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Weighted estimates for operators associated to the Bergman-Besov kernels

Estimations pondérées pour les opérateurs associés aux noyaux de Bergman-Besov

David Békollè[1], Adriel R. Keumo[1], Edgar L. Tchoundja[1], and Brett D. Wick[2]

[1]Department of Mathematics, Faculty of Science, University of Yaounde I; PO. Box 812 Yaounde-Cameroon
dbekolle@gmail.com, keumo.adriel@gmail.com, tchoundjaedgar@yahoo.fr
[2]Department of Mathematics, Washington University - St. Louis. One Brookings Drive, St. Louis, MO 63130-4899 USA
wick@math.wustl.edu

ABSTRACT. We characterize the weights for which we have the boundedness of standard weighted integral operators induced by the Bergman-Besov kernels acting between two weighted Lebesgue classes on the unit ball of $\mathbb{C}^N$ in terms of Békollè - Bonami type condition on the weights. To accomplish this we employ the proof strategy originated by Békollè.
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1. Introduction

Weighted inequalities appeared almost simultaneously with the birth of singular integrals that stimulated their development. In particular, a natural question is the characterization of positive functions ω for which a singular integral maps $L^p(\omega d\mu)$ to itself. A famous example of a singular integral is the Bergman projection, whose boundedness problem, solved elsewhere by Békollè and Bonami, is historically linked to the duality problem for Bergman spaces.

The inner product and the norm in $\mathbb{C}^N$ are $\langle z, w\rangle = z_1\overline{w_1} + \cdots + z_N\overline{w_N}$ and $|z| = \sqrt{\langle z, z\rangle}$. We let $d\mu_q(z) = c_q(1-|z|^2)^q d\mu(z)$ where $q > -1$ and μ be the Lebesgue (volume) measure on the unit ball $\mathbb{B} = \{z \in \mathbb{C}^N : |z| < 1\}$ of $\mathbb{C}^N = \mathbb{R}^{2N}$, and c_q is the normalized constant, that is $\mu_q(\mathbb{B}) = 1$. Take $(c_q = \frac{\Gamma(n+q+1)}{N!\Gamma(q+1)})$. When $N = 1$, $\mathbb{B}$ is the unit disc $\mathbb{D}$. For $a > -1$, it is a well-known result of Békollè and Bonami that the Bergman projection T_a defined by

$$T_a f(z) := \int_{\mathbb{B}} \frac{f(x)}{(1-\langle z, x\rangle)^{N+1+a}} d\mu_a(x)$$

is bounded on $L^p(\omega d\mu_a)$ if and only if the weight ω belongs to the so-called Békollè - Bonami class [3]. The Bergman projection can be extended to all a less than or equal to -1. Therefore a natural question is whether the Békollè - Bonami result can be generalized. In this paper we work with more general operators than the extended Bergman projection, and more generally we characterize weights for which we have the boundedness between two weighted Lebesgue classes on the unit ball of $\mathbb{C}^N$.

We set $L^p_q := L^p(d\mu_q)$, the Lebesgue space on $\mathbb{B}$ relative to μ_q with $1 \leq p \leq +\infty$. Let $H(\mathbb{B})$ denote the space of holomorphic functions in the unit ball $\mathbb{B}$. For $q > -1$, a function $f \in H(\mathbb{B})$ belongs to the weighted Bergman space A^p_q whenever $f \in L^p(d\mu_q)$. The norm $\|f\|_{A^p_q}$ is simply the L^p_q norm of f.

Besov spaces extend weighted Bergman spaces to all q. To define them, we first take a radial differential operator D_s^t of order t for any $s, t \in \mathbb{R}$ defined on $H(\mathbb{B})$. Let $f \in H(\mathbb{B})$ be given by its convergent homogeneous expansion $f = \sum_{k=0}^{\infty} f_k$ in which f_k is a homogeneous polynomial in $z_1, \dots, z_N$ of degree k. We define, for $s, t \in \mathbb{R}$

$$D_s^t f := \sum_{k=0}^{\infty} d_k(s,t) f_k = \sum_{k=0}^{\infty} \frac{c_k(s+t)}{c_k(s)} f_k$$

where

$$c_k(a) = \begin{cases} \frac{(N+1+a)_k}{k!} & \text{if} \quad a > -(N+1), \\ \frac{k!}{(1-N-a)_k} & \text{if} \quad a \leq -(N+1). \end{cases}$$

Consider the linear transformation I_s^t defined for $f \in H(\mathbb{B})$ by

$$I_s^t f(z) = (1-|z|^2)^t D_s^t f(z).$$

We say that a function $f \in H(\mathbb{B})$ belongs to the Besov space B_q^p whenever $I_s^t f \in L_q^p$ for some s, t satisfying

$$\begin{cases} q + pt > -1 & \text{if } 1 \leq p < \infty \\ t > 0 & \text{if } p = \infty. \end{cases}$$

It is well known [9] that the L_q^p-norm, $\|I_s^t f\|_{L_q^p}$, of any one of the functions $I_s^t f$ is an equivalent norm for $\|f\|_{B_q^p}$, the norm of f in B_q^p. When $q > -1$ we have $A_q^p = B_q^p$. The space B_q^2 is a Hilbert space with reproducing kernel K_q (see [9] or [2, Theorem 1.9] or [16]) defined by

$$K_q(z,w) = \begin{cases} \frac{1}{(1-\langle z,w\rangle)^{N+1+q}} = \sum_{k=0}^{\infty} \frac{(N+1+q)_k}{k!} \langle z,w\rangle^k, & \text{if } q > -(N+1) \\ {}_2F_1(1,1;1-(N+q);\langle z,w\rangle) = \sum_{k=0}^{\infty} \frac{k!}{(1-N-q)_k} \langle z,w\rangle^k, & \text{if } q \leq -(N+1), \end{cases}$$

where ${}_2F_1 \in H(\mathbb{D})$ is the Gauss hypergeometric function and $(u)_v$ is the Pochhammer symbol defined by $(u)_v = \frac{\Gamma(u+v)}{\Gamma(u)}$ with Γ the gamma function. Namely, for a number s satisfying $q + 1 < p(s+1)$, if t satisfies $q + pt > -1$ then for $f \in B_q^2$ (see [9, Theorem 1.2])

$$(P_s \circ I_s^t) f = \frac{N!}{(1+s+t)_N} f,$$

where

$$P_s f(z) = \int_{\mathbb{B}} K_s(z,w) f(w) (1-|w|^2)^s d\mu(w),$$

is the extended Bergman projection (s may be smaller than or equal to -1).

For $a, b, s, t \in \mathbb{R}$ the operators that we are interested in are defined by (reproducing) Bergman-Besov kernels. For $f \in L^p(d\mu_q)$ we define

$$T_{a,b}^q f(z) \;:=\; T_{a,b} f(z) = \int_{\mathbb{B}} K_a(z,w) f(w) (1-|w|^2)^{b-q} d\mu_q(w),$$

$$S^q_{a,b}f(z) := S_{a,b}f(z) = \int_{\mathbb{B}} |K_a(z,w)||f(w)|(1-|w|^2)^{b-q}d\mu_q(w),$$

$$P^q_{s,t}f(z) := P_{s,t}f(z) = (1-|z|^2)^t \int_{\mathbb{B}} K_{s+t}(z,w)f(w)(1-|w|^2)^{s-q}d\mu_q(w).$$

Throughout the paper $b > -1$ and $s > -1$ because we want our operator to be well defined (see for example Lemma 5.1). Note that

$$P_{s,t}f(z) = (1-|z|^2)^t T_{s+t,s}f(z) \tag{1.1}$$

and

$$T_{a,b}f(z) = (1-|z|^2)^{b-a}P_{b,a-b}f(z). \tag{1.2}$$

Our main motivation comes from the operators $P_{s,0}$, and $P^+_{s,N+1+s}$, which are the Bergman projection and Berezin transform respectively, where $P^+_{s,N+1+s}f(z) = (1-|z|^2)^t S_{s+N+1+s,s}f(z)$. The operators $P_{s,t}$, $T_{a,b}$ and $S_{a,b}$ are important in the study of function-theoretic operator theory, see for example [17] when $q = -N-1$.

The boundedness of the operators $T^q_{a,b}$ was already studied by Kaptanoglu and Ureyen [10] in the cases where the operators $T^q_{a,b}$ act from L^p_q to L^P_Q, with $q \in \mathbb{R}, 1 \le p, P \le \infty, Q > -1$.

Theorem 1.1. *[10, Theorem 1.2] Let $a,b,q,Q \in \mathbb{R},\ 1 \le p \le P \le \infty$, and assume $Q > -1$ when $P < \infty$. Then, the following three conditions are equivalent.*

1. $T_{a,b} : L^p_q \to L^P_Q$;
2. $S_{a,b} : L^p_q \to L^P_Q$;
3. (a) $\frac{1+q}{p} < 1+b$ *and* $a \le b + \frac{1+N+Q}{P} - \frac{1+N+q}{p}$ *for* $1 < p \le P < \infty$;

   (b) $\frac{1+q}{p} \le 1+b$ *and* $a \le b + \frac{1+N+Q}{P} - \frac{1+N+q}{p}$ *for* $1 = p \le P \le \infty$, *but at least one inequality must be strict;*

   (c) $\frac{1+q}{p} < 1+b$ *and* $a < b + \frac{1+N+Q}{P} - \frac{1+N+q}{p}$ *for* $1 < p \le P = \infty$.

This result is useful for our work, especially for the case $p = P$ and $q = Q$, to investigate the case where these operators $P_{s,t}$ and $T_{a,b}$ are bounded from L^1_q to $L^{1,\infty}_q$. Here the weak Lebesgue space, $L^{1,\infty}_q$, is the space of measurable functions f for which there exists $A > 0$ such that for all $\lambda > 0$, $\lambda\mu_q\{z \in \mathbb{B} : |f(z)| > \lambda\} \le A$. Our main result in this direction is the following.

Theorem 1.2. *In the case $q = s$, $s + 2t > -1$ and $s + t > -1$ with $s > -1$ the operators $P_{s,t}$ are bounded from L^1_q to $L^{1,\infty}_q$ and not from L^1_q to L^1_q.*

In this paper we are mainly interested on the weighted estimates for the operators $T_{a,b}$ or $P_{s,t}$ from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$. Here and throughout the paper ω is a locally integrable positive function called a weight. It follows from (1.1) and (1.2) that it will be enough to study weighted estimates for one of these two operators since the $(L^p(\omega d\mu_q), L^p(\omega d\mu_Q))$ inequality for the family $T_{a,b}$ is equivalent to the $(L^p(\omega d\mu_q), L^p(\omega d\mu_{Q+pt}))$ inequality for the family $P_{s,t}$ and conversely. In the special case of the Bergman projection $T_{a,0}$, Békollè [3] obtained the characterisation of the weights ω in terms of the Békollè -Bonami condition.

Let d be the pseudo-distance in $\overline{\mathbb{B}}$ defined by

$$d(z,w) = \begin{cases} ||z|-|w|| + \left|1 - \langle \frac{z}{|z|}, \frac{w}{|w|}\rangle\right| & z, w \in \overline{\mathbb{B}}^* \\ |z| + |w| & z = 0 \, or \, w = 0. \end{cases}$$

Definition 1.3 (Békollè - Bonami class)**.** Let $a > -1$. Let ω be a weight on $\mathbb{B}$. We say that $\omega d\mu_a$ belongs to (B_p^a), $1 < p < \infty$, if there is a constant $B_p^a(\omega)$ such that for every ball B (with respect to the pseudo-distance d) of $\mathbb{B}$ that intersects the boundary of $\mathbb{B}$, we have

$$\left(\frac{1}{\mu_a(B)} \int_B \omega(z) d\mu_a(z)\right) \left(\frac{1}{\mu_a(B)} \int_B \omega^{\frac{-1}{p-1}}(z) d\mu_a(z)\right)^{p-1} \leq B_p^a(\omega).$$

For $a > -1$, let

$$T_a f(z) := T_{a,0} f(z) := \int_{\mathbb{B}} \frac{f(x)}{(1 - \langle z, x\rangle)^{N+1+a}} d\mu_a(x)$$

be the Bergman projection. Békollè showed in [3] that

Theorem 1.4. *Let ω be a weight on $\mathbb{B}$. The operator T_a, $a > -1$, is well defined and continuous on $L^p(\omega d\mu_a)$, $1 < p < \infty$, if and only if $\omega d\mu_a \in (B_p^a)$.*

The results we obtain depend upon the values of $s+t$, q and Q. In the case $s+t < -(N+1)$ we have the following two main results.

