A Result on Bruck Conjecture Related to Shift Polynomials

Un résultat sur la Conjecture de Bruck sur les Polynômes Shift

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ABSTRACT. This paper mainly concerns about establishing the Bruck conjecture for differential-difference polynomial generated by an entire function. The polynomial considered is of finite order and involves the entire function $f(z)$ and its shift $f(z+c)$ where $c \in \mathbb{C}$. Suitable examples are given to prove the sharpness of sharing exceptional values of Borel and Nevanlinna.
2020 Mathematics Subject Classification. 30D35
KEYWORDS. Entire Function, Exceptional Values, Differential-Difference Polynomial, Value Sharing

1. Introduction and Definitions

Throughout this article, the phrase "entire function" means that the function is analytic everywhere in $\mathbb{C}$. The fundamentals of Nevanlinna theory and standard notations can be read in [5, 6, 11]. The notation $E = \{x : x \in \mathbb{R}^+\}$ set of positive real numbers of finite linear measure. Let $\mathcal{F} = \{f : f\ is\ non-constant\ meromorphic\ function\ in\ \mathbb{C}\}$. For $f, g \in \mathcal{F}$ and $b \in \mathbb{C} \cup \{\infty\}$, if $f - b$ and $g - b$ have the identical zeros including multiplicities then f and g share b CM (counting multiplicities), if the multiplicities are ignored, then f and g share b IM and if $1/f$ and $1/g$ share 0 CM then, f and g share ∞ CM [13]. $N(r, \frac{1}{f-b})$ denotes the counting function of f whose b-points are counted according to multiplicity and the corresponding reduced counting function when multiplicity is ignored is denoted by $\overline{N}(r, \frac{1}{f-b})$. For $\phi(z) \in \mathcal{F}$, if $T(r, \phi) = S(r, f)$ then ϕ is called a "small function" of f where $T(r, \phi)$ is the Nevanlinna characteristic function and $S(r, f) = o(T(r, f)),\ as\ r \to \infty,\ r \notin E$. Some of the definitions which are essential for this paper are given below:

Definition 1.1. *[9] Let $f \in \mathcal{F}$. The order $\sigma(f)$ and exponent of convergence of zeros $\lambda(f)$ is defined as*

$$\sigma(f) = \limsup_{r\to\infty} \frac{log^+ T(r, f)}{log\, r} \quad \lambda(f) = \limsup_{r\to\infty} \frac{log^+ N(r, \frac{1}{f-b})}{log\, r}.$$

Definition 1.2. *[11] Let $f \in \mathcal{F}$. The deficiency of the value point b with respect to $f(z)$ is defined by*

$$\delta(b, f) = 1 - \limsup_{r\to\infty} \frac{N(r, \frac{1}{f-b})}{T(r, f)}.$$

It is proved that $\delta(b, f) \in [0, 1]$. If $\delta(b, f) > 0$ then b is called a Nevanlinna exceptional value of $f(z)$.

Definition 1.3. *[9] Let $f \in \mathcal{F}$. The value point b is called a Borel exceptional value of $f(z)$ if*

$$\limsup_{r\to\infty} \frac{log^+ N(r, \frac{1}{f-b})}{log\, r} < \sigma(f).$$

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R. Bruck in his classical work [1] proposed a conjecture that, for $f \in \mathcal{F}$ of hyper order $\rho_2(f) \notin \mathbb{Z}^+,\ \rho_2(f) \neq \infty$, if f and f' share a finite value a CM, then $\dfrac{f'-a}{f-a} = c$, where c is a non-zero constant. He proved the conjecture for $a = 0$ and for $a \neq 0$, imposed a condition that $N(r, 0; f') = S(r, f)$. From then on the conjecture has been extended in multiple dimensions. Other than the original condition of hyper order, the concept of finite order has also been implemented which can be seen in the works [2, 3, 4]. The sharing values in Bruck's conjecture has also been revised for instance, in [7] to sharing a polynomial, in [8, 10, 12] to function sharing meromorphic and entire small functions. On the other hand the first derivative in the conjecture has been extended to higher derivatives. Also different analogue shift functions are considered to establish the conjecture.

