \text{in } \Omega, \\ u = \Delta u = 0, \quad \text{on } \partial\Omega. \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $\gamma : \overline{\Omega} \to (0,1)$ be a continuous function, $f \in C^1(\overline{\Omega}\times\mathbb{R})$, $p : \overline{\Omega} \longrightarrow (1,\infty)$ and a is a function that is almost everywhere positive in Ω . Using variational techniques combined with the theory of the generalized Lebesgue-Sobolev spaces, we prove the existence at least one nontrivial weak solution.
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1. Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ and $\gamma : \overline{\Omega} \longrightarrow (0,1)$ be a continuous function. In this paper, we are concerned with the existence of solutions for the following biharmonic problem

$$(\mathbf{P}_\lambda)\begin{cases} \Delta\Big(|x|^{p(x)}|\Delta u|^{p(x)-2}\Delta u\Big) = a(x)u^{-\gamma(x)} + \lambda f(x,u), & \text{in } \Omega, \\ u = \Delta u = 0, \quad \text{on } \partial\Omega, \end{cases}$$

where $f \in C^1(\overline{\Omega}\times\mathbb{R})$, $p : \overline{\Omega} \to \mathbb{R}$ is a Lipschitz continuous function satisfying

$$1 < p^- := \inf_{x\in\Omega} p(x) \leq p^+ := \sup_{x\in\Omega} p(x) < N,$$

and a is a function that is almost everywhere positive in Ω provided that

$$a \in L^{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}}(\Omega), \quad \text{with } p^*(x) = \frac{Np(x)}{N-p(x)}.$$

The operator $\Delta_{p(x)}$ is defined as $\Delta_{p(x)}u = -div(|\nabla u|^{p(x)-2})\nabla u$ and called the $p(x)$-Laplace operator. This operator is a natural generalization of the p-Laplace operator $\Delta_p u = -div(|\nabla u|^{p-2})\nabla u$. However, the p(x)-Laplace operator operator possesses more complicated non-linearity than p-Laplace operator, for example, it is inhomogeneous, this fact implies some difficulties, for example, we can not use the Lagrange Multiplier.

In recent years, the study of problems involving biharmonic, p-biharmonic and $p(x)$-biharmonic operators has been widely approached. These problems are interesting in applications, for example in nonlinear elasticity theory and in modelling electrorheological fluids (See [2, 5, 7, 12, 20, 21]), and raise many difficult mathematical problems.

p-biharmonic and $p(x)$-biharmonic equations with Navier boundary conditions, which are investigated on function spaces with variable exponents, have been extensively studied by many authors see for example [1, 3, 6, 10, 11] and references therein. In particular, Li et al. [6] considered the fourth-order quasilinear elliptic equation

$$\begin{cases} -\Delta^2_{p(x)} + |u|^{p(x)-2}u(x) = \lambda f(x, u(x)), \text{ in } \Omega, \\ u = \Delta u = 0, \text{ on } \partial\Omega, \end{cases}$$

where λ is a nonnegative real number and f is a Carathéodory function, based on a three critical points theorem by Ricceri, they showed the existence of at least three solutions to the above problem. We also refere to the paper [3] in which based on a minimax method, the existence of at least one nontrivial solution for a class of singular $p(x)$-Kirchhoff equation coupled with Navier boundary conditions was established.

Recently, Mousaviankhatir and Alimohammady [18] in 2017, have considered the following type of boundary value problems

$$\begin{cases} \Delta\Big(|x|^{p(x)}|\Delta u(x)|^{p(x)-2}\Delta u(x)\Big) = \lambda|u(x)|^{q(x)-1}u(x), \quad \text{in } \Omega, \\ u = \Delta u = 0, \quad \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Using a variant of the mountain pass theorem, they proved that for λ small enough, problem (1.1) has a nontrivial solution.

Finally, In Kefi and Rădulescu [13], the authors have studied the following $p(x)$-biharmonic problem:

$$\begin{cases} -\Delta^2_{p(x)} + a(x)|u|^{p(x)-2}u = \lambda(V_1(x)|u|^{q(x)-2}u - V_2(x)|u|^{\alpha(x)-2}u), \text{ in } \Omega, \\ u = \Delta u = 0, \text{ on } \partial\Omega, \end{cases}$$

The main result in [13] establishes the existence of at least one nontrivial weak solution for all $\lambda > 0$.