Theorem 1.5. *In the case $s+t < -(N+1)$, there are no weights ω such that $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ for $Q \leq q$.*

Theorem 1.6. *Let ω be a weight on $\mathbb{B}$. In the case $s+t < -(N+1)$, if $Q > q$, then $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ if and only if*

$$\left(\int_{\mathbb{B}} \omega(z) d\mu_{Q+pt}(z)\right) \left(\int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z)\right)^{p-1} < \infty.$$

Moreover

$$\|P_{s,t}\|^p \simeq \left(\int_{\mathbb{B}} \omega(z) d\mu_{Q+pt}(z)\right) \left(\int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z)\right)^{p-1}.$$

In order to give our necessary condition for the boundedness of $T_{a,b}$ when $a > -(1+N)$ we introduce a Békollè -Bonami type class of weights denoted by $(B_p^{a,b,q,Q})$.

Definition 1.7. Let ω be a weight on $\mathbb{B}$. For $Q \leq q$ and $a > -1$, we say that $\omega \in (B_p^{a,b,q,Q})$ $(b > -1)$ if

$$\sup_{B:B\cap\partial\mathbb{B}\neq\varnothing} \left(\frac{\mu_b(B)}{\mu_a^2(B)} \int_B \omega(z) d\mu_Q(z)\right) \left(\frac{\mu_b(B)}{\mu_a^2(B)} \int_B (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(b-q)}(z)\right)^{p-1} < \infty$$

where the supremum is taken over the pseudo-balls B.

For $Q \leq q$ and $a > -N-1$, we say that $\omega \in (B_p^{a,b,q,Q})$ $(b > -1)$ if

$$\sup_{B:B\cap\partial\mathbb{B}\neq\varnothing} \left(\frac{\mu_b(B)}{R_B^{2(N+1+a)}} \int_B \omega(z) d\mu_Q(z)\right) \left(\frac{\mu_b(B)}{R_B^{2(N+1+a)}} \int_B (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(b-q)}(z)\right)^{p-1} < \infty$$

where the supremum is taken over the pseudo-balls B with radius R_B.

For $Q > q$ and $a > -1$, we say that $\omega \in (B_p^{a,b,q,Q})$ $(b > -1)$ if

$$\sup_{B:B\cap\partial\mathbb{B}\neq\varnothing} \left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)} \int_B \omega(z) d\mu_Q(z)\right) \left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)} \int_B (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(b-q)}(z)\right)^{p-1} < \infty$$

where the supremum is taken over the pseudo-balls B.

For $Q > q$ and $a > -N-1$, we say that $\omega \in (B_p^{a,b,q,Q})$ $(b > -1)$ if

$$\sup_{B:B\cap\partial\mathbb{B}\neq\varnothing} \left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{R_B^{2(N+1+a)}} \int_B \omega(z) d\mu_Q(z)\right) \left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{R_B^{2(N+1+a)}} \int_B (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(b-q)}(z)\right)^{p-1} < \infty$$

where the supremum is taken over the pseudo-balls B with radius R_B.

A necessary condition for the boundedness of $P_{s,t}$ when $-1 > s+t > -(1+N)$ is the following.

Theorem 1.8. *In the case both $-(N+1) < s+t < -1$ and $s+t+\frac{Q-q}{p} \leq -1$ hold, and in the case both $s+t > -1$ and $Q < q$ hold, there are no weights ω such that $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$.*

For the remaining cases, we introduce another Békollè -Bonami type class of weights, $(K_p^{s,t,q,Q})$, in order to give our necessary condition for the boundedness of $P_{s,t}$ when $s+t > -(1+N)$.

Definition 1.9. Let ω be a weight on $\mathbb{B}$. For $s+t+\frac{Q-q}{p} > -1$ and $-1 > s+t > -N-1$, we say that $\omega \in (K_p^{s,t,q,Q})$ $(s > -1)$ if

$$\sup_{B:B\cap\partial\mathbb{B}\neq\varnothing} \left(\frac{R_B^{\frac{Q-q}{p}}}{R_B^{N+1+s+t}} \int_B \omega(z) d\mu_{Q+pt}(z)\right) \left(\frac{R_B^{\frac{Q-q}{p}}}{R_B^{N+1+s+t}} \int_B (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z)\right)^{p-1} < \infty$$

where the supremum is taken over the pseudo-balls B with radius R_B.

For $Q \geq q$ and $s+t > -1$, we say that $\omega \in (K_p^{s,t,q,S})$ $(s > -1)$ if

$$\sup_{B: B\cap\partial\mathbb{B}\neq\varnothing} \left(\frac{R_B^{\frac{Q-q}{p}}}{\mu_{s+t}(B)} \int_B \omega(z) d\mu_{Q+pt}(z) \right) \left(\frac{R_B^{\frac{Q-q}{p}}}{\mu_{s+t}(B)} \int_B (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z) \right)^{p-1} < \infty$$

where the supremum is taken over the pseudo-balls B with radius R_B.

A necessary condition for the boundedness of $P_{s,t}$ when $s+t > -(1+N)$ is the following.

Theorem 1.10. *In the case both $s+t+\frac{Q-q}{p} > -1$ and $-1 > s+t > -N-1$ hold, and in the case both $s+t > -1$ and $Q \geq q$ hold, if $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$, then $\omega \in (K_p^{s,t,q,Q})$.*

We introduce a maximal and a fractional maximal operator that will be used to establish a good lambda inequality in order to give sufficient conditions for the boundedness of $P_{s,t}$.

If $a > -1$ we set

$$m_{a,b}f(z) = \sup_{\zeta\in\mathbb{B}, R>1-|\zeta|: z\in B(\zeta,R)} \frac{1}{\mu_a(B(\zeta,R))} \int_{B(\zeta,R)} |f(w)| d\mu_b(w),$$

more generally if $a > -1-N$ we set

$$m'_{a,b}f(z) = \sup_{\zeta\in\mathbb{B}, R>1-|\zeta|: z\in B(\zeta,R)} \frac{1}{R^{N+1+a}} \int_{B(\zeta,R)} |f(w)| d\mu_b(w).$$

Before giving our good lambda inequality, we introduce here $(D_p^{s,t,q,Q})$ a Békollè-Bonami type class of weights.

Definition 1.11. Let ω be a weight on $\mathbb{B}$. For $s+t+\frac{Q-q}{p} > -1$ and $-1 > s+t > -N-1$, we say that $\omega \in (D_p^{s,t,q,Q})$ $(s > -1)$ if

$$\sup_{B: B\cap\partial\mathbb{B}\neq\varnothing} \left(\frac{1}{R_B^{N+1+s+t+\frac{Q-q}{p}}} \int_B \omega(z) d\mu_{Q+pt}(z) \right) \left(\frac{1}{R_B^{N+1+s+t+\frac{Q-q}{p}}} \int_B (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z) \right)^{p-1} < \infty$$

where the supremum is taken over the pseudo-balls B with radius R_B.

For $Q \geq q$ and $s+t > -1$, we say that $\omega \in (D_p^{s,t,q,Q})$ $(s > -1)$ if

$$\sup_{B: B\cap\partial\mathbb{B}\neq\varnothing} \left(\frac{1}{\mu_{s+t+\frac{Q-q}{p}}(B)} \int_B \omega(z) d\mu_{Q+pt}(z) \right) \left(\frac{1}{\mu_{s+t+\frac{Q-q}{p}}(B)} \int_B (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z) \right)^{p-1} < \infty$$

where the supremum is taken over the pseudo-balls B with radius R_B. In each case we denote by $D_p^{s,t,q,Q}(\omega)$ the expression in the left hand side.

Remark 1.12. Constants and standard weights ($\omega(z) = (1-|z|^2)^\eta$) are in $(D_p^{s,t,q,Q})$. We also have $(D_p^{s,t,q,q}) \subseteq (D_p^{s,t,q,Q}) \subseteq (K_p^{s,t,q,Q})$. For $Q = q$ we have $(K_p^{s,t,q,Q}) = (D_p^{s,t,q,Q})$.

Here is our good lambda inequality.

Theorem 1.13. *Suppose that* $1 < p < \infty$. *Let* $\omega \in (D_p^{s,t,q,Q})$ *where both* $s+t+\frac{Q-q}{p} > -1$ *and* $-1 > s+t > -N-1$ *hold, or both* $s+t > -1$ *and* $Q \geq q$ *hold. There are two positive constants* C *and* β *such that for all* γ *sufficiently small,* $\lambda > 0$ *and for all positive locally integrable functions* f, *if* $-N-1 < s+t$ *and* $s+t+\frac{Q-q}{p} > -1$, *then*

$$\omega d\mu_{Q+pt}(\{z \in \mathbb{B} : S_{s+t,s}f(z) > 2\lambda, m'_{s+t,s}f(z) \leq \gamma\lambda\}) \leq \\ CD_p^{s,t,q,Q}(\omega)\gamma^{\beta}\omega d\mu_{Q+pt}(\{z \in \mathbb{B} : S_{s+t,s}f(z) > \lambda\}). \quad (1.3)$$

To show that $(D_p^{s,t,q,Q})$ is sufficient for the boundedness of $P_{s,t}$ when $s+t > -1$, we introduce the following maximal and fractional maximal operator. If $s+t > -1$ we set

$$O_{s,t}f(z) = (1-|z|^2)^t m_{s+t,s}f(z);$$

more generally if $s+t > -1-N$ we set

$$O'_{s,t}f(z) = (1-|z|^2)^t m'_{s+t,s}f(z).$$

The following theorem shows together with the good lambda inequality that $(D_p^{s,t,q,Q})$ is sufficient for the boundedness of $P_{s,t}$ from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ when $-N-1 < s+t$ and $s+t+\frac{Q-q}{p} > -1$.

Theorem 1.14. *For* $-N-1 < s+t$ *and* $s+t+\frac{Q-q}{p} > -1$, *if* $\omega d\mu_q \in (D_p^{s,t,q,Q})$, *there is a constant* $C_{s,t,p,q,Q} > 0$ *such that for all* $f \in L^p(\omega d\mu_q)$,

$$\int_{\mathbb{B}} (O_{s,t}f(z))^p \omega(z) d\mu_q(z) \leq C_{s,t,p,q,Q} \int_{\mathbb{B}} |f(z)|^p \omega(z) d\mu_q(z).$$

Then for the case $s+t > -1$ and $q = Q$ we have

Corollary 1.15. *Let* ω *be a weight on* $\mathbb{B}$. *Then for* $s+t > -1, s > -1$ *the following assertions are equivalent.*

1. $P_{s,t}$ *is well defined and continuous from* $L^p(\omega d\mu_q)$ *to* $L^p(\omega d\mu_q)$;
2. $T_{s+t,s}$ *is well defined and continuous from* $L^p(\omega d\mu_q)$ *to* $L^p(\omega d\mu_{q+pt})$;
3. $S_{s+t,s}$ *is well defined and continuous from* $L^p(\omega d\mu_q)$ *to* $L^p(\omega d\mu_{q+pt})$;
4. $\omega \in (K_p^{s,t,q,Q})$.

Rahm, the third and the fourth authors settled in [11] the particular case of the operators $P_{s,t}$ for $s+t > -1, s > -1, Q = q = s$. To this aim, they used dyadic methods that have been initiated by Aleman, Pott and Reguera in the unit disk [1]. It might be interesting to see if the dyadic methods used in those papers extend to the situation studied in this paper.

The outline of the paper is as follows. In Section 2 we briefly give requisite background information. We prove Theorem 1.2 in Section 3. From Section 4 on, we look at weighted estimates; there we show Theorem 1.5 and Theorem 1.6. The proof of Theorem 1.8, Theorem 1.10, Theorem 1.14 are in Section 5. The proof of Theorem 1.13 is in Section 6. Corollary 1.15 appears in Section 7.

2. Main tools

2.1. *Complex Analysis Tools*

Throughout this paper d is the pseudo-distance in $\overline{\mathbb{B}}$ defined by

$$d(z,w) = \begin{cases} ||z| - |w|| + \left|1 - \langle \frac{z}{|z|}, \frac{w}{|w|} \rangle\right| & z, w \in \overline{\mathbb{B}}^* \\ |z| + |w| & z = 0 \; or \; w = 0. \end{cases}$$

Throughout this paper K will be a constant such that

$$d(x,y) \leq K(d(x,z) + d(z,y)) \tag{2.1}$$

for all x, y and z in $\mathbb{B}$. We will consider pseudo-balls in $\mathbb{B}$, $B(z,r)$, as points w of $\mathbb{B}$ such that $d(z,w) < r$ and we say that $B(z,r)$ touches the boundary of $\mathbb{B}$ if $r > 1-|z|$. When $B(z,r)$ is such that $r > k(1-|z|)$ for some absolute value $k < 1$, we say that $B(z,r)$ almost touches the boundary of $\mathbb{B}$.

One can find the following two results in [3, 15].