In this paper we investigate the sharing of exceptional values and prove the conjecture by considering the following differential polynomial involving shift function. Define

$$\psi(f) := \sum_{j_1\in J_1} A_{j_1}(z)f^{(k_{j_1})}(z) + \sum_{j_2\in J_2} B_{j_2}(z)f^{(k_{j_2})}(z+b_{j_2}) + \sum_{j_3\in J_3} C_{j_3}(z)f(z+c_{j_3}), \tag{1.1}$$

where $A_{j_1}(z),\ B_{j_2}(z),\ C_{j_3}(z)$ are entire small functions of $f(z)$, $\{k_{j_1},\ k_{j_2}\} > 0 \in \mathbb{Z}^+$, b_{j_2}, c_{j_3} are complex constants and $j_m \in J_m,\ m = \{1,2,3\}$ are finite indexed sets. Define

$$\begin{aligned} w_1(f) &:= \sum_{i_1\in I_1} D_{i_1}(z)f^{(l_{i_1})}(z) + \sum_{i_2\in I_2} E_{i_2}(z)f^{(l_{i_2})}(z+d_{i_2}) + \sum_{i_3\in I_3} F_{i_3}(z)f(z+e_{i_3}),\\ w_2(f) &:= \sum_{j_1\in J_1} A_{j_1}(z)f^{(k_{j_1})}(z) + \sum_{j_2\in J_2} B_{j_2}(z)f^{(k_{j_2})}(z+b_{j_2}) + \sum_{j_3\in J_3} C_{j_3}(z)f(z+c_{j_3}), \end{aligned} \tag{1.2}$$

where $A_{j_1}(z),\ B_{j_2}(z),\ C_{j_3}(z),\ D_{i_1}(z),\ E_{i_2}(z),\ F_{i_3}(z)$ are entire small functions of $f(z)$, $\{k_{j_1},\ k_{j_2},\ l_{i_1},\ l_{i_2}\} > 0 \in \mathbb{Z}^+$, $b_{j_2},\ c_{j_3},\ d_{i_2},\ e_{i_3}$ are complex constants and $i_n \in I_n$, $j_m \in J_m,\ n,\ m = \{1,2,3\}$ are finite indexed sets. The main results of the paper is as follows.

Theorem 1.1. *Let $f(z)$ be a transcendental entire function of finite order and $\psi(f)$ be as defined in* (1.1) *such that $\sum_{j_3\in J_3} C_{j_3} \equiv 0$. Suppose that $\psi(f)$ and $f(z)$ share the finite value a CM and $f(z)$ has exceptional value $\alpha(\neq a)$,*

(i) If $a \neq 0$ and α is a Nevanlinna exceptional value of $f(z)$ then

$$\frac{\psi(f)-a}{f-a} = \tau\ (\neq 0).$$

(ii) If α is a Borel exceptional value of $f(z)$ then

$$\frac{\psi(f)-a}{f-a} = \frac{a}{a-\alpha}.$$

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Theorem 1.2. *Let $f(z)$ be a transcendental entire function of finite order and $w_1(f),\ w_2(f)$ be two differential-difference polynomials with entire small function coefficients as defined in* (1.2)*. Suppose $w_1(f)$ and $w_2(f)$ share the finite value $a\ (\neq 0)$ CM, $\sum_{i_3\in I_3} F_{i_3}(z) \equiv 0$ and $f(z)$ has a finite exceptional value $\alpha\ (\neq a)$,* *(i) If α is a Nevanlinna exceptional value of $f(z)$ then*

$$\frac{w_1(f)-a}{w_2(f)-a}=\tau\ (\neq 0).$$

(ii) If α is a Borel exceptional value of $f(z)$, $\sum_{j_3\in J_3} C_{j_3}-1\equiv 0$ and $\alpha\sum_{j_3\in J_3} C_{j_3}-a\neq 0$ then