Motivated by the above-mentioned papers, the problem ($\mathbf{P}_\lambda$) is a new variant of $p(x)$-biharmonic equations due to the singular term and the presence of the weight. The aim of this work is to show the existence of nontrivial solutions of problem ($\mathbf{P}_\lambda$).

Throughout this paper, we assume the following conditions:

(H1) There exists $\Omega_1 \subset\subset \Omega$, with $|\Omega_1| > 0$ and a nonnegative function h on Ω_1 with $h \in L^{s(x)}(\Omega)$ and

$$\lim_{|t|\to 0} \frac{f(x,t)}{h(x)|t|^{r(x)-1}} = 0 \text{ for } x \in \Omega \text{ uniformly.}$$

(H2) There exists a function $h_1 \in L^{s_1(x)}(\Omega)$ such that

$$\lim_{|t|\to \infty} \frac{f(x,t)}{h_1(x)|t|^{r(x)-1}} = 0 \quad \text{for } x \in \Omega \text{ uniformly,}$$

where s, s_1 and r are continuous functions on $\overline{\Omega}$ satisfying:

$$\begin{aligned} &1 < r(x) < p(x) < N < min(s(x), s_1(x)), \ \forall x \in \Omega, \\ &1 - \inf_{x\in\Omega} \gamma(x) < \inf_{x\in\Omega} r(x). \end{aligned} \tag{1.2}$$

(H3) There exists $A > 0$ such that

$$\int_\Omega F(x,t)dx > 0, \forall\, t > A,$$

where $F(x,t) = \int_0^t f(x,s)ds$.

Remark 1.1. *Using assumption* $(\mathbf{H1})$, f *leads to the so-called Euler identity*

$$uf(x,u) = rF(x,u),$$

$$F(x,u) \leq K|u|^r \quad \textit{for some constant } K, \tag{1.3}$$

and $f(x,0) = 0 = \frac{\partial f}{\partial t}(x,0)$ *for every* $t \in \mathbb{R}$.

The paper is organized as follows. In Section 2, we recall the definition of variable exponent Lebesgue spaces $L^{p(x)}(\Omega)$, as well as of Sobolev spaces $W^{k,p(x)}(\Omega)$. Moreover, some properties of these spaces will be also exhibited to be used later. In Section 3, we give the main result and their proofs.

2. Abstract setting

In this section, we recall some definitions and basic properties of the generalized Lebesgue Sobolev spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$, for more details concerning these spaces we refer the reader to [9], [15], [16] and references therein.

Put

$$C_+(\overline{\Omega}) := \{h : h \in C(\overline{\Omega}), h(x) > 1, \text{ for all } x \in \overline{\Omega}\},$$

and let $p \in C_+(\overline{\Omega})$ such that

$$1 < p^- := \inf_{x\in\Omega} p(x) \leq p(x) \leq p^+ := \sup_{x\in\Omega} p(x) < +\infty. \tag{2.1}$$

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined by:

$$L^{p(x)}(\Omega) = \{u : u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}dx < \infty\},$$

and equiped with the so-called Luxemburg norm defined as:

$$|u|_{p(x)} = \inf\left\{\mu > 0 : \int_\Omega |\frac{u(x)}{\mu}|^{p(x)}dx \leq 1\right\}.$$

Variable exponent Lebesgue spaces is like classical Lebesgue spaces in many respects: they are Banach spaces, they are reflexive if and only if

$$1 < p^- \leq p^+ < \infty,$$

moreover, the set of continuous functions is dense in these spaces if $p^+ < \infty$. Also, the inclusion between Lebesgue spaces is generalized naturally, that is if p_1 and p_2 are variable exponents so that $p_1(x) \leq p_2(x)$ a.e. $x \in \Omega$, then there exists a continuous embedding $L^{p_2}(x)(\Omega) \hookrightarrow L^{p_1}(x)(\Omega)$.