Lemma 2.1. *For each $z \in \mathbb{B}$ and r_0, $0 < r_0 < 1$, if we set $z^0 = (r_0, 0, \cdots, 0)$, then we have*

1. $|1 - z_1 r_0| \geq \frac{1}{4} d(z, z^0)$;
2. $|z_1 - r_0| \leq d(z, z^0)$;
3. $|z - z^0| \leq d(z, z^0)$;
4. $\sum_{k=2}^{N} |z_k|^2 \leq 2d(z, z^0)$.

Proposition 2.2. *There is a constant $C_1 > 0$ so that for all $z, w, w_0 \in \mathbb{B}$ with $d(z, w_0) > C_1 d(w, w_0)$ we have*

$$|\langle z, w_0\rangle - \langle z, w\rangle| \leq \frac{1}{2}|1 - \langle z, w_0\rangle|.$$

Then

$$|1 - \langle z, w\rangle| \geq \frac{1}{2}|1 - \langle z, w_0\rangle|.$$

We recall that $d\mu_q(z) = c_q(1-|z|^2)^q d\mu(z)$ where $q > -1$ and μ be the Lebesgue (volume) measure on the unit ball $\mathbb{B} = \{z \in \mathbb{C}^N : |z| < 1\}$ of $\mathbb{C}^N = \mathbb{R}^{2N}$, and c_q is the normalized constant, that is $\mu_q(\mathbb{B}) = 1$. When $q \leq -1$, we simply write $d\mu_q(z) = (1-|z|^2)^q d\mu(z)$. We then obtain the following result that will be heavily used throughout the paper. This extends to all q the result given in [3].

Lemma 2.3. *For each $w \in \mathbb{B}$, $0 < |w| = r < 1$ and $0 < R < 2$*

$$\mu_q(B(w,R)) \simeq R^{N+1}[\max(R, 1-r)]^q \; if \; q > -1.$$

Then for $q > -1$, $(\mathbb{B}, d, \mu_q)$ is a homogeneous space in the sense of [5].

However, if $B(w,R)$ is away from the boundary ($R < \frac{1-|w|}{2}$), the equivalence remains true if $q \leq -1$, ie

$$\mu_q(B(w,R)) \simeq R^{N+1}(1-|w|)^q.$$

Proof. Assume $q \leq -1$. Since $z \in B(w, R)$, we have $|w| - |z| < R$. Hence $1 - |z| < 1 - |w| + R$ and $1 - |w| < 1 - |z| + R$. Since $R < \frac{1-|w|}{2}$, we have $1 - |w| < 2(1 - |z|)$ so that

$$\frac{1}{2}(1 - |w|) \leq 1 - |z| \leq 1 - |w| + R \leq \frac{3}{2}(1 - |w|). \tag{2.2}$$

Hence, for all $R \in (0, \frac{1-|w|}{2})$, we have

$$\int\limits_{B(w,R)} (1 - |z|)^q d\mu(z) \simeq R^{N+1}(1 - |w|)^q.$$

□

We recall the following well-known estimates in [12].

Proposition 2.4. *Let*

$$I = \int\limits_{\mathbb{B}} \frac{(1 - |w|^2)^d}{|1 - \langle z, w\rangle|^{1+N+c}} d\mu(w)$$

for $d > -1$ *and* $c \in \mathbb{R}$. *We have, when* $|z| \to 1^-$,

(i) $I \sim 1$ *if* $c < d$*;*

(ii) $I \sim \frac{1}{|z|^2} \log \frac{1}{1-|z|^2}$ *if* $c = d$*;*

(iii) $I \sim (1 - |z|^2)^{-(c-d)}$ *if* $c > d$.

The following results can be found in [9] and [10] respectively.

Theorem 2.5. *Let* $q \in \mathbb{R}$. *Let* $t, s \in \mathbb{R}$ *such that* $q + 2t > -1$. *Equipped with the following equivalent scalar product*

$${}_q\langle f, g\rangle_s^t = \int\limits_{\mathbb{B}} I_s^t f(z) \overline{I_s^t g(z)} d\mu_q(z),$$

B_q^2 *is a Hilbert space with reproducing kernel given by*

$$K_q(z, w) = \begin{cases} \frac{1}{(1-\langle z,w\rangle)^{N+1+q}} = \sum_{k=0}^{\infty} \frac{(N+1+q)_k}{k!} \langle z, w\rangle^k, & \text{if } q > -(N+1) \\ {}_2F_1(1, 1; 1 - (N+q); \langle z, w\rangle) = \sum_{k=0}^{\infty} \frac{k!}{(1-N-q)_k} \langle z, w\rangle^k, & \text{if } q \leq -(N+1). \end{cases}$$

Lemma 2.6.

1. *For* $q < -(N + 1)$, *each* $|K_q(z, w)|$ *is bounded above as* z, w *vary in* $\mathbb{B}$.
2. *For each* $q \in \mathbb{R}$,
   (a) $|K_q(z, w)|$ *is bounded below by a positive constant as* z, w *vary in* $\mathbb{B}$. *In particular,* $K_q(z, w)$ *is zero free in* $\mathbb{B} \times \mathbb{B}$.
   (b) there is a $\rho_0 < 1$ *such that for* $|z| \leq \rho_0$ *and all* $w \in \mathbb{B}$, *we have* $\Re K_q(z, w) \geq \frac{1}{2}$.

2.2. Harmonic Analysis Tools

The following result can be found in [5] and will be helpful in the proof of Theorem 3.1.

Theorem 2.7 ([5])**.** *Let (X, d, μ) be a homogeneous space and let $K(x, y)$ be a function such that $K(x, .) : y \to K(x, y) \in L^2(X)$. Suppose the operator T defined by*

$$Tf(x) = \int_X K(x, y)f(y)d\mu(y),$$

satisfies the following two conditions

1. *there is a constant C_1 so that $\|Tf\|_2 \leq C_1\|f\|_2$;*
2. *there are two constants C_2 and C_3 so that for all y, y_0 we have*

$$\int_{d(x,y_0)>C_2d(y,y_0)} |K(x, y) - K(x, y_0)|d\mu(x) < C_3 \text{ (Hörmander Condition)}.$$

Then for all p, $1 \leq p \leq 2$, there is a constant A_p depending only on $C_i, i = 1, 2, 3$, so that for all $f \in L^2 \bigcap L^p$ we have $\|Tf\|_p \leq A_p\|f\|_p$ if $p > 1$, and $\forall\lambda > 0$

$$\mu(\{x \in X : |Tf(x)| > \lambda\}) \leq A_1\frac{\|f\|_1}{\lambda}.$$

One can find the following result in [7].

Theorem 2.8 (Marcinkiewicz Interpolation Theorem)**.** *Let p_0, p_1 be so that $1 \leq p_0 < p_1 \leq \infty$. Let T be a sublinear operator defined from $L^{p_0} + L^{p_1}$ to the space of measurable functions. Assume that T is simultaneously of weak type (p_0, p_0) with operator norm A_{p_0,p_0} and of weak type (p_1, p_1) with operator norm A_{p_1,p_1}. Then for every $0 < t < 1$, T is of (strong) type (p_t, p_t) where*

$$\frac{1}{p_t} = \frac{t}{p_0} + \frac{1-t}{p_1}.$$

Moreover, if $p_1 < \infty$, then $\|Tf\|_{p_t} \leq A_{p_t,p_t}\|f\|_{p_t}$ with

$$A_{p_t,p_t} = 2\left[p_t(\frac{A_{p_0,p_0}^{p_0}}{p_t - p_0} - \frac{A_{p_1,p_1}^{p_1}}{p_1 - p_t})\right]^{\frac{1}{p_t}}.$$

If $p_1 = \infty$, we can take

$$A_{p_t,p_t} = 2\left[p_t\frac{A_{p_0,p_0}^{p_0}}{p_t - p_0}\right]^{\frac{1}{p_t}}.$$

The fractional maximal function is defined as follows

$$M_\gamma f(z) = \sup_{B:z\in B} \frac{1}{\nu^{1-\gamma}(B)} \int_B |f(w)|d\nu(w), \quad \gamma \in [0, 1).$$

When $\gamma = 0$ it is the Hardy-Littlewood maximal operator. The following result will be used in Section 5 in the study of our maximal and fractional maximal function. One can find their proof in [6] or in [13].

Theorem 2.9. *Let X be a homogeneous space, $0 \leq \gamma < 1$, $1 < p \leq r < \infty$ and a pair of weights (u, v), then the following are equivalent.*

(i) There exists a constant $C_1 > 0$ so that

$$\left(\int_X [M_\gamma f(x)]^r v(x) d\nu(x) \right)^{\frac{1}{r}} \leq C_1 \left(\int_X |f(x)|^p u(x) d\nu(x) \right)^{\frac{1}{p}}$$

for any $f \in L^p(X, ud\nu)$;

(ii) There exists a constant $C_2 > 0$ such that

$$\left(\int_B [M_\gamma(\chi_B u^{1-p'})(x)]^r v(x) d\nu(x) \right)^{\frac{1}{r}} \leq C_2 \left(\int_B u^{1-p'}(x) d\nu(x) \right)^{\frac{1}{p}}$$

for any ball $B \subset X$.

We will also make use of the following class, in Section 5, in the study of our maximal and fractional maximal function and to establish the good lambda inequality.

Definition 2.10. A measure $\omega d\mu_\alpha$ is in the (A_p, α) $(1 < p < \infty)$ class if there is a constant $C_p(\omega)$ so that for all pseudo-balls $B := B(\zeta, R)$ we have

$$\left(\frac{1}{\mu_\alpha(B)} \int_B \omega(z) d\mu_\alpha(z) \right) \left(\frac{1}{\mu_\alpha(B)} \int_B (\omega(z))^{\frac{-1}{p-1}} d\mu_\alpha(z) \right)^{p-1} \leq C_p(\omega).$$

Definition 2.11. A measure $\omega d\mu_\alpha$is a Muckenhoupt weight or is in the (A_∞, α) class if there exist δ, β with $0 < \delta, \beta < 1$, so that for all pseudo-balls B of $\mathbb{B}$ and for all measurable subset E of B we have

$$\mu_\alpha(E) \leq \delta\mu_\alpha(B) \Rightarrow \omega d\mu_\alpha(E) \leq \beta\omega d\mu_\alpha(B).$$

We give now two properties of Muckenhoupt weight that we will need later (see [8]).

Lemma 2.12. *If $\sigma \in (A_\infty, \alpha)$ then there are two positive constants A and β_0 so that for all balls B and a measurable subset E of B we have*

$$\int_E \sigma(x) d\mu_\alpha(x) \leq A \left(\frac{d\mu_\alpha(E)}{d\mu_\alpha(B)} \right)^{\beta_0} \int_B \sigma(x) d\mu_\alpha(x).$$

Theorem 2.13. *The Hardy-Littlewood maximal operator is bounded on $L^p(\omega d\mu_\alpha)$, $1 < p < \infty$, if and only if $\omega d\mu_\alpha \in (A_p, \alpha)$.*

The following known lemma [14, Lemma 2-Chapter IV] will be used in Section 6.

Lemma 2.14. *Let $(X, \mathcal{A}, \mu)$ a measure space. Let f and g be two positive measurable functions so that for all $t > 0$*

$$\mu(\{x \in X : f(x) > t, g(x) \leq ct\}) \leq a\mu(\{x \in X : f(x) > bt\}),$$

where a, b and c are positive constants such that $a < b^p$ $(1 < p < \infty)$. Then

$$\|f\|_p^p \leq \frac{c^{-p}}{1 - ab^{-p}} \|g\|_p^p.$$

We will use the following lemma in Section 5 to show Lemma 5.1. One can find it in [3, 15].

Lemma 2.15. *For $a > -1-N$, there are two constants C_1, C_2 $(C_1 > 0)$ so that, for $z, w, w_0 \in \mathbb{B}$ with $|1-\langle z, w_0\rangle| > C_1 d(w, w_0)$, we have*

$$\left|\frac{1}{(1-\langle z,w\rangle)^{N+1+a}} - \frac{1}{(1-\langle z,w_0\rangle)^{N+1+a}}\right| \leq C_2 \frac{d(w,w_0)}{|1-\langle z,w_0\rangle|^{N+a+2}}.$$

3. Weak Type L^1 Inequality for $P_{s,t}$ and $T_{a,b}$.

In this section, we will prove Theorem 1.2 that we first recall here.

Theorem 3.1. *In the case $q = s$, $s+2t > -1$ and $s+t > -1$ with $s > -1$ the operators $P_{s,t}$ are bounded from L^1_q to $L^{1,\infty}_q$ and not from L^1_q to L^1_q.*

Proof. The kernel of $P_{s,t}$ is $H_{s,t}(z,w) = \frac{(1-|z|^2)^t}{(1-\langle z,w\rangle)^{N+1+s+t}}$. We are going to proceed in three steps.