$$\frac{w_1(f)-a}{w_2(f)-a}=\frac{a}{a-\alpha}.$$

Example 1.3. *Let $f(z)=e^z+1$ and $\psi(f)=3zf''(z)+zf'(z)+5zf(z)-3zf'(z+log\ 4)-2zf(z+log\ 3)+4zf(z+log\ 2)+(z+1)f(z+log\ 1)$. Clearly 0 is a Borel exceptional value of $f(z)$. Now f and ψ share the value 1 CM thus $\frac{\psi-1}{f-1}=1$.*

Example 1.4. *Let $f(z)=e^z$ and $\psi(f)=(6z+1)f''(z)+(3z-2)f'(z)-(1-2z)f(z)-3zf'(z+log\ 4)+(z+1)f(z+log\ 2)+(z-2)f(z+log\ 3)$. Clearly 1 is a Borel exceptional value of $f(z)$. Now f and ψ share the value 2 CM thus $\frac{\psi-2}{f-2}=1$.*

Example 1.5. *Let $f(z)=e^z+1$. Clearly 1 is a Borel exceptional value of $f(z)$. Now let $w_1(f)=(8z+2)f''(z)-3zf'(z+log\ 2)+2zf(z+log\ 3)-2zf(z+log\ 4)$ and $w_2(f)=(18z-4)f''(z)-5zf'(z+log\ 3)+(3z+1)f(z+log\ 5)-3zf(z+log\ 6)$, so w_1 and w_2 share the value 2 CM hence $\frac{w_2-2}{w_1-2}=2$.*

Example 1.6. *Let $f(z)=e^z$. Clearly, $f(z)$ has 0 as Borel exceptional value. Now let $w_1(f)=(4z^2+1)f''(z)+(1-2z)f'(z)+z^2f(z)-z^3f'(z+log\ 2)-zf(z+log\ 3)-z^2f(z+log\ 4)$ and $w_2(f)=(z^2+z)f''(z)+(1-2z)f'(z)+z^2f(z)+(z^2-z-1)f'(z+log\ 5)+(z-z^2)f(z+log\ 6)+(6-z^2)f(z+log\ 1)$, so w_1 and w_2 share the value 1 CM, hence $\frac{w_2-1}{w_1-1}=1$.*

In the proof section without specifically mentioning we use the well known lemma of logarithmic derivative and its difference analogue and the readers are referred to lemma (2.1) and (2.2) of [4], lemma (1.4) and theorem (1.22) of [11] for the same.

2. Proof of Theorems

Theorem 1.1 We first prove the case(i). Since $f(z)$ is a transcendental entire function with finite order, using lemma 2.2 of [4], we see that $\psi(f)$is also of finite order. As $\psi(f)$ and $f(z)$ share a CM, we can write,

$$\frac{\psi(f)-a}{f-a}=e^{\phi(z)}, \tag{2.1}$$

where $\phi(z)$ is a polynomial. We claim that $\phi(z)$ is a constant. Suppose if $deg(\phi(z))\geq 1$, differentiating (2.1), we get

$$\phi'(z)=\frac{\psi'(z)}{\psi(z)-a}-\frac{f'(z)}{f(z)-a}. \tag{2.2}$$

Clearly

$$m(r,\phi'(z)) \le m\left(r,\frac{\psi'}{\psi-a}\right)+m\left(r,\frac{f'}{f-a}\right) \le S(r,f). \tag{2.3}$$

Now

$$\begin{aligned}\phi' &= (f-\alpha)\left[\frac{\psi'}{(\psi-a)(f-\alpha)}-\frac{f'}{(f-a)(f-\alpha)}\right]\\ &= (f-\alpha)\left[\frac{F_1}{a}-\frac{F_2}{\alpha-a}\right],\end{aligned} \tag{2.4}$$

$$where \quad F_1=\frac{\psi'}{\psi-a}\frac{\psi}{f-\alpha}-\frac{\psi'}{f-\alpha} \quad and \quad F_2=\frac{f'}{f-\alpha}-\frac{f'}{f-a}.$$