For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the Hölder inequality

$$|\int_{\Omega} uvdx| \leq (\frac{1}{p^-} + \frac{1}{(p')^-})|u|_{p(x)}|v|_{p'(x)}, \tag{2.2}$$

holds true (See [9] and [15]), where $L^{p'(x)}(\Omega)$ is the conjugate space of $L^{p(x)}(\Omega)$ and $p'(x)$ is such that $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

The modular on the space $L^{p(x)}(\Omega)$ is the map $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) := \int_{\Omega} |u|^{p(x)} dx,$$

and it satisfies the following proposition.

Proposition 2.1. *(See [15]) For all $u, v \in L^{p(x)}(\Omega)$, we have*

1. $|u|_{p(x)} < 1(resp. = 1, > 1) \Leftrightarrow \rho_{p(x)}(u) < 1(resp. = 1, > 1)$.
2. $\min(|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+}) \leq \rho_{p(x)}(u) \leq \max(|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+})$.
3. $\rho_{p(x)}(u - v) \to 0 \Leftrightarrow |u - v|_{p(x)} \to 0$.

Proposition 2.2. *(See [8]) Assume that p and q are two measurable functions such that*

$$p \in L^{\infty}(\Omega) \text{ and } 1 \leq p(x)q(x) \leq \infty \text{ for } a.e. x \in \Omega.$$

If $u \in L^{q(x)}(\Omega)$ with $u \neq 0$, then we have

$$min(|u|_{p(x)q(x)}^{p^+}, |u|_{p(x)q(x)}^{p^-}) \leq ||u|^{p(x)}|_{q(x)} \leq max(|u|_{p(x)q(x)}^{p^-}, |u|_{p(x)q(x)}^{p^+}).$$

In order to check the Sobolev embedding, the following Lemma plays an essential role.

Lemma 2.3. *Let s, s_1 and r be continuous functions satisfying condition (1.2), then, we have*

$$s'r < p^* \quad \text{and} \quad s_1'r < p^*,$$

where s' and s_1' are such that $\frac{1}{s} + \frac{1}{s'} = 1$ and $\frac{1}{s_1} + \frac{1}{s_1'} = 1$.

Proof. Assume that condition (1.2) holds true, then , $N < ps$. On the other hand, we have

$$s'p - p^* = \frac{ps(N-p) - Np(s-1)}{(s-1)(N-p)} = \frac{p(N-ps)}{(s-1)(N-p)} < 0.$$

Hence $s'r < s'p < p^*$.

Similar arguments schow that $s_1'r < s_1'p < p^*$. This completes the proof. □

For a multi-index $\alpha = (\alpha_1, \alpha_2, ...\alpha_N)$ and $p(x) \in C_+(\overline{\Omega})$, we denote by:

$$p_k^*(x) = \begin{cases} \dfrac{Np(x)}{N - kp(x)}, & \text{if } p(x) < \dfrac{N}{k}, \\ +\infty, & \text{if } p(x) \geq \frac{N}{k}, \end{cases}$$

and we define the Lebesgue-Sobolev space $W^{k,p(x)}(\Omega)$ as:

$$W^{k,p(x)}(\Omega) := \left\{ u \in L^{p(x)}(\Omega) : D^{\alpha}u \in L^{p(x)}(\Omega), \mid \alpha \mid \leq k \right\},$$

where $D^{\alpha} = \frac{\partial^{|\alpha_1|}}{\partial x_1{}^{\alpha_1} \ldots \partial x_1{}^{\alpha_N}}$ and $|\alpha| = \sum_{i=1}^{N} \alpha_i$.

Endowed with the norm

$$\|u\| = \sum_{|\alpha| \leq k} \|D^{\alpha}u\|_{p(x)},$$

the space $W^{k,p(x)}(\Omega)$ is a separable reflexive Banach space.

For $p, q \in C_{+}(\Omega)$ with $s(x) < p_k^*(x)$ for all $x \in \overline{\Omega}$, there is a continuous and compact embedding $W^{k,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^{\infty}(\Omega)$ in $W^{k,p(x)}(\Omega)$ with respect to the norm $\|u\|$. For more information about these spaces we refer the reader to [14] and [17].