Step 1: Show that $H_{s,t}(z,.) \in L^2_q, \forall z \in \mathbb{B}$.

Indeed, we have

$$\int_{\mathbb{B}} |H_{s,t}(z,w)|^2 d\mu_q(w) = \int_{\mathbb{B}} \frac{(1-|z|^2)^{2t}}{|1-\langle z,w\rangle|^{2(N+1+s+t)}} d\mu_q(w)$$
$$\leq \frac{(1-|z|^2)^{2t}}{(1-|z|)^{2(N+1+s+t)}} \int_{\mathbb{B}} (1-|w|^2)^q d\mu(w)$$

where in the second inequality, the member of the right hand side is finite because $q = s > -1$.

Step 2: Show that $P_{s,t}$ is bounded from L^2_q to L^2_q.

We have to show the boundedness of $T_{s+t,s}$ from L^2_q to L^2_{q+2t}. By Kaptanoglu and Ureyen, for $a = s+t$, $b = s$, $p = P = 2$ and $Q = q+2t$ (this is the reason $q+2t > -1$ is needed), this holds.

Step 3: Show that there are two constants C_1 and C_2 so that $\forall w, w_0 \in \mathbb{B}$ we have

$$\int_{d(z,w_0) > C_1 d(w,w_0)} |H_{s,t}(z,w) - H_{s,t}(z,w_0)| d\mu_q(z) < C_2.$$

This was already done in [3] (see the proof of [3, Proposition 1] choose $a = q+t+1$).

Because of *Step 1*, *Step 2* and *Step 3* we have by using Theorem 2.7 that the operators $P_{s,t}$ are bounded from L^1_q to $L^{1,\infty}_q$. Observe that $P_{s,t}$ is bounded from L^1_q to L^1_q if and only if $T_{s+t,s}$ is bounded from L^1_q to L^1_{q+t}; and by Theorem 1.1, $T_{s+t,s}$ is not bounded from L^1_q to L^1_{q+t} because $q = s$. □

Remark 3.2. In the case $a > -(N+1)$, the operators $T^q_{a,b}$ are bounded from L^1_q to $L^{1,\infty}_q$ if we have the following two conditions

i) $a \leq b$

ii) $-1 < q \leq b$.

The case $a = b = q > -1$ is due to Békollè in [3]. The remaining cases are obtained by using Theorem 1.1.

Remark 3.3. In the special case $b = q$, $T^q_{a,b}$ is self adjoint and bounded from L^p_q to itself for $1 < p < \infty$. Indeed, let $f \in L^p_q$ and $g \in L^{p'}_q$. Then

$$\begin{aligned}
\langle T^q_{a,b} f, g\rangle_{L^2_q} &= \int_{\mathbb{B}} \int_{\mathbb{B}} K_a(z,w) f(w)(1-|w|^2)^{b-q} d\mu_q(w) \overline{g(z)} (1-|z|^2)^q d\mu(z) \\
&= \int_{\mathbb{B}} f(w)(1-|w|^2)^{b-q} \overline{\int_{\mathbb{B}} K_a(w,z) g(z)(1-|z|^2)^q d\mu(z)} (1-|w|^2)^q d\mu(w) \\
&= \int_{\mathbb{B}} f(w) \overline{T^*_{a,b} g(w) d\mu_q(w)} \\
&= \langle f, (T^q_{a,b})^* g\rangle_{L^2_q},
\end{aligned}$$

where

$$(T^q_{a,b})^* g(w) = (1-|w|^2)^{b-q} \int_{\mathbb{B}} K_a(w,z) g(z)(1-|z|^2)^q d\mu(z).$$

Observe that when $b = q$, $(T^q_{a,b})^* = T^q_{a,b}$ and since $T^q_{a,b}$ is bounded from L^p_q to L^p_q when $1 < p < 2$, then $T^q_{a,b} = \left(T^q_{a,b}\right)^*$ is bounded from $L^{p'}_q$ to $L^{p'}_q$ with $2 < p' < \infty$.

Remark 3.4. By Remark 3.2 we have that $T^q_{a,b}$ is of weak type $(1,1)$ and let $A_{1,1}$ be the operator norm. By Theorem 1.1 we have that $T^q_{a,b}$ is of strong type $(2,2)$ and let $A_{2,2}$ be the operator norm. Applying Theorem 2.8 leads us, with a better estimation of the operator norm

$$A_{p,p} = 2\left[p\left(\frac{A_{1,1}}{p-1} - \frac{A^2_{2,2}}{2-p}\right)\right]^{\frac{1}{p}},$$

to a new way to have the boundedness of $T^q_{a,b}$ from L^p_q to L^p_q when $1 < p < 2$. In the special case $b = q$ we have the boundedness from L^p_q to L^p_q when $1 < p < \infty$ because in this case $T^q_{a,b}$ is self adjoint.

4. Weighted estimates: Preliminary necessary conditions and the case where $s + t < -N - 1$

In this section we will give a proof of our criterion for the weights that provide boundedness of $P_{s,t}$ when $s + t < -N - 1$) (respectively $T_{a,b}$ when $a < -N - 1$). We start first with some general necessary conditions.

4.1. *Preliminary Necessary Conditions*

Lemma 4.1. *Let ω be a weight. For $q, Q \in \mathbb{R}$, if $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$, then ω must be in $L^1(d\mu_{Q+pt})$.*

Proof. Let $f(w) = (1-|w|^2)^{-s}\chi_{B(0,R)}(w)$ where $B(0,R)$ is the Euclidean ball. Then

$$\begin{aligned}
P_{s,t}f(w) &= (1-|w|^2)^t \int_{\mathbb{B}} K_{s+t}(w,z)(1-|z|^2)^{-s}\chi_{B(0,R)}(z)(1-|z|^2)^s d\mu(z) \\
&= (1-|w|^2)^t \int_{B(0,R)} K_{s+t}(w,z)d\mu(z) \\
&= (1-|w|^2)^t \overline{\int_{B(0,R)} K_{s+t}(z,w)d\mu(z)} \\
&= (1-|w|^2)^t \overline{K_{s+t}(0,w)}\mu(B(0,R)) \\
&= (1-|w|^2)^t \mu(B(0,R)).
\end{aligned}$$

Since $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$,

$$\int_{\mathbb{B}} |P_{s,t}f(z)|^p \omega(z)d\mu_Q(z) < \infty,$$

so that

$$\mu^p(B(0,R)) \int_{\mathbb{B}} \omega(z)d\mu_{Q+pt}(z) < \infty,$$

then

$$\int_{\mathbb{B}} \omega(z)d\mu_{Q+pt}(z) < \infty,$$

and $\omega \in L^1(d\mu_{Q+pt})$. □

Lemma 4.2. *Let ω be a weight and $q, Q \in \mathbb{R}$. If $T_{a,b}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$, then $\omega^{\frac{-1}{p-1}} \in L^1(d\mu_{q+p'(b-q)})$.*

Proof. Assume that $T_{a,b}$ is well defined, that is for any $z \in \mathbb{B}$

$$S_{a,b}f(z) = \int_{\mathbb{B}} |K_a(z,w)||f(w)|d\mu_b(w) < \infty,$$

and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$. We want to show that $\omega^{\frac{-1}{p-1}} \in L^1(d\mu_{q+p(p'-1)(b-q)})$, in other words we want to show $\omega^{-1} \in L^{p'}(\omega d\mu_{q+p(p'-1)(b-q)})$. Assume that ω^{-1} is not in $L^{p'}(\omega d\mu_{q+p(p'-1)(b-q)})$, then by the Riesz representation theorem there exists a positive function h in $L^p(\omega d\mu_{q+p(p'-1)(b-q)})$ so that

$$\langle h, \omega^{-1}\rangle_{\omega,q+p(p'-1)(b-q)} = \infty.$$

This means that

$$\infty = \int_{\mathbb{B}} h(z)d\mu_{q+p(p'-1)(b-q)}(z)$$

$$= \int_{\mathbb{B}} h(z)(1-|z|^2)^{p(p'-1)(b-q)} d\mu_q(z)$$
$$= \int_{\mathbb{B}} h(z)(1-|z|^2)^{[p(p'-1)-1](b-q)} d\mu_b(z)$$
$$= \int_{\mathbb{B}} h(z)(1-|z|^2)^{[p'-1](b-q)} d\mu_b(z).$$

Since $h \in L^p(\omega d\mu_{q+p(p'-1)(b-q)})$, then $g \in L^p(\omega d\mu_q)$, where $g(z) = h(z)(1-|z|^2)^{(p'-1)(b-q)}, \ \forall z \in \mathbb{B}$. So

$$T_{a,b}g(0) = \int_{\mathbb{B}} h(z)(1-|z|^2)^{[p'-1](b-q)} d\mu_b(z) = \infty,$$

contradicting the fact that $T_{a,b}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$. □

Proposition 4.3. *In the case $s+t \leq -1$ and $Q \leq q$, or in the case $s+t+\frac{Q-q}{p} \leq -1$, for any pseudo-ball B that touches the boundary, there are no weights ω so that, both conditions $\omega \in L^1(B, d\mu_{Q+pt})$ and $\omega^{\frac{-1}{p-1}} \in L^1(B, d\mu_{q+p'(s-q)})$ hold at the same time.*

Proof. Let $s+t \leq -1$, $Q \leq q$ and B be a pseudo-ball that touches the boundary. Then

$$\infty = \int_B d\mu_{s+t}(z)$$
$$= \int_B (1-|z|^2)^{s+t} d\mu(z)$$
$$= \int_B (1-|z|^2)^{s-\frac{q}{p}+\frac{q}{p}+t} d\mu(z)$$
$$= \int_B \omega^{\frac{1}{p}}(z)(1-|z|^2)^{\frac{q+pt}{p}} \omega^{-\frac{1}{p}}(z)(1-|z|^2)^{s-\frac{q}{p}} d\mu(z)$$
$$\leq \left(\int_B \omega(z)(1-|z|^2)^{q+pt} d\mu(z) \right)^{\frac{1}{p}} \left(\int_B \omega^{-\frac{1}{p-1}}(z)(1-|z|^2)^{q+p'(s-q)} d\mu(z) \right)^{\frac{1}{p'}}$$
$$\leq \left(\int_B \omega(z)(1-|z|^2)^{Q+pt} d\mu(z) \right)^{\frac{1}{p}} \left(\int_B \omega^{-\frac{1}{p-1}}(z)(1-|z|^2)^{q+p'(s-q)} d\mu(z) \right)^{\frac{1}{p'}}$$

so that necessarily $\omega \notin L^1(d\mu_{Q+pt})$ or $\omega^{\frac{-1}{p-1}} \notin L^1(d\mu_{q+p'(s-q)})$.

Let $s+t+\frac{Q-q}{p} \leq -1$, and B be a pseudo-ball that touches the boundary. Then

$$\infty = \int_B d\mu_{s+t+\frac{Q-q}{p}}(z)$$
$$= \int_B (1-|z|^2)^{s+t+\frac{Q-q}{p}} d\mu(z)$$

$$= \int_B (1-|z|^2)^{s-\frac{q}{p}+\frac{Q}{p}+t} d\mu(z)$$

$$= \int_B \omega^{\frac{1}{p}}(z)(1-|z|^2)^{\frac{Q+pt}{p}} \omega^{-\frac{1}{p}}(z)(1-|z|^2)^{s-\frac{q}{p}} d\mu(z)$$

$$\leq \left(\int_B \omega(z)(1-|z|^2)^{Q+pt} d\mu(z) \right)^{\frac{1}{p}} \left(\int_B \omega^{-\frac{1}{p-1}}(z)(1-|z|^2)^{q+p'(s-q)} d\mu(z) \right)^{\frac{1}{p'}}$$

so that necessarily $\omega \notin L^1(d\mu_{Q+pt})$ or $\omega^{\frac{-1}{p-1}} \notin L^1(d\mu_{q+p'(s-q)})$.

□

4.2. *The case where* $s+t < -N-1$

Here we are going to characterize the boundedness of the operators $P_{s,t}$ from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$, where ω is a weight and $s+t < -N-1$.