In other words, (2.4) can be written as

$$\frac{1}{f-\alpha}=\frac{1}{\phi'}\left[\frac{F_1}{a}-\frac{F_2}{\alpha-a}\right]. \tag{2.5}$$

From (2.4),

$$\begin{aligned}m(r,F_1) &\le m\left(r,\frac{\psi'}{\psi-a}\right)+m\left(r,\frac{\psi}{f-\alpha}\right)+m\left(r,\frac{\psi'}{f-\alpha}\right)+S(r,f),\\ &= m\left(r,\frac{\psi}{f-\alpha}\right)+S(r,f).\end{aligned}$$

Substituting ψ from (1.1) we get

$$\begin{aligned}m(r,F_1) &\le m\left(r,\frac{\sum\limits_{j_1\in J_1}A_{j_1}(z)f^{(k_{j_1})}(z)}{f-\alpha}\right)+m\left(r,\frac{\sum\limits_{j_2\in J_2}B_{j_2}(z)f^{(k_{j_2})}(z+b_{j_2})}{f-\alpha}\right)\\ &\quad +m\left(r,\frac{\sum\limits_{j_3\in J_3}C_{j_3}(z)f(z+c_{j_3})-\alpha+\alpha}{f-\alpha}\right)+S(r,f),\\ &\le \sum_{j_1\in J_1}\left[m\left(r,A_{j_1}(z)\right)+m\left(r,\frac{f^{(k_{j_1})}(z)}{f-\alpha}\right)\right]+\sum_{j_2\in J_2}\left[m\left(r,B_{j_2}(z)\right)+m\left(r,\frac{f^{(k_{j_2})}(z+b_{j_2})}{f-\alpha}\right)\right]\\ &\quad +\sum_{j_3\in J_3}\left[m\left(r,C_{j_3}(z)\right)+m\left(r,\frac{f(z+c_{j_3})}{f-\alpha}\right)\right]+S(r,f).\end{aligned}$$

We know that $A_{j_1}(z), B_{j_2}(z)$ and $C_{j_3}(z)$ are small functions of $f(z)$ and using the condition $\sum\limits_{j_3\in J_3}C_{j_3}(z)=0$ we get,

$$m(r,F_1)\le S(r,f). \tag{2.6}$$

From (2.4),

$$m(r,F_2)\le m\left(r,\frac{f'}{f-\alpha}\right)+m\left(r,\frac{f'}{f-a}\right)+S(r,f)\le S(r,f). \tag{2.7}$$

From (2.5),

$$m\left(r,\frac{1}{f-\alpha}\right)\leq m\left(r,\frac{1}{\phi'}\right)+m\left(r,\frac{F_1}{\alpha}\right)+m\left(r,\frac{F_2}{\alpha-a}\right).$$

As $\phi'\not\equiv 0$, using (2.6) and (2.7), we conclude that

$$m\left(r,\frac{1}{f-\alpha}\right)=S(r,f). \tag{2.8}$$

Using the definition of $\delta(\alpha,f)$, from (2.8) we see that $\delta(\alpha,f)=0$ which implies that α is not a Nevanlinna exceptional value which is a contradiction hence $\phi(z)$ is a constant.