In the sequel, we denote by $D_0^{2,p(x)}(\Omega)$ the closure of $C_c^2(\Omega)$ with respect to the norm

$$\|u\| = \big| |x||\Delta u| \big|_{p(x)}.$$

We recall (See [18]) that $\left(D_0^{2,p(x)}(\Omega), \|.\|\right)$ is a reflexive Banach space. Moreover, if $1 < q^- \leq q \leq q^+ < \frac{2Np^-}{2N+p^-} < p^-$, then, there is a continuous compact embedding from $D_0^{2,p(x)}(\Omega)$ into $L^{q(x)}(\Omega)$.

3. The main result and its proofs

Our main result in this paper is the following.

Theorem 3.1. *Under the assumptions* $(\mathbf{H1})$, $(\mathbf{H2})$ *and* $(\mathbf{H3})$, *if* $\lambda < 0$, *then, problem* $(\boldsymbol{P}_{\lambda})$ *has at least one nontrivial weak solution with negative energy.*

Definition 3.2. *We say that* $u \in D_0^{2,p(x)}(\Omega)$ *is weak solution of* $(\boldsymbol{P}_{\lambda})$ *if*

$$\int_{\Omega} |x|^{p(x)} |\Delta u|^{p(x)-2} \Delta u \Delta v dx - \int_{\Omega} a(x)|u|^{-\gamma(x)} uv dx - \lambda \int_{\Omega} f(x,u) v dx = 0,$$

for any $v \in D_0^{2,p(x)}(\Omega)$.

The energy functional corresponding to problem $(\mathbf{P}_{\lambda})$ is defined on $u \in D_0^{2,p(x)}(\Omega)$ as:

$$J_{\lambda}(u) = I(u) - \Phi_{\lambda}(u),$$

where

$$I(u) = \int_{\Omega} \frac{|x|^{p(x)}}{p(x)} |\Delta u|^{p(x)} dx$$

and

$$\Phi_{\lambda}(u) = \int_{\Omega} \frac{a(x)}{1-\gamma(x)} |u|^{1-\gamma(x)} dx + \lambda \int_{\Omega} F(x, u(x)) dx.$$

Remarks 3.3. *If p is a Lipschitz continous function, then, under assymptions* $(\mathbf{H1})$*,* $(\mathbf{H2})$ *and* $(\mathbf{H3})$*, we have:*

1. *Since* $(1-\gamma(x))p(x) < p(x) < p^*(x), \forall x \in \overline{\Omega}$*, then, the embedding*
   $$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{(1-\gamma(x))p(x)}(\Omega),$$
   is compact and continuous.
2. *From Lemma 2.3, the embeddings*
   $$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{s'(x)r(x)}(\Omega) \text{ and } W_0^{1,p(x)}(\Omega) \hookrightarrow L^{s_1'(x)r(x)}(\Omega),$$
   are compact and continuous.
3. *The embedding* $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p^*(x)}(\Omega)$ *is continuous.*

It is important to mention that, using remark 3.3, J_λ is well defined but not Fréchet differentiable due to the singular term.

Proof of Theorem 3.1 To prove theorem 3.1, we need to prove several lemmas. first, we notes that from remark 3.3, there exist positive constants C and C_1 such that

$$|u|_{(1-\gamma(x))p(x)} \leq C\|u\| \text{ and } |u|_{s'(x)r(x)} \leq C_1\|u\|, \quad \forall\, u \in W_0^{1,p(x)}(\Omega). \tag{3.1}$$

Lemma 3.4. J_λ *is weakly lower semi-continuous.*

Proof. The proof is divided into three steps.