Theorem 4.4. *In the case $s+t < -(N+1)$, there are no weights ω such that $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ for $Q \leq q$.*

Proof. This is due to Proposition 4.3, Lemma 4.1 and Lemma 4.2. □

Theorem 4.5. *In the case $s+t < -(N+1)$, if $Q > q$, then $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ if and only if*

$$\left(\int_{\mathbb{B}} \omega(z) d\mu_{Q+pt}(z) \right) \left(\int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z) \right)^{p-1} < \infty.$$

Moreover,

$$\|P_{s,t}\|^p \simeq \left(\int_{\mathbb{B}} \omega(z) d\mu_{Q+pt}(z) \right) \left(\int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z) \right)^{p-1}.$$

Proof. Assume that

$$\left(\int_{\mathbb{B}} \omega(z) d\mu_{Q+pt}(z) \right) \left(\int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z) \right)^{p-1} < \infty.$$

We have

$$\int_{\mathbb{B}} |P_{s,t}f(z)|^p \omega(z) d\mu_Q(z) \lesssim \int_{\mathbb{B}} (1-|z|^2)^{pt} \left(\int_{\mathbb{B}} |f(v)| d\mu_s(v) \right)^p \omega(z) d\mu_Q(z)$$

$$= \int_{\mathbb{B}} \left(\int_{\mathbb{B}} |f(v)| d\mu_s(v) \right)^p \omega(z) d\mu_{Q+pt}(z)$$

$$= \left(\int_{\mathbb{B}} \omega(z) d\mu_{Q+pt}(z) \right) \left(\int_{\mathbb{B}} |f(z)| d\mu_s(z) \right)^p ,$$

where the first inequality is due to the fact that K_{s+t} is bounded when $s + t < -(N + 1)$. We also have

$$\begin{aligned} \left(\int_{\mathbb{B}} |f(z)| d\mu_s(z) \right)^p &= \left(\int_{\mathbb{B}} |f(z)| (\omega(z))^{\frac{-1}{p}} (\omega(z))^{\frac{1}{p}} (1 - |z|^2)^{s-q} d\mu_q(z) \right)^p \\ &\leq \left(\int_{\mathbb{B}} |f(z)|^p \omega(z) d\mu_q(z) \right) \left(\int_{\mathbb{B}} ((\omega(z))^{\frac{-1}{p}})^{p'} |(1 - |z|^2)^{p'(s-q)} d\mu_q(z) \right)^{\frac{p}{p'}} . \end{aligned}$$

Then $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ when

$$\left(\int_{\mathbb{B}} \omega(z) d\mu_{Q+pt}(z) \right) \left(\int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z) \right)^{p-1} < \infty.$$

Now assume that $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$. Let ρ_0 be as in Lemma 2.6, then for positive functions f we have

$$\begin{aligned} \int_{\mathbb{B}} |P_{s,t} f(z)|^p \omega(z) d\mu_Q(z) &= \int_{\mathbb{B}} \left| \int_{\mathbb{B}} K_{s+t}(z, w) f(w) d\mu_s(w) \right|^p \omega(z) d\mu_{Q+pt}(z) \\ &\geq \int_{|z| \leq \rho_0} | \int_{\mathbb{B}} \Re K_{s+t}(z, w) f(w) d\mu_s(w)|^p \omega(z) d\mu_{Q+pt}(z) \\ &= \frac{1}{2^p} \left(\int_{\mathbb{B}} f(w) d\mu_s(w) \right)^p \int_{|z| \leq \rho_0} \omega(z) d\mu_{Q+pt}(z). \end{aligned}$$

By continuity of $P_{s,t}$ there exists a constant $C_{s,t,p,q,Q} > 0$ such that

$$\int_{\mathbb{B}} |P_{s,t} f(z)|^p \omega(z) d\mu_Q(z) \leq C_{s,t,p,q,Q} \int_{\mathbb{B}} |f(z)|^p \omega(z) d\mu_q(z),$$

for all $f \in L^p(\omega d\mu_q)$. Hence,

$$\frac{1}{2^p} \left(\int_{|z| \leq \rho_0} \omega(z) d\mu_{Q+pt}(z) \right) \left(\int_{\mathbb{B}} |f(z)| d\mu_s(z) \right)^p \leq C_{s,t,p,q,Q} \int_{\mathbb{B}} |f(z)|^p \omega(z) d\mu_q(z)$$

for all positive $f \in L^p(\omega d\mu_q)$. Let $f(z) = (\omega(z))^{\frac{-1}{p-1}} (1 - |z|^2)^{(p'-1)(s-q)}, \ \forall z \in \mathbb{B}$. Then $f \in L^p(\omega d\mu_q)$ by Lemma 4.2. Replacing this choice of f in the last inequality we obtain

$$\left(\int_{|z| \leq \rho_0} \omega(z) d\mu_{Q+pt}(z) \right) \left(\int_{\mathbb{B}} (\omega(z)^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z) \right)^{p-1} \leq 2^p C_{s,t,p,q,Q} < \infty.$$

Then

$$\left(\int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z)\right)^{p-1} < \infty.$$

Using Lemma 4.1, we get

$$\left(\int_{\mathbb{B}} \omega(z) d\mu_{Q+pt}(z)\right)\left(\int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z)\right)^{p-1} < \infty.$$

□

5. Weighted estimates: The case where $a > -N-1$ and $s+t > -N-1$

Here we are going to start the study of the boundedness of the operators $T_{a,b}$ and $P_{s,t}$ from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ in the case $a > -(N+1)$ and $s+t > -(N+1)$ respectively.

5.1. *Necessary Conditions*

To obtain our necessary conditions, we are going to use the following lemma. This is an extension to the analogue lemma in [3].

Lemma 5.1. *Let $B_1 = B(w_0, R)$ be a pseudo-ball of sufficiently small radius R touching the boundary of $\mathbb{B}$. There is a pseudo-ball B_2 with the same radius touching the boundary of $\mathbb{B}$, sufficiently far from B_1, but such that $d(B_1, B_2) \simeq R$, for which for all non negative functions f with support in B_i we have if $a > -1-N$*

$$|T_{a,b}f(z)| \geq C_{a,b}\frac{1}{R^{N+1+a}}\int_{B_i} f(w)d\mu_b(w),$$

for all $z \in B_j$, $i \neq j$, $i,j = 1,2$. In particular, if $a > -1$ then

$$|T_{a,b}f(z)| \geq C_{a,b}\frac{1}{\mu_a(B_i)}\int_{B_i} f(w)d\mu_b(w)$$

for all $z \in B_j$, $i \neq j$, $i,j = 1,2$. The constant $C_{a,b}$ does not depend of B_i, B_j or f.

Proof. Let w_0 be the center of B_i. If R is sufficiently small, we take B_j such that for all z in B_j and for all w in B_i, we have $d(z, w_0) \geq C_1 d(w, w_0)$ where C_1 is as in Lemma 2.15. Let f be a non negative function with support in B_i and let $z \in B_j$, we have

$$T_{a,b}f(z) = \frac{1}{(1-\langle z, w_0\rangle)^{N+1+a}}\int_{B_i} f(w)d\mu_b(w) + \int_{B_i}\left[\frac{1}{(1-\langle z, w\rangle)^{N+1+a}} - \frac{1}{(1-\langle z, w_0\rangle)^{N+1+a}}\right] f(w)d\mu_b(w).$$

Then

$$|T_{a,b}f(z)| \geq \frac{1}{|1-\langle z,w_0\rangle|^{N+1+a}} \int\limits_{B_i} f(w)d\mu_b(w)$$
$$- \int\limits_{B_i} \left| \frac{1}{(1-\langle z,w\rangle)^{N+1+a}} - \frac{1}{(1-\langle z,w_0\rangle)^{N+1+a}} \right| f(w)d\mu_b(w)$$

By Lemma 2.15 and Proposition 2.2 we have

$$\left| \frac{1}{(1-\langle z,w\rangle)^{N+1+a}} - \frac{1}{(1-\langle z,w_0\rangle)^{N+1+a}} \right| \leq \frac{1}{2} \frac{1}{|1-\langle z,w_0\rangle|^{N+a+1}},$$

so that

$$|T_{a,b}f(z)| \geq \frac{1}{2} \frac{1}{|1-\langle z,w_0\rangle|^{N+1+a}} \int\limits_{B_i} f(w)d\mu_b(w).$$

Since our pseudo-balls touch the boundary and since $d(B_i, B_j) \simeq R$, we have

$$|1-\langle z,w_0\rangle| \lesssim R.$$

Hence

$$|T_{a,b}f(z)| \gtrsim \frac{1}{2} \frac{1}{R^{N+1+a}} \int\limits_{B_i} f(w)d\mu_b(w).$$

By Lemma 2.3 and [15, Lemma 2.8], if $a > -1$ and because B_i touches the boundary we have $\mu_a(B_i) \simeq R^{N+1+a}$, so that

$$|T_{a,b}f(z)| \gtrsim \frac{1}{2} \frac{1}{\mu_a(B_i)} \int\limits_{B_i} f(w)d\mu_b(w).$$

□

We are now ready to prove Theorem 1.9.

Theorem 5.2. *If $T_{a,b}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ for $Q \leq q$ then if $a > -1$ we have*

$$\sup_{pseudo-balls\ B: B\cap\partial\mathbb{B}\neq\varnothing} \left(\frac{\mu_b(B)}{\mu_a^2(B)} \int\limits_B \omega(z)d\mu_Q(z) \right) \left(\frac{\mu_b(B)}{\mu_a^2(B)} \int\limits_B (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(b-q)}(z) \right)^{p-1} < \infty.$$

More generally if $a > -(N+1)$, then

$$\sup_{pseudo-balls\ B: B\cap\partial\mathbb{B}\neq\varnothing} \left(\frac{\mu_b(B)}{R_B^{2(N+1+a)}} \int\limits_B \omega(z)d\mu_Q(z) \right) \left(\frac{\mu_b(B)}{R_B^{2(N+1+a)}} \int\limits_B (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(b-q)}(z) \right)^{p-1} < \infty,$$

where R_B is the radius of B.

Proof. Assume that $a > -(N+1)$ and $T_{a,b}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ for $Q \leq q$. Then there exists a constant $C_{a,b,p,q,Q} > 0$ such that

$$\int_{\mathbb{B}} |T_{a,b}f(z)|^p \omega(z) d\mu_Q(z) \leq C_{a,b,p,q,Q} \int_{\mathbb{B}} |f(z)|^p \omega(z) d\mu_q(z).$$

Let f be a positive function with support in B_i (we take B_i, B_j as in Lemma 5.1). By Lemma 5.1, we then have in the case $a > -1$

$$C_{a,b}^p \int_{B_j} \frac{1}{\mu_a^p(B_i)} \left(\int_{B_i} f(w) d\mu_b(w) \right)^p \omega(z) d\mu_Q(z) \leq C_{a,b,p,q,Q} \int_{B_i} |f(z)|^p \omega(z) d\mu_q(z),$$

hence

$$\begin{aligned} \frac{1}{\mu_a^p(B_i)} \left(\int_{B_i} f(w) d\mu_b(w) \right)^p \left(\int_{B_j} \omega(z) d\mu_Q(z) \right) & \leq C'_{a,b,p,q,Q} \int_{B_i} |f(z)|^p \omega(z) d\mu_q(z) \\ & \leq C'_{a,b,p,q,Q} \int_{B_i} |f(z)|^p \omega(z) d\mu_Q(z). \end{aligned} \tag{5.1}$$

Choosing $f = \frac{\mu_a(B_i)}{\mu_b(B_i)} \chi_{B_i}$ in the last inequality we get

$$\omega(B_j) \leq C'_{a,b,p,q,Q} \frac{\mu_a^p(B_i)}{\mu_b^p(B_i)} \omega(B_i),$$

where $\omega(B_k) = \int_{B_k} \omega(z) d\mu_Q(z)$, $k = i, j$. As B_i and B_j touch the boundary of $\mathbb{B}$ and have the same radius, by Lemma 2.3 we have

$$\frac{\mu_a^p(B_i)}{\mu_b^p(B_i)} \simeq \frac{\mu_a^p(B_j)}{\mu_b^p(B_j)}.$$

We then have

$$\omega(B_j) \leq C''_{a,b,p,q,Q} \frac{\mu_a^p(B_j)}{\mu_b^p(B_j)} \omega(B_i).$$

Interchanging B_i and B_j (See Lemma 5.1) we get

$$\omega(B_i) \leq C''_{a,b,p,q,Q} \frac{\mu_a^p(B_i)}{\mu_b^p(B_i)} \omega(B_j),$$

so that

$$\frac{\mu_b^p(B_i)}{\mu_a^p(B_i)} \omega(B_i) \leq C''_{a,b,p,q,Q} \omega(B_j)$$

which together with (5.1) leads to

$$\frac{\mu_b^p(B_i)}{\mu_a^{2p}(B_i)}\left(\int_{B_i} f(w)d\mu_b(w)\right)^p\left(\int_{B_i}\omega(z)d\mu_Q(z)\right)\le C'''_{a,b,p,q,Q}\int_{B_i}|f(z)|^p\omega(z)d\mu_q(z).$$

Then choosing $f(z)=\omega^{\frac{-1}{p-1}}(z)(1-|z|^2)^{(p'-1)(b-q)}\chi_{B_i}(z)$ ($f\in L^p(\omega d\mu_q)$ by Lemma 4.2) in that last inequality we obtain

$$\left(\frac{\mu_b(B_i)}{\mu_a^2(B_i)}\int_{B_i}\omega(z)d\mu_Q(z)\right)\left(\frac{\mu_b(B_i)}{\mu_a^2(B_i)}\int_{B_i}(\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(b-q)}(z)\right)^{p-1}\le C'''_{a,b,p,q,Q}.$$

When $-1-N<a<-1$ it is sufficient to replace $\mu_a(B_i)$ by R^{N+1+a} in the proof. □

Theorem 5.3. *If $T_{a,b}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ for $Q>q$ then if $a>-1$ we have*

$$\sup_{B:B\cap\partial\mathbb{B}\neq\varnothing}\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)}\int_B\omega(z)d\mu_Q(z)\right)\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)}\int_B(\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(b-q)}(z)\right)^{p-1}<\infty.$$

where the supremum is taken over the pseudo-balls B.