Now we prove case(ii) Suppose $a=0$, since $\psi(f)$ and $f(z)$ share 0 CM, we have

$$\frac{\psi(f)}{f(z)}=e^{h(z)}, \tag{2.9}$$

where $h(z)$ is a polynomial. As $\alpha\ (\neq 0)$ is a Borel exceptional value, using Hadamard's factorization theorem, $f(z)$ can be written as

$$f(z)=A(z)\ e^{p(z)}+\alpha, \tag{2.10}$$

where $A(z)\neq 0$ is an entire function, $p(z)$ is a polynomial with $deg(p(z))\geq 1$ and $A(z),p(z)$ satisfy

$$\lambda(A)=\sigma(A)=\lambda(f-\alpha)<\sigma(f)=deg(p(z)). \tag{2.11}$$

Let $k_{j_1}=1$ in (1.1), using (2.10), we get

$$f^{(1)}(z)=e^{p(z)}\ T_1(z),\quad T_1(z)=A(z)\ p'(z)+A'(z).$$

If $k_{j_1}=2$,

$$f^{(2)}(z)=e^{p(z)}\ T_2(z),\quad T_2(z)=T_1(z)\ p'(z)+T_1'(z).$$

Similarly $\forall\ k_{j_1},j_1\in J_1$,

$$f^{(k_{j_1})}(z)=e^{p(z)}\ T_{j_1}(z),\quad T_{j_1}(z)=T_{j_1-1}(z)\ p'(z)+T'_{j_1-1}(z). \tag{2.12}$$

On the same lines $\forall\ k_{j_2},j_2\in J_2$ and $\forall\ k_{j_3},j_3\in J_3$ one can write

$$\begin{aligned} f^{(k_{j_2})}(z+b_{j_2})&=e^{p(z+b_{j_2})}\ T_{j_2}(z+b_{j_2}),\\ T_{j_2}(z+b_{j_2})&=T_{j_1-1}(z+b_{j_2})\ p'(z+b_{j_2})+T'_{j_1-1}(z+b_{j_2}),\end{aligned} \tag{2.13}$$

where $T_{j_1}(z),\ T_{j_2}(z+b_{j_2})$ are differential polynomials formed by $A(z),\ p(z)$ and its derivatives. From (2.11), we see that $T_{j_1}(z),\ T_{j_2}(z+b_{j_2})$ are small functions of $f(z)$. In other words

$$\{\sigma(T_{j_1}(z)),\ \sigma(T_{j_2}(z+b_{j_2}))\}<\sigma(f)=deg(P(z)). \tag{2.14}$$

Substituting (2.10), (2.12), (2.13) in (2.9), we get

$$\frac{\begin{array}{l}\sum\limits_{j_1\in J_1}A_{j_1}(z)T_{j_1}(z)e^{p(z)}+\sum\limits_{j_2\in J_2}B_{j_2}(z)T_{j_2}(z+b_{j_2})e^{p(z+b_{j_2})}\\ \qquad\qquad+\sum\limits_{j_3\in J_3}C_{j_3}(z)[A(z+b_{j_3})e^{p(z+c_{j_3})}+\alpha]\end{array}}{A(z)\ e^{p(z)}+\alpha}=e^{h(z)}.$$

The above equation can also be written as

$$\frac{\sum_{j_1\in J_1} A_{j_1}(z)T_{j_1}(z) + \sum_{j_2\in J_2} B_{j_2}(z)T_{j_2}(z+b_{j_2})e^{p(z+b_{j_2})-p(z)} + \sum_{j_3\in J_3} C_{j_3}(z)\left[A(z+c_{j_3})e^{p(z+c_{j_3})-p(z)} + \alpha\, e^{-p(z)}\right]}{A(z)+\alpha\, e^{-p(z)}} = e^{h(z)}. \tag{2.15}$$

In equation (2.15), we see that $deg\ (p(z+b_{j_2})-p(z)) = deg\ (p(z+c_{j_3})-p(z)) < \sigma(f)$, $A_{j_1}(z), B_{j_2}(z)$ and $C_{j_3}(z)$ are small functions of $f(z)$ hence the order of the numerator is less than $\sigma(f)$ and the order of denominator is $\sigma(f) = deg(p(z))$. Noting that $\alpha \neq 0$ is a Borel exceptional value of $f(z)$, we get $\overline{N}\left(r, \frac{1}{f-\alpha}\right) = S(r,f)$. In addition $f(z)$ is an entire function hence from second fundamental theorem, we have

$$T(r,f) \leq \overline{N}(r,f) + N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f-\alpha}\right) + S(r,f) \leq N\left(r,\frac{1}{f}\right). \tag{2.16}$$