Step 1: The functional $I : D_0^{2,p(x)}(\Omega) \to \mathbb{R}$ is convex. Indeed, since for any $\theta > 1$, the function $t \longmapsto t^\theta$, is convex on $[0,\infty)$, then, for each $x \in \Omega$ we have

$$\left|\frac{\xi+\psi}{2}\right|^{p(x)} \leq \left(\frac{|\xi|+|\psi|}{2}\right)^{p(x)} \leq \frac{1}{2}|\xi|^{p(x)} + \frac{1}{2}|\psi|^{p(x)} \;\; \forall\, \xi,\psi \in \mathbb{R}^N.$$

So, for all $u, v \in D_0^{2,p(x)}(\Omega)$, one has:

$$\left|\frac{\Delta u+\Delta v}{2}\right|^{p(x)} \leq \left(\frac{|\Delta u|+|\Delta v|}{2}\right)^{p(x)} \leq \frac{1}{2}|\Delta u|^{p(x)} + \frac{1}{2}|\Delta v|^{p(x)}. \tag{3.2}$$

Multiplying (3.2) by $\frac{|x|^{p(x)}}{p(x)}$ and integrating over Ω, we get:

$$I\left(\frac{u+v}{2}\right) \leq \frac{1}{2}I(u) + \frac{1}{2}I(v) \;\; \forall\, u,v \in D_0^{2,p(x)}(\Omega).$$

That is I is convex.

Step 2: I is weakly lower semi continuous on $D_0^{2,p(x)}(\Omega)$.

From Step 1 and Corollary 3.8 in [4] it is enough to show that I is strongly lower semi continuous on $D_0^{2,p(x)}(\Omega)$. On the other hand, let $\epsilon > 0$, $u \in D_0^{2,p(x)}(\Omega)$ and $v \in D_0^{2,p(x)}(\Omega)$ such that

$$|u-v|_{p(x)} < \frac{\epsilon}{\left||\Delta u|^{p(x)-1}\right|_{\frac{p(x)}{p(x)-1}}}.$$

Then, since the functional I is convex and using inequality (2.2), we obtain

$$\begin{aligned} I(v) &\geq I(u) + \langle I'(u), v-u\rangle, \\ &\geq I(u) - \int\limits_{\Omega} |\Delta u|^{p(x)-1} |\Delta(v-u)| dx, \\ &\geq I(u) - c_1 \left| |\Delta u|^{p(x)-1} \right|_{\frac{p(x)}{p(x)-1}} |\Delta(u-v)|_{p(x)}, \\ &\geq I(u) - c_2 \|u-v\|_{p(x)}, \\ &\geq I(u) - \epsilon, \end{aligned}$$

for some positive constants c_1 and c_2.

It follows that I is strongly lower semi continuous and convex, so, also from corollary 3.8 in [4] we deduce that the functional I is weakly lower semi continuous.

Step 3: J_λ is weakly lower semi-continuous.

Let $\{u_n\}$ be a sequence which is weakly converges to u in $D_0^{2,p(x)}(\Omega)$. Then, from step 2, we have

$$I(u) \leq \liminf_{n\to+\infty} I(u_n). \tag{3.3}$$

On the other hand, by Vital's theorem (see [15] pp:113), we can claim that

$$\lim_{n\to\infty} \int\limits_{\Omega} a(x)|u_n|^{1-\gamma(x)} dx = \int\limits_{\Omega} a(x)|u|^{1-\gamma(x)} dx. \tag{3.4}$$

Indeed, we only need to prove that

$$\left\{ \int\limits_{\Omega} a(x)|u_n|^{1-\gamma(x)} dx, n \in \mathbb{N} \right\},$$

is equi-absolutely-continuous, which means that (see [19]) for any $\varepsilon > 0$ there exists $\delta > 0$ such that for each $\Omega_1 \subset \Omega$. If $mes(\Omega_1) < \delta$, then $| \int_\Omega f_n(x)dx | < \varepsilon$.

We begin by mentioned that if $\{u_n\}$ is bounded, then by the Sobolev embedding theorem, there exits $C > 0$, such that $|u_n|_{p^*(x)} \leq C$.