More generally if $a>-(N+1)$, then

$$\sup_{B:B\cap\partial\mathbb{B}\neq\varnothing}\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{R_B^{2(N+1+a)}}\int_B\omega(z)d\mu_Q(z)\right)\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{R_B^{2(N+1+a)}}\int_B(\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(b-q)}(z)\right)^{p-1}<\infty$$

where the supremum is taken over the pseudo-balls B with radius R_B.

Proof. The proof is similar to the proof of Theorem 5.2, except that we choose the first testing function to be $f(z)=(1-|z|^2)^{\frac{Q-q}{p}}\chi_{B_i}(z)$. □

Theorem 5.4. *In the case $-(N+1)<s+t<-1$ and $Q\le q$, or in the case $s+t+\frac{Q-q}{p}\le -1$, there are no weights ω such that $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$.*

Proof. This is due to Proposition 4.3, Lemma 4.1 and Lemma 4.2. □

Theorem 5.5. *In the case $-(N+1)<s+t<-1$, with $s+t+\frac{Q-q}{p}>-1$ if $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ then*

$$\sup_{B:B\cap\partial\mathbb{B}\neq\varnothing}\left(\frac{R_B^{\frac{Q-q}{p}}}{R_B^{N+1+s+t}}\int_B\omega(z)d\mu_{Q+pt}(z)\right)\left(\frac{R_B^{\frac{Q-q}{p}}}{R_B^{N+1+s+t}}\int_B(\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(s-q)}(z)\right)^{p-1}<\infty$$

where the supremum is taken over the pseudo-balls B with radius R_B.

Proof. Assume that $-1>s+t>-(N+1)$, with $s+t+\frac{Q-q}{p}>-1$ and $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$. Then there exists a constant $C_{s,t,p,q,Q}>0$ such that

$$\int_{\mathbb{B}}|P_{s,t}f(z)|^p\omega(z)d\mu_Q(z)\le C_{s,t,p,q,Q}\int_{\mathbb{B}}|f(z)|^p\omega(z)d\mu_q(z).$$

Let f be a positive function with support in B_i (we take B_i, B_j of radius R as in Lemma 5.1). By Lemma 5.1, we then have

$$C_{s,t}^p \int_{B_j} \frac{1}{R^{(N+1+s+t)p}} \left(\int_{B_i} f(w) d\mu_s(w) \right)^p \omega(z) d\mu_{Q+pt}(z) \leq C_{s,t,p,q} \int_{B_i} |f(z)|^p \omega(z) d\mu_q(z),$$

hence

$$\frac{1}{R^{(N+1+s+t)p}} \left(\int_{B_i} f(w) d\mu_s(w) \right)^p \left(\int_{B_j} \omega(z) d\mu_{Q+pt}(z) \right) \leq C'_{s,t,p,q} \int_{B_i} |f(z)|^p \omega(z) d\mu_q(z). \qquad (5.2)$$

Choosing $f(z) = (1-|z|^2)^{\frac{Q-q}{p}+t} \chi_{B_i}(z)$ in the last inequality we get, because $N+s+t > -1$,

$$R^{Q-q} \int_{B_j} \omega(z) d\mu_{Q+pt}(z) \leq \int_{B_i} \omega(z) d\mu_{Q+pt}(z).$$

Interchanging B_i and B_j (See Lemma 5.1) we get

$$R^{Q-q} \int_{B_i} \omega(z) d\mu_{Q+pt}(z) \lesssim \int_{B_j} \omega(z) d\mu_{Q+pt}(z),$$

which together with (5.2) lead to

$$\frac{R^{Q-q}}{R^{p(N+1+s+t)}} \left(\int_{B_i} f(w) d\mu_s(w) \right)^p \left(\int_{B_i} \omega(z) d\mu_{Q+pt}(z) \right) \leq C'_{s,t,p,q,Q} \int_{B_i} |f(z)|^p \omega(z) d\mu_q(z).$$

Then choosing $f(z) = \omega^{\frac{-1}{p-1}}(z)(1-|z|^2)^{(p'-1)(s-q)} \chi_{B_i}(z)$ ($f \in L^p(\omega d\mu_q)$ by Lemma 4.2) in that last inequality, we obtain

$$\left(\frac{R^{\frac{Q-q}{p}}}{R^{N+1+s+t}} \int_{B_i} \omega(z) d\mu_{Q+pt}(z) \right) \left(\frac{R^{\frac{Q-q}{p}}}{R^{N+1+s+t}} \int_{B_i} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z) \right)^{p-1} \leq C'''_{s,t,p,q,Q}.$$

□

In the same way we have

Theorem 5.6. *If $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ for $Q \geq q$, then if $s+t > -1$ we have*

$$\sup_{pseudo-balls\ B: B\cap\partial\mathbb{B}\neq\varnothing} \left(\frac{R_B^{\frac{Q-q}{p}}}{\mu_{s+t}(B)} \int_B \omega(z) d\mu_{Q+pt}(z) \right) \left(\frac{R_B^{\frac{Q-q}{p}}}{\mu_{s+t}(B)} \int_B (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z) \right)^{p-1} < \infty$$

where R_B is the radius of B.

Proof. The proof is similar to the proof of Theorem 5.5, we choose the first testing function to be $f(z) = (1-|z|^2)^{\frac{Q-q}{p}+t}\chi_{B_i}$. □

Theorem 5.7. *In the case $s+t > -1$ there are no weights ω such that $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ for $Q < q$.*

Proof. We are going to proceed in two steps.

Step 1: Show that in the case $s+t > -1$, if $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$, for $Q < q$, then we have

$$\sup_{pseudo-balls\ B: B\cap\partial\mathbb{B}\neq\emptyset} \left(\frac{1}{\mu_{s+t}(B)}\int_B \omega(z)d\mu_{Q+pt}(z)\right)\left(\frac{1}{\mu_{s+t}(B)}\int_B (\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(s-q)}(z)\right)^{p-1} < \infty.$$

Assume that $s+t > -1$ and $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ for $Q < q$. There exists a constant $C_{s,t,p,q,Q} > 0$ such that

$$\int_{\mathbb{B}} |P_{s,t}f(z)|^p\omega(z)d\mu_Q(z) \leq C_{s,t,p,q,Q}\int_{\mathbb{B}} |f(z)|^p\omega(z)d\mu_q(z), \quad f\in L^p(\omega d\mu_q).$$

Let f be a positive function with support in B_i (we take B_i, B_j as in Lemma 5.1). By Lemma 5.1, we then have

$$\frac{1}{\mu^p_{s+t}(B_i)}\left(\int_{B_i} f(w)d\mu_s(w)\right)^p\left(\int_{B_j}\omega(z)d\mu_{Q+pt}(z)\right) \leq C'_{s,t,p,q,Q}\int_{B_i}|f(z)|^p\omega(z)d\mu_q(z) \tag{5.3}$$
$$\leq C'_{s,t,p,q,Q}\int_{B_i}|f(z)|^p\omega(z)d\mu_Q(z).$$

Choosing $f(z) = (1-|z|^2)^t\chi_{B_i}(z)$ in the last inequality we get

$$\int_{B_j}\omega(z)d\mu_{Q+pt}(z) \leq \int_{B_i}\omega(z)d\mu_{Q+pt}(z).$$

Interchanging B_i and B_j (see Lemma 5.1) we get

$$\int_{B_i}\omega(z)d\mu_{Q+pt}(z) \leq \int_{B_j}\omega(z)d\mu_{Q+pt}(z),$$

which together with (5.3) lead to

$$\frac{1}{\mu^p_{s+t}(B_i)}\left(\int_{B_i} f(w)d\mu_s(w)\right)^p\left(\int_{B_i}\omega(z)d\mu_{Q+pt}(z)\right) \leq C'_{s,t,p,q,Q}\int_{B_i}|f(z)|^p\omega(z)d\mu_q(z).$$

Then choosing $f(z) = (\omega^{\frac{-1}{p-1}}(z))(1-|z|^2)^{(p'-1)(s-q)}\chi_{B_i}(z)$ ($f \in L^p(\omega d\mu_q)$ by Lemma 4.2) in that last inequality we obtain

$$\left(\frac{1}{\mu_{s+t}(B)}\int_B \omega(z)d\mu_{Q+pt}(z)\right)\left(\frac{1}{\mu_{s+t}(B)}\int_B (\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(s-q)}(z)\right)^{p-1} \le C'''_{s,t,p,q,Q}.$$

Step 2: Show that

$$\sup_{pseudo-balls\ B:B\cap\partial\mathbb{B}\neq\varnothing}\left(\frac{1}{\mu_{s+t}(B)}\int_B \omega(z)d\mu_{Q+pt}(z)\right)\left(\frac{1}{\mu_{s+t}(B)}\int_B (\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(s-q)}(z)\right)^{p-1} = \infty.$$

Let

$$II = \left(\frac{1}{\mu_{s+t}(B)}\int_B \omega(z)d\mu_{Q+pt}(z)\right)\left(\frac{1}{\mu_{s+t}(B)}\int_B (\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(s-q)}(z)\right)^{p-1}.$$

Let B be a pseudo-ball that touches the boundary and R_B its radius. By Hölder's inequality we have

$$\begin{aligned}
\mu_{s+t}(B) &= \int_B d\mu_{s+t}(z) \\
&= \int_B (1-|z|^2)^{s+t}d\mu(z) \\
&= \int_B (1-|z|^2)^{s-\frac{q}{p}+\frac{q}{p}+t}d\mu(z) \\
&= \int_B \omega^{\frac{1}{p}}(z)(1-|z|^2)^{\frac{q+pt}{p}}\omega^{-\frac{1}{p}}(z)(1-|z|^2)^{s-\frac{q}{q}}d\mu(z) \\
&\le \left(\int_B \omega(z)(1-|z|^2)^{q+pt}d\mu(z)\right)^{\frac{1}{p}}\left(\int_B \omega^{-\frac{1}{p-1}}(z)(1-|z|^2)^{q+p'(s-q)}d\mu(z)\right)^{\frac{1}{p'}}.
\end{aligned}$$

Note that for $z \in B$ we have $1-|z| < 2R_B$. Then

$$\begin{aligned}
\mu^p_{s+t}(B) &\le \left(\int_B \omega(z)(1-|z|^2)^{q+pt}d\mu(z)\right)\left(\int_B \omega^{-\frac{1}{p-1}}(z)(1-|z|^2)^{q+p'(s-q)}d\mu(z)\right)^{p-1} \\
&= \left(\int_B \omega(z)(1-|z|^2)^{q-Q+Q+pt}d\mu(z)\right)\left(\int_B \omega^{-\frac{1}{p-1}}(z)(1-|z|^2)^{q+p'(s-q)}d\mu(z)\right)^{p-1} \\
&\le (2R_B)^{q-Q}\left(\int_B \omega(z)(1-|z|^2)^{Q+pt}d\mu(z)\right)\left(\int_B \omega^{-\frac{1}{p-1}}(z)(1-|z|^2)^{q+p'(s-q)}d\mu(z)\right)^{p-1}.
\end{aligned}$$

Hence

$$(2R_B)^{Q-q} \le \frac{1}{\mu_{s+t}^p(B)} \left(\int_B \omega(z)(1-|z|^2)^{Q+pt} d\mu(z) \right) \left(\int_B \omega^{-\frac{1}{p-1}}(z)(1-|z|^2)^{q+p'(s-q)} d\mu(z) \right)^{p-1}$$
$$= II.$$

Taking the $\sup$ over smaller radii, we get $\sup II = \infty$. □

5.2. *The Associated Maximal and Fractional Maximal Functions*

We introduce for $b > -1$ and $s > -1$ the following maximal functions. If $a > -1$ we set

$$m_{a,b}f(z) = \sup_{\zeta\in\mathbb{B}, R>1-|\zeta|: z\in B(\zeta,R)} \frac{1}{\mu_a(B(\zeta,R))} \int_{B(\zeta,R)} |f(w)| d\mu_b(w), \tag{5.4}$$

more generally if $a > -1-N$ we set

$$m'_{a,b}f(z) = \sup_{\zeta\in\mathbb{B}, R>1-|\zeta|: z\in B(\zeta,R)} \frac{1}{R^{N+1+a}} \int_{B(\zeta,R)} |f(w)| d\mu_b(w). \tag{5.5}$$

If $a > -1$ we set

$$M_{a,b}f(z) = \sup_{B:z\in B} \frac{1}{\mu_a(B)} \int_B |f(w)| d\mu_b(w), \tag{5.6}$$

and more generally if $a > -1-N$ we set

$$M'_{a,b}f(z) = \sup_{B:z\in B} \frac{1}{R^{N+1+a}} \int_B |f(w)| d\mu_b(w). \tag{5.7}$$

If $s+t > -1$ we set

$$O_{s,t}f(z) = (1-|z|^2)^t \sup_{\zeta\in\mathbb{B}, R>1-|\zeta|: z\in B(\zeta,R)} \frac{1}{\mu_{s+t}(B(\zeta,R))} \int_{B(\zeta,R)} |f(w)| d\mu_s(w), \tag{5.8}$$

more generally if $s+t > -1-N$ we set

$$O'_{s,t}f(z) = (1-|z|^2)^t \sup_{\zeta\in\mathbb{B}, R>1-|\zeta|: z\in B(\zeta,R)} \frac{1}{R^{N+1+s+t}} \int_{B(\zeta,R)} |f(w)| d\mu_s(w). \tag{5.9}$$

Let finally define the following fractional maximal function

$$M_\gamma f(z) = \sup_{B:z\in B} \frac{1}{\mu_b^{1-\gamma}(B)} \int_B |f(w)| d\mu_b(w), \quad \gamma \in (0,1). \tag{5.10}$$

Note that for $a < b$ we get by Lemma 2.3 $M_{a,b} \sim M_\gamma$ with $\gamma = 1 - \frac{N+1+a}{N+1+b}$.