From (2.16), we see that $\sigma(f) \leq \lambda(f)$. By the fact that $\lambda(f) \leq \sigma(f)$, we get $\sigma(f) = \lambda(f)$ which is a contradiction to (2.15) hence $a \neq 0$. By the conclusion of case(i) $\phi(z)$ is a constant and for $a \neq 0$, (2.1) can be written as

$$\frac{\phi(f)-a}{f(z)-a} = \tau\ (\neq 0). \tag{2.17}$$

Since α is a Borel exceptional value of $f(z)$, we arrive at (2.10) and the condition of (2.11) follows. Substituting (2.10), (2.12), (2.13) and (2.14) in (2.17) we get

$$\frac{\sum_{j_1\in J_1} A_{j_1}(z)T_{j_1}(z) + \sum_{j_2\in J_2} B_{j_2}(z)T_{j_2}(z+b_{j_2})e^{p(z+b_{j_2})-p(z)} + \sum_{j_3\in J_3} C_{j_3}(z)\left[A(z+c_{j_3})e^{p(z+c_{j_3})-p(z)} - \alpha\, e^{-p(z)}\right] - a\, e^{-p(z)}}{A(z)+\alpha\, e^{-p(z)} - a\, e^{-p(z)}} = \tau.$$

The above equation can be rearranged as

$$\sum_{j_2\in J_2} B_{j_2}(z)T_{j_2}(z+b_{j_2})e^{p(z+b_{j_2})-p(z)} + \sum_{j_3\in J_3} C_{j_3}(z)A(z+c_{j_3})e^{p(z+c_{j_3})-p(z)} = (\alpha\tau - a\tau + a)e^{-p(z)} + \tau A(z) - \sum_{j_1\in J_1} A_{j_1}(z)T_{j_1}(z). \tag{2.18}$$

From (2.18), we see that order of left-hand side is less than $\sigma(f)$ and order of right hand side is equal to $\sigma(f)$ which is a contradiction, hence the coefficient of $e^{-p(z)}$ must be zero i.e.,

$$\tau = \frac{a}{a-\alpha}. \tag{2.19}$$

Theorem 1.2 We first prove case(i). Since, $w_1(f)$ and $w_2(f)$ share a CM, we can write,

$$\frac{w_1(f)-a}{w_2(f)-a} = e^{\phi(z)}, \tag{2.20}$$

where $\phi(z)$ is a polynomial such that $deg(\phi(z)) \geq 1$. Suppose $\phi'(z) \not\equiv 0$, differentiating (2.1), we get

$$\phi'(z) = \frac{w_1'(f)}{w_1-a} - \frac{w_2'(f)}{w_2-a}. \tag{2.21}$$

Clearly from (2.21),

$$m(r,\phi'(z)) \leq m\left(r,\frac{w_1'}{w_1-a}\right)+m\left(r,\frac{w_2'}{w_2-a}\right) \leq S(r,f). \tag{2.22}$$

Now

$$\begin{aligned} \phi' &= (f-\alpha)\left[\frac{w_1'}{(w_1-a)(f-\alpha)}-\frac{w_2'}{(w-a)(f-\alpha)}\right], \\ &= (f-\alpha)\left[\frac{A_*}{a}-\frac{B_*}{\alpha-a}\right], \quad where \end{aligned} \tag{2.23}$$

$$A_*=\frac{w_1'}{w_1-a}\frac{w_1}{w_1-a}-\frac{w_1'}{f-\alpha} \quad and \quad B_*=\frac{w_2'}{f-\alpha}-\frac{w_2'}{w_2-a}-\frac{(w_2-f)}{f-\alpha}\frac{w_2'}{w_2-a}.$$