Now, let $\varepsilon > 0$, then, using Proposition 2.1 and the absolutely-continuity of $\int\limits_{\Omega_2} |a(x)|^{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}} dx$, there exist two positive constants ζ and ξ such that

$$\begin{aligned} |a|^{\zeta}_{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}} &\leq \int\limits_{\Omega_2} |a(x)|^{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}} dx \\ &\leq \varepsilon^{\zeta} \text{ for every } \Omega_2 \subset \Omega \text{ with } |\Omega_2| < \xi. \end{aligned}$$

Consequently, by the Hölder inequality and Propositin 2.2 we have

$$\begin{aligned} \int\limits_{\Omega_2} |a(x)| \mid u_n \mid^{1-\gamma(x)} dx &\leq |a|_{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}} \left| |u_n|^{1-\gamma(x)} \right|_{p^*(x)} \\ &\leq |a|_{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}} |u_n|_{(1-\gamma(x))p^*(x)}. \end{aligned}$$

Since $(1-\gamma(x))p^*(x) < p^*(x)$, then there exists C_2 such that

$$|u_n|_{(1-\gamma(x))p^*(x)} \leq C_2|u_n|_{p^*(x)},$$

and so,

$$\begin{aligned}\int_{\Omega_2} |a(x)| \mid u_n \mid^{1-\gamma(x)} dx &\leq |a|_{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}} ||u_n|^{1-\gamma(x)}|_{p^*(x)} \\ &\leq |a|_{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}} |u_n|_{(1-\gamma(x))p^*(x)} \\ &< C_2\varepsilon^{\zeta} \, |u_n|_{p^*(x)}.\end{aligned}$$

Finally, the fact that $|u_n|_{p^*(x)}$ is bounded implies that claim (3.4) is valid.

In what follows, we remark, using assumptions **(H1)** and **(H2)**, that for all $\varepsilon > 0$, there exists C_ε such that

$$|F(x,u(x))| \leq \varepsilon \frac{C}{r^-}|h(x)||u|^{r(x)} + C_\varepsilon \frac{C_1}{r^-}|h_1(x)||u|^{r(x)}. \tag{3.5}$$

Then, by the Hölder inequality, we get

$$\int_\Omega |F(x,u(x)| \leq \varepsilon \frac{C}{r^-}|h|_{s(x)}||u|^{r(x)}|_{s'(x)} + C_\varepsilon \frac{C_1}{r^-}|h_1|_{s_1(x)}||u|^{r(x)}|_{{s_1}'(x)}.$$

Besides, if $u_n \rightharpoonup u$ in $D_0^{2,p(x)}(\Omega)$, then we have strong convergence in $L^{s'(x)r(x)}(\Omega)$ and $L^{{s_1}'(x)r(x)}(\Omega)$. So the Lebesgue dominated convergence theorem and proposition 2.2 makes it possible to write that the function

$$u \mapsto \lambda \int_\Omega F(x,u(x))dx,$$

is a weakly continuous. Which yields to

$$\lim_{n\to+\infty} \Phi_\lambda(u_n) = \Phi_\lambda(u). \tag{3.6}$$

Combining (3.3), (3.4) and (3.6), we deduce that J_λ is weakly lower semi-continuous. □

Lemma 3.5. *J_λ is bounded from below and coercive.*

Proof. From assumptions ($\mathbf{H1}$)-($\mathbf{H3}$), Remark 3.3, Proposition 2.1 and Proposition 2.2, for any $u \in D_0^{2,p(x)}(\Omega)$ with $\|u\| > max(1,A)$, we get

$$\begin{aligned} J_\lambda(u) &\geq \frac{1}{p^+}\int_\Omega |x|^{p(x)}|\Delta u|^{p(x)}dx - C_1|a|_{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}}||u||^{1-\gamma^-} - \lambda\int_\Omega F(x,u)dx, \\ &\geq \frac{1}{p^+}\|u\|^{p^-} - C_1|a|_{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}}||u||^{1-\gamma^-}.\end{aligned}$$

Since $1-\gamma^- < p^-$, we infer that $J_\lambda(u) \to \infty$ as $\|u\| \to \infty$, in other words J_λ is is bounded from below and coercive. □

Lemma 3.6. *There exist $\varphi \in D_0^{2,p(x)}(\Omega)$ such that $\varphi \neq 0$ and $J_\lambda(\varphi) < 0$.*