For all $k \in (0,1)$, we define the operator of regularisation R_k^b

$$R_k^b f(z) = \frac{1}{\mu_b(B_k(z))} \int\limits_{B_k(z)} f(\zeta) d\mu_b(\zeta), \tag{5.11}$$

where $B_k(z) = \{w \in \mathbb{B} : d(z,w) < k(1-|z|)\}$.

We will need the following lemmas to show Theorem 1.14 (see [3]).

Lemma 5.8. *Let $k \in (0, \frac{1}{2})$. If $z' \in B_k(z)$, then $z \in B_{k'}(z')$, where $k' = \frac{k}{1-k}$.*

Proof. We have

$$||z| - |z'|| \leq d(z,z') < k(1-|z|).$$

Therefore $1 - |z| \leq \frac{1}{1-k}(1-|z'|)$, so that $d(z,z') < k(1-|z|) \leq \frac{k}{1-k}(1-|z'|)$. □

Lemma 5.9. *If $B := B(x,R)$ touches the boundary then if we take $B' = B(x, K(1+2k_1)R)$, then for all $w \in B$, $B_{k_1}(w) \subset B'$ (See (2.1) for the definition of K).*

Proof. Let $w \in B$ and $z \in B_{k_1}(w)$. We have

$$\begin{aligned} d(x,z) &\leq K(d(x,w) + d(w,z)) \\ &\leq K\left(R + k_1(1-|w|)\right) \\ &\leq K\left(R + k_1(2R)\right) = K(1+2k_1)R. \end{aligned}$$

□

The next three lemmas are generalizations of their analogues in [3] to the maximal functions $m'_{a,b}$ and $O'_{s,t}$. For the sake of completeness and for the reader convenience, we include their proofs.

Lemma 5.10. *For all $k \in (0,1)$, there is a constant C_k such that for all positive locally integrable function f we have if $a > -1$*

$$m_{a,b} f \leq C_k m_{a,b}(R_k^b f),$$

and more generally if $a > -1 - N$

$$m'_{a,b} f \leq C_k m'_{a,b}(R_k^b f).$$

Proof. We have to show that for all z and all pseudo-balls B containing z which touch the boundary of $\mathbb{B}$, there is a pseudo-ball B' containing z, which touches the boundary of $\mathbb{B}$ so that

$$\frac{1}{\mu_a(B)} \int\limits_B f(w) d\mu_b(w) \leq C_k \frac{1}{\mu_a(B')} \int\limits_{B'} \left[\frac{1}{\mu_b(B_k(w))} \int\limits_{B_k(w)} f(\zeta) d\mu_b(\zeta) \right] d\mu_b(w).$$

By Lemma 5.8, $\chi_{B_k(w)}(\zeta) \geq \chi_{B_{k_1}(\zeta)}(w)$, where $k_1 = \frac{k}{k+1}$. If $B = B(x,R)$ $(R > 1 - |x|)$, by Lemma 5.9, for $B' = B(x, K(1+2k_1)R)$, we have $\forall w \in B, B_{k_1}(w) \subset B'$. Note that

$$\mu_b(B_{k_1}(\zeta)) \simeq \mu_b(B_k(w)) \ \text{ when } \ \mathrm{w} \in \mathrm{B}_{\mathrm{k}_1}(\zeta). \tag{5.12}$$

Hence

$$\begin{aligned}
\int_{B'} R_k^b f(w) d\mu_b(w) &= \int_{B'} \left[\frac{1}{\mu_b(B_k(w))} \int_{B_k(w)} f(\zeta) d\mu_b(\zeta) \right] d\mu_b(w) \\
&= \int_{B'} \left[\frac{1}{\mu_b(B_k(w))} \int_{B} f(\zeta) \chi_{B_k(w)}(\zeta) d\mu_b(\zeta) \right] d\mu_b(w) \\
&\geq \int_{B'} \left[\frac{1}{\mu_b(B_k(w))} \int_{B} f(\zeta) \chi_{B_{k_1}(\zeta)}(w) d\mu_b(\zeta) \right] d\mu_b(w) \\
&= \int_{B} \left[\int_{B'} \frac{1}{\mu_b(B_k(w))} \chi_{B_{k_1}(\zeta)}(w) d\mu_b(w) \right] f(\zeta) d\mu_b(\zeta).
\end{aligned}$$

Using (5.12), we get

$$\begin{aligned}
\int_{B'} R_k^b f(w) d\mu_b(w) &\gtrsim \int_{B} \frac{1}{\mu_b(B_{k_1}(\zeta))} \left[\int_{B'} \chi_{B_{k_1}(\zeta)}(w) d\mu_b(w) \right] f(\zeta) d\mu_b(\zeta) \\
&= \int_{B} f(\zeta) d\mu_b(\zeta).
\end{aligned}$$

Since μ_a is a homogeneous measure we have

$$\frac{1}{\mu_a(B)} \int_{B} f(w) d\mu_b(w) \lesssim \frac{1}{\mu_a(B')} \int_{B'} R_k^b f(w) d\mu_b(w).$$

For $m'_{a,b}$ it is sufficient to observe that B and B' have equivalent radii. □

The following lemma appears as a corollary of the preceding one by observing that

$$O_{s,t} f(z) = (1 - |z|^2)^t m_{s+t,s} f(z)$$

and

$$O'_{s,t} f(z) = (1 - |z|^2)^t m'_{s+t,s} f(z).$$

Lemma 5.11. *For all $k \in (0,1)$, there is a constant C_k such that for all positive locally integrable functions f we have if $s + t > -1$*

$$O_{s,t} f \leq C_k O_{s,t}(R_k^s f),$$

and more generally if $s + t > -1 - N$

$$O'_{s,t} f \leq C_k O'_{s,t}(R_k^s f).$$

One can find the following lemma in [3] but for $b = Q$.

Lemma 5.12. *For all $k \in (0, \frac{1}{2})$, there are two constants C and $k' < 1$ depending only on k, b, Q, N such that for all $f, g \in L^1(d\mu_b)$, $f \geq 0, g \geq 0$*

$$\int_{\mathbb{B}} f(z)[R_k^b g(z)]d\mu_Q(z) \leq C \int_{\mathbb{B}} g(z)[R_{k'}^{b,Q} f(z)]d\mu_b(z),$$

where

$$R_{k'}^{b,Q} f(z) = \frac{1}{\mu_b(B_{k'}(z))} \int_{B_{k'}(z)} f(\zeta)d\mu_Q(\zeta). \tag{5.13}$$

Proof. By Lemma 5.8, $\chi_{B_k(z)}(w) \leq \chi_{B_{k'}(w)}(z)$, where $k' = \frac{k}{1-k}$. Because of (5.12) there is a constant C such that

$$\frac{1}{\mu_b(B_k(z))}\chi_{B_k(z)}(w) \leq \frac{C}{\mu_b(B_{k'}(w))}\chi_{B_{k'}(w)}(z).$$

We want to form the quantity $f(z)[R_k^b g(z)]$ on the left while controlling it on the right in order to use Fubini's theorem to bring out the quantity $g(z)[R_{k'}^{b,Q} f(z)]$. Then, for $w \in B_k(z)$

$$\frac{1}{\mu_b(B_k(z))}\chi_{B_k(z)}(w)g(w) \leq \frac{C}{\mu_b(B_{k'}(w))}\chi_{B_{k'}(w)}(z)g(w).$$

We form $R_k^b g(z)$ on the left

$$\int_{B_k(z)} \frac{1}{\mu_b(B_k(z))}\chi_{B_k(z)}(w)g(w)d\mu_b(w) \leq \int_{B_k(z)} \frac{C}{\mu_b(B_{k'}(w))}\chi_{B_{k'}(w)}(z)g(w)d\mu_b(w),$$

by a multiplication by $f(z)$ we have

$$f(z)R_k^b g(z) \leq Cf(z) \int_{B_k(z)} \frac{1}{\mu_b(B_{k'}(w))}\chi_{B_{k'}(w)}(z)g(w)d\mu_b(w).$$

After integration, we obtain

$$\int_{\mathbb{B}} f(z)R_k^b g(z)d\mu_Q(z) \leq C \int_{\mathbb{B}} \left[\int_{B_k(z)} \frac{1}{\mu_b(B_{k'}(w))}\chi_{B_{k'}(w)}(z)g(w)f(z)d\mu_b(w)\right] d\mu_Q(z).$$

Recall that $(z \in \mathbb{B}$ and $w \in B_k(z)) \Longrightarrow (z \in B_{k'}(w)$ and $w \in \mathbb{B})$, hence using Fubini's theorem

$$\int_{\mathbb{B}} f(z)R_k^b g(z)d\mu_Q(z) \leq C \int_{\mathbb{B}} g(w) \left[\int_{B_{k'}(w)} \frac{1}{\mu_b(B_{k'}(w))}\chi_{B_{k'}(w)}(z)f(z)d\mu_Q(z)\right] d\mu_b(w),$$

hence

$$\int_{\mathbb{B}} f(z)R_k^b g(z)d\mu_Q(z) \leq C \int_{\mathbb{B}} g(w) \left[\frac{1}{\mu_b(B_{k'}(w))} \int_{B_{k'}(w)} f(z)d\mu_Q(z)\right] d\mu_b(w),$$

then

$$\int_{\mathbb{B}} f(z)[R_k^b g(z)]d\mu_Q(z) \leq C \int_{\mathbb{B}} g(z)[R_{k'}^{b,Q} f(z)]d\mu_b(z).$$

□

The following result will be used in the proof of Theorem 1.14. This extends Lemma 9 in [3]. We give a proof for the reader convenience.

Lemma 5.13. *Let $k \in (0,1)$. There are two constants c, C depending only on a, b, N, k such that for all positive locally integrable functions g if $a > -1$*

$$c\, m_{a,b} g \leq R_k^b(m_{a,b} g) \leq C m_{a,b} g,$$

and more generally if $a > -1 - N$

$$c\, m'_{a,b} g \leq R_k^b(m'_{a,b} g) \leq C m'_{a,b} g.$$

Proof. It is sufficient to show that there are two constants $0 < c < C$ such that $\forall w \in B_k(z)$

$$c\, m_{a,b} g(z) \leq m_{a,b} g(w) \leq C m_{a,b} g(z).$$

We are going to show the two inequalities. More precisely we are going to show that there are two constants $0 < c < C$ such that, for each pseudo-ball B containing z and touching the boundary, there is a pseudo-ball B' containing w and touching the boundary so that

$$\frac{c}{\mu_a(B)} \int_B |g(\zeta)| d\mu_b(\zeta) \leq \frac{1}{\mu_a(B')} \int_{B'} |g(\zeta)| d\mu_b(\zeta)$$

and show for each pseudo-ball B containing z touching the boundary, there is a pseudo-ball B' containing w touching the boundary so that

$$\frac{1}{\mu_a(B)} \int_B |g(\zeta)| d\mu_b(\zeta) \leq \frac{C}{\mu_a(B')} \int_{B'} |g(\zeta)| d\mu_b(\zeta).$$

In each case, by Lemma 5.9, it is sufficient if $B = B(x, R)$ to take $B' = B(x, K(1 + 2kK)R)$. For the result with $m'_{a,b}$ it is sufficient to notice that B and B' have equivalent radii. □

In the same way as Lemma 5.13, since $O_{s,t} f(z) := (1 - |z|^2)^t m_{s+t,s} f(z)$ and $O'_{s,t} f(z) := (1 - |z|^2)^t m'_{s+t,s} f(z)$, we obtain the following result.