In other words, (2.23) can be written as

$$\frac{1}{f-\alpha}=\frac{1}{\phi'}\left[\frac{A_*}{a}-\frac{B_*}{\alpha-a}\right]. \tag{2.24}$$

From (2.23),

$$\begin{aligned} m(r,A_*) &\leq m\left(r,\frac{w_1'}{w_1-a}\right)+m\left(r,\frac{w_1}{f-\alpha}\right)+m\left(r,\frac{w_1'}{f-\alpha}\right)+S(r,f) \\ &= m\left(r,\frac{w_1}{f-\alpha}\right)+S(r,f) \\ &\leq m\left(r,\frac{\sum\limits_{i_1\in I_1} D_{i_1}(z)f^{(l_{i_1})}(z)}{f-\alpha}\right)+m\left(r,\frac{\sum_{i_2\in I_2} E_{i_2}(z)f^{(l_{i_2})}(z+d_{i_2})}{f-\alpha}\right) \\ &\quad +m\left(r,\frac{\sum\limits_{i_3\in I_3} F_{i_3}(z)f(z+e_{i_3})-\alpha+\alpha}{f-\alpha}\right)+S(r,f) \\ &\leq \sum_{i_1\in I_1}\left[m\left(r,D_{i_1}(z)\right)+m\left(r,\frac{f^{(l_{i_1})}(z)}{f-\alpha}\right)\right] \\ &\quad +\sum_{i_2\in I_2}\left[m\left(r,E_{i_2}(z)\right)+m\left(r,\frac{f^{(l_{i_2})}(z+d_{i_2})}{f-\alpha}\right)\right] \\ &\quad +\sum_{i_3\in I_3}\left[m\left(r,F_{i_3}(z)\right)+m\left(r,\frac{f(z+e_{i_3})-\alpha}{f-\alpha}\right)\right]+S(r,f). \end{aligned}$$

We know that $D_{i_1}(z), E_{i_2}(z)$ and $F_{i_3}(z)$ are small functions of $f(z)$ and using the condition $\sum\limits_{i_3\in I_3} F_{i_3}(z) \equiv 0$ we get,

$$m(r,A_*) \leq S(r,f). \tag{2.25}$$

From (2.23),

$$\begin{aligned} m(r,B_*) &\leq m\left(r,\frac{w_2'}{f-\alpha}\right)+m\left(r,\frac{w_2'}{w_2-a}\right)+m\left(r,\frac{w_2-f}{f-\alpha}\right)+m\left(r,\frac{w_2'}{w_2-a}\right)+S(r,f), \\ &= m\left(r,\frac{w_2-f}{f-\alpha}\right)+S(r,f). \end{aligned}$$

Substituting for w_2 from (1.2) we get,

$$m(r,B_*)\le m\left(r,\frac{\sum\limits_{j_1\in J_1}A_{j_1}(z)f^{(k_{j_1})}(z)}{f-\alpha}\right)+m\left(r,\frac{\sum\limits_{j_2\in J_2}B_{j_2}(z)f^{(k_{j_2})}(z+b_{j_2})}{f-\alpha}\right)$$
$$+m\left(r,\frac{\sum\limits_{j_3\in J_3}C_{j_3}(z)f(z+c_{j_3})-f}{f-\alpha}\right)+S(r,f),$$
$$\le\sum_{j_1\in J_1}\left[m\left(r,A_{j_1}(z)\right)+m\left(r,\frac{f^{(k_{j_1})}(z)}{f-\alpha}\right)\right]+\sum_{j_2\in J_2}\left[m\left(r,B_{j_2}(z)\right)+m\left(r,\frac{f^{(k_{j_2})}(z+b_{j_2})}{f-\alpha}\right)\right]$$
$$+\sum_{j_3\in J_3}\left[m\left(r,C_{j_3}(z)\right)+m\left(r,\frac{f(z+c_{j_3})-\alpha}{f-\alpha}\right)-m\left(r,\frac{f-\alpha}{f-\alpha}\right)\right]+S(r,f).$$