Proof. **Proof.** Let $\varphi \in C_0^\infty(\Omega)$ and Ω' be such that $\Omega' \subset supp(\varphi) \subset \Omega_1 \subset \Omega$, where Ω_1 is given by hypothesis $(\mathbf{H1})$. Assume that $0 \leq \varphi \leq 1$ in Ω_1, and $\varphi = 1$ in Ω', then, we have

$$\begin{aligned} J_\lambda(t\varphi) &= \int_\Omega \frac{|x|^{p(x)} t^{p(x)}}{p(x)} |\Delta\varphi|^{p(x)} dx - \int_\Omega \frac{t^{1-\gamma(x)}}{1-\gamma(x)} a(x)|\varphi|^{1-\gamma(x)} dx - \lambda \int_\Omega F(x, t\varphi) dx, \\ &\leq \frac{t^{p^-}}{p^-} \int_\Omega |x|^{p(x)} |\Delta\varphi|^{p(x)} dx - \frac{t^{1-\gamma^-}}{1-\gamma^-} \int_\Omega a(x)|\varphi|^{1-\gamma(x)} dx - \lambda C_1 t^{r^-} \int_{\Omega_1} h(x)|\varphi|^{r(x)} dx, \\ &\leq t^{min(p^-, r^-)} \left[\frac{1}{p^-} \max(\|\varphi\|^{p^-}, \|\varphi\|^{p^+}) - \lambda C_1 \int_{\Omega_1} h(x)|\varphi|^{r(x)} dx \right] \\ &- \frac{t^{1-\gamma^-}}{1-\gamma^-} \int_\Omega a(x)|\varphi|^{1-\gamma(x)} dx. \end{aligned}$$

So,

$$J_\lambda(t\varphi) < 0 \quad \text{for } t < \psi^{\frac{1}{min(p^-, r^-) - (1-\gamma^-)}}, \tag{3.7}$$

with

$$0 < \psi < min \left\{ 1, \frac{\frac{1}{1-\gamma^-} \int_\Omega a(x)|\varphi|^{1-\gamma(x)} dx}{\frac{C}{p^-} \max(\|u\|^{p^-}, \|u\|^{p^+}) - \lambda C_1 \int_{\Omega_1} h(x)|\varphi|^{r(x)} dx} \right\}.$$

Finally, we point out that

$$\frac{C}{(p^-)^\delta} \max(\|\varphi\|^{p^-}, \|\varphi\|^{p^+}) - \lambda C_1 \int_{\Omega_1} h(x)|\varphi|^{r(x)} dx > 0.$$

Indeed, if not if

$$\frac{C}{(p^-)^\delta} \max(\|\varphi\|^{p^-}, \|\varphi\|^{p^+}) - \lambda C_1 \int_{\Omega_1} h(x)|\varphi|^{r(x)} dx = 0$$

then, $\|\varphi\| = 0$ and so, $\varphi = 0$, which is a contradiction. □

Now, we are in a position to prove Theorem 3.1. Indeed, from Lemma 3.5, we can define

$$m_\lambda = \inf_{v \in D_0^{2,p(x)}(\Omega)} J_\lambda(v).$$

Let $\{u_n\}$ be a minimizing sequence, that is $J_\lambda(u_n) \to m_\lambda$ as n tends to infinity. we claim that u_n is bounded. If not up to a subsequence, we may assume that $\|u_n\| \to \infty$, this yields to $J_\lambda(u_n) \to \infty$ which is a contradiction with the fact that $\{u_n\}$ be a minimizing sequence. Since $D_0^{2,p(x)}(\Omega)$ is a reflexive Banach space, then there exists a subsequence still denoted by u_n and $u_\lambda \in D_0^{2,p(x)}(\Omega)$ such that $u_n \rightharpoonup u_\lambda$ weakly in $D_0^{2,p(x)}(\Omega)$. From Lemma 3.4, we see that

$$J_\lambda(u_\lambda) \leq \lim \inf_{n\to\infty} J_\lambda(u_n) = m_\lambda.$$

On the other hand, from the definition of m_λ, we have $m_\lambda \leq J_\lambda(u_\lambda)$. Therefore, u_λ is a global minimum for J_λ which is a weak solution for problem $(\mathbf{P}_\lambda)$. Finally, from (3.7) it follows that $u_\lambda \neq 0$. This ends the proof of Theorem 3.1.

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