Lemma 5.14. *Let $k \in (0,1)$. There are two constants c, C depending only on s, t, N, k such that for all locally integrable function g if $s + t > -1$*

$$c\, O_{s,t} g \leq R_k^b(O_{s,t} g) \leq C O_{s,t} g,$$

and more generally if $s + t > -1 - N$

$$c\, O'_{s,t} g \leq R_k^b(O'_{s,t} g) \leq C O'_{s,t} g.$$

Now we give a useful characterization of elements in $(B_p^{a,b,q,Q})$, see Definition 1.7.

Lemma 5.15. *For $a > -1$ and $Q > q$, $\omega \in (B_p^{a,b,q,Q})$ $(b > -1)$ if and only if there is a constant $C_{a,b,p,q,Q} > 0$ such that*

$$\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)}\int_B f(z)d\mu_b(z)\right)^p \int_B \omega(z)d\mu_Q(z) \leq C_{a,b,p,q,Q}\int_B f^p(z)\omega(z)d\mu_q(z).$$

More generally if $a > -1-N$ and $Q > q$, $\omega \in (B_p^{a,b,q,Q})$ $(b > -1)$ if and only if there is a constant $C_{a,b,p,q,Q} > 0$ such that

$$\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{R^{2(N+1+a)}}\int_B f(z)d\mu_b(z)\right)^p \int_B \omega(z)d\mu_Q(z) \leq C_{a,b,p,q,Q}\int_B f^p(z)\omega(z)d\mu_q(z),$$

for all positive $f \in L^p(\omega d\mu_q)$ and all pseudo-balls B of radius R that touch the boundary.

Proof. Assume $\omega \in (B_p^{a,b,q,Q})$. Let $0 \leq f \in L^p(\omega d\mu_q)$ and B be a pseudo-ball of radius R such that $R > 1-|z|$. Then

$$\begin{aligned}\left(\int_B f(z)d\mu_b(z)\right)^p &= \left(\int_{\mathbb{B}} f(z)(1-|z|^2)^{b-q}d\mu_q(z)\right)^p\\ &= \left(\int_B f(z)(\omega(z))^{\frac{-1}{p}}(\omega(z))^{\frac{1}{p}}(1-|z|^2)^{b-q}d\mu_q(z)\right)^p\\ &\leq \left(\int_B f^p(z)\omega(z)d\mu_q(z)\right)\left(\int_B ((\omega(z))^{\frac{-1}{p}})^{p'}(1-|z|^2)^{p'(b-q)}d\mu_q(z)\right)^{\frac{p}{p'}}\\ &= \left(\int_B f^p(z)\omega(z)d\mu_q(z)\right)\left(\int_B \omega^{\frac{-1}{p-1}}(z)d\mu_{q+p'(b-q)}(z)\right)^{p-1}.\end{aligned}$$

Hence

$$\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)}\int_B f(z)d\mu_b(z)\right)^p \omega(B) \leq$$
$$\left(\int_B f^p(z)\omega(z)d\mu_q(z)\right)\left[\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)}\omega(B)\right]\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)}\int_B \omega^{\frac{-1}{p-1}}(z)d\mu_{q+p'(b-q)}(z)\right)^{p-1}$$

and because $\omega \in B_p^{a,b,q,Q}$, there is a constant $C_{a,b,p,q,Q} > 0$ such that

$$\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)}\int_B f(z)d\mu_b(z)\right)^p \omega(B) \leq C_{a,b,p,q,Q}\int_B f^p(z)\omega(z)d\mu_q(z).$$

For the general case it is sufficient to replace $\mu_a(B)$ with R^{N+1+a}.

If we assume that there is a constant $C_{a,b,p,q,Q} > 0$ such that

$$\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{R^{2(N+1+a)}} \int_B f(z) d\mu_b(z) \right)^p \omega(B) \leq C_{a,b,p,q,Q} \int_B f^p(z) \omega(z) d\mu_q(z),$$

for all positive $f \in L^p(\omega d\mu_q)$ and all pseudo-balls B of radius R such that $R > 1 - |z|$ it is sufficient to take $f(z) = (1 - |z|^2)^{(p'-1)(b-q)} \omega^{\frac{-1}{p-1}}(z) \chi_B(z)$ to get $\omega \in (B_p^{a,b,q,Q})$. □

Remark 5.16. The result remains true even if B almost touches the boundary.

In the same way, for $(D_p^{s,t,q,Q})$ (see Definition 1.11), we have the following lemma.

Lemma 5.17. *For $Q \geq q$ and $s + t > -1$, $\omega \in (D_p^{s,t,q,Q})$ ($s > -1$) if and only if there is a constant $C_{s,t,p,q,Q} > 0$ such that*

$$\left(\frac{1}{\mu_{s+t+\frac{Q-q}{p}}(B)} \int_B f(z) d\mu_s(z) \right)^p \int_B \omega(z) d\mu_{Q+pt}(z) \leq C_{s,t,p,q,Q} \int_B f^p(z) \omega(z) d\mu_q(z),$$

for all positive $f \in L^p(\omega d\mu_q)$ and all pseudo-balls B that touch the boundary.

For $s + t + \frac{Q-q}{p} > -1$ and $-1 > s + t > -1 - N$, $\omega \in (D_p^{s,t,q,Q})$ ($s > -1$) if and only if there is a constant $C_{s,t,p,q,Q} > 0$ such that

$$\left(\frac{1}{\mu_{s+t+\frac{Q-q}{p}}(B)} \int_B f(z) d\mu_s(z) \right)^p \int_B \omega(z) d\mu_{Q+pt}(z) \leq C_{s,t,p,q,Q} \int_B f^p(z) \omega(z) d\mu_q(z),$$

for all positive $f \in L^p(\omega d\mu_q)$ and all pseudo-balls B that touch the boundary.

Remark 5.18. The result remains true even if B almost touches the boundary.

Corollary 5.19. *For $C_1 > 1$, if $\omega \in D_p^{s,t,q,Q}$ then there exists a constant $C_2 > 0$ such that for any pseudo-ball $B := B(y, r)$ which touches or almost touches the boundary, we have*

$$\int_{B(y,C_1 r)} \omega(\zeta) d\mu_{Q+pt}(\zeta) \leq C_2 \int_{B(y,r)} \omega(\zeta) d\mu_{Q+pt}(\zeta).$$

Proposition 5.20. *Let X be an homogeneous space. Let w be a weight in X. For $-N - 1 < a \leq b$, $Q > q$ and $k \in (0, \frac{1}{2})$, assume that there exists a constant $C_2 > 0$ such that*

$$\left(\int_B [M_\gamma(\chi_B u^{1-p'})(x)]^r v(x) d\nu(x) \right)^{\frac{1}{r}} \leq C_2 \left(\int_B u^{1-p'}(x) d\nu(x) \right)^{\frac{1}{p}} \tag{5.14}$$

for any pseudo-ball $B \subset X$, where $v(z) = R_{k'}^{b,Q}\omega(z)$, $u(z) = R_{k'}^{b,Q}\omega(z)(1-|z|^2)^{2p(b-a)+Q-q}$, $d\nu = d\mu_b$ $p = r$ and $\gamma = 1 - \frac{N+1+a}{N+1+b}$. If $\omega \in (B_p^{a,b,q,Q})$, there is a constant $C_{a,b,p,q,Q} > 0$ such that for all $f \in L^p(\omega d\mu_q)$,

$$\int_{\mathbb{B}} (m_{a,b}f(z))^p \, \omega(z) d\mu_Q(z) \le \int_{\mathbb{B}} |f(z)|^p \omega(z) d\mu_q(z).$$

Proof. Let set

$$III = \int_{\mathbb{B}} (m_{a,b}f(z))^p \, \omega(z) d\mu_Q(z).$$

Using in this order Lemma 5.10, Lemma 5.13, Hölder's inequality and Lemma 5.12 we have,

$$\begin{aligned}
III &\le C_k^p \int_{\mathbb{B}} (m_{a,b}R_k^b f(z))^p \omega(z) d\mu_Q(z) \\
&\le C_k^p A^p \int_{\mathbb{B}} \left[R_k^b(m_{a,b}R_k^b f(z))\right]^p \omega(z) d\mu_Q(z) \\
&\le C_k^p A^p \int_{\mathbb{B}} R_k^b \left[(m_{a,b}R_k^b f(z))^p\right] \omega(z) d\mu_Q(z) \\
&\le C_k^p A^p C \int_{\mathbb{B}} (m_{a,b}R_k^b f(z))^p R_{k'}^{b,Q}\omega(z) d\mu_b(z).
\end{aligned}$$

For $a \le b$, assume that there exists a constant $C_2 > 0$ such that

$$\left(\int_B [M_\gamma(\chi_B u^{1-p'})(x)]^r v(x) d\nu(x)\right)^{\frac{1}{r}} \le C_2 \left(\int_B u^{1-p'}(x) d\nu(x)\right)^{\frac{1}{p}}$$

for any ball $B \subset X$, where $v(z) = R_{k'}^{b,Q}\omega(z)$, $u(z) = R_{k'}^{b,Q}\omega(z)(1-|z|^2)^c$, $d\nu = d\mu_b$ $p = r$ and $\gamma = 1 - \frac{N+1+a}{N+1+b}$, with c to be determined. Then we have

$$\begin{aligned}
\int_{\mathbb{B}} (m_{a,b}f(z))^p \omega(z) d\mu_Q(z) &\le C_k^p A^p C \int_{\mathbb{B}} (m_{a,b}R_k^b f(z))^p R_{k'}^{b,Q}\omega(z) d\mu_b(z) \\
&\le C_k^p A^p C \int_{\mathbb{B}} (M_{a,b}R_k^b f(z))^p R_{k'}^{b,Q}\omega(z) d\mu_b(z) \\
&\lesssim C_k^p A^p C \int_{\mathbb{B}} (M_\gamma R_k^b f(z))^p R_{k'}^{b,Q}\omega(z) d\mu_b(z) \\
&\le C_k^p A^p C' \int_{\mathbb{B}} (R_k^b f(z))^p (1-|z|^2)^c R_{k'}^{b,Q}\omega(z) d\mu_b(z),
\end{aligned}$$

where for the last inequality we used Theorem 2.9. Now let us control $(R_k^b f(z))^p R_{k'}^{b,Q}\omega(z)$. We have

$$(R_k^b f(z))^p R_{k'}^{b,Q}\omega(z) = \left(\frac{1}{\mu_b(B_k(z))} \int_{B_k(z)} f(\zeta) d\mu_b(\zeta)\right)^p \left(\frac{1}{\mu_b(B_{k'}(z))} \int_{B_{k'}(z)} \omega(\zeta) d\mu_Q(\zeta)\right)$$

$$\leq \left(\frac{1}{\mu_b(B_k(z))} \int\limits_{B_{k'}(z)} f(\zeta) d\mu_b(\zeta) \right)^p \left(\frac{1}{\mu_b(B_{k'}(z))} \int\limits_{B_{k'}(z)} \omega(\zeta) d\mu_Q(\zeta) \right)$$

$$\lesssim \left(\frac{1}{\mu_b(B_{k'}(z))} \int\limits_{B_{k'}(z)} f(\zeta) d\mu_b(\zeta) \right)^p \left(\frac{1}{\mu_b(B_{k'}(z))} \int\limits_{B_{k'}(z)} \omega(\zeta) d\mu_Q(\zeta) \right),$$

where the second inequality is because $k' > k$, the third one is because $\mu_b(B_{k'}(z)) \backsimeq \mu_b(B_k(z))$. Then

$$(R_k^b f(z))^p R_{k'}^{b,Q} \omega(z) \lesssim$$

$$\frac{\mu_a^{2p}(B_{k'}(z))}{\mu_b^{p+1}(B_{k'}(z)) \mu_{b+\frac{Q-q}{p}}^p (B_{k'}(z))} \left(\frac{\mu_{b+\frac{Q-q}{p}}(B_{k'}(z))}{\mu_a^2(B_{k'}(z))} \int\limits_{B_{k'}(z)} f(\zeta) d\mu_b(\zeta) \right)^p \left(\int\limits_{B_{k'}(z)} \omega(\zeta) d\mu_Q(\zeta