Using the condition $\sum\limits_{j_3\in J_3}C_{j_3}(z)-1\equiv 0$ the above equation becomes

$$m(r,B_*)\le S(r,f). \tag{2.26}$$

From (2.24), (2.25),(2.26) and (2.22)

$$m\left(r,\frac{1}{f-\alpha}\right)\le m\left(r,\frac{1}{\phi'}\right)+m\left(r,\frac{A_*}{a}\right)+m\left(r,\frac{B_*}{\alpha-a}\right)+S(r,f)=S(r,f). \tag{2.27}$$

Using the definition of $\delta(\alpha,f)$, from (2.27) we see that $\delta(\alpha,f)=0$ which implies that α is not a Nevanlinna exceptional value which is a contradiction hence $\phi(z)$ should be a constant.

Now we shall prove case (ii). Since $w_1(f)$ and $w_2(f)$ share a CM, from case (i)(2.20) can be written as

$$\frac{w_1(f)-a}{w_2(f)-a}=\tau(\neq 0). \tag{2.28}$$

Let α be the Borel exceptional value of $f(z)$. By Hadamard's factorization theorem, $f(z)$ can be written in the form

$$f(z)=A(z)\,e^{p(z)}+\alpha, \tag{2.29}$$

where $A(z)$ and $p(z)$ are as defined in (2.10) and the condition of (2.11) follows. Proceeding on the same lines as in case (ii) of proof of Theorem (1.1), (2.29) can be written in the form where $S_{i_1}(z),\ S_{i_2}(z+d_{1_2}),\ V_{j_1}(z),\ V_{j_2}(z+b_{j_2})$ are differential polynomials formed by $A(z),\ p(z)$ and its derivatives The above equation can be written as

$$\frac{\sum\limits_{i_1\in I_1}D_{i_1}(z)S_{i_1}(z)+M_*-ae^{-p(z)}}{\sum\limits_{j_1\in J_1}A_{j_1}(z)V_{j_1}(z)+N_*-ae^{-p(z)}}=\tau. \tag{2.30}$$

where

$$M_*=\sum_{i_2\in I_2}E_{i_2}(z)S_{i_2}(z+d_{i_2})e^{p(z+d_{i_2})-p(z)}+\sum_{i_3\in I_3}F_{i_3}(z)A(z+e_{i_3})e^{p(z+e_{i_3})-p(z)}.$$
$$N_*=\sum_{j_2\in J_2}B_{j_2}(z)V_{j_2}(z+b_{j_2})e^{p(z+b_{j_2})-p(z)}+\sum_{j_3\in J_3}C_{j_3}(z)A(z+c_{j_3})e^{p(z+c_{j_3})-p(z)}.$$

In other words

$$M_* - \tau N_* = \left(a + \tau\left(\alpha \sum_{j_3\in J_3} C_{j_3}(z) - a\right)\right) e^{-p(z)} - \sum_{i_1\in I_1} D_{i_1}(z)S_{i_1}(z) + \sum_{j_1\in J_1} A_{j_1}(z)V_{j_1}(z). \tag{2.31}$$

From (2.31) we see that the order of left-hand side is less than growth order of f whereas the order of right-hand side is equal to the growth order of f and also $\alpha \sum_{j_3\in J_3} C_{j_3}(z) - a$ which implies that the coefficient of $e^{-p(z)}$ must be zero, hence

$$a + \tau\left(\alpha \sum_{j_3\in J_3} C_{j_3}(z) - a\right) = 0 \quad (or) \quad \tau = \frac{a}{a - \alpha \sum_{j_3\in J_3} C_{j_3}(z)}. \tag{2.32}$$

We have $\sum_{j_3\in J_3} C_{j_3}(z) = 1$ hence

$$\tau = \frac{a}{a-\alpha}. \tag{2.33}$$

■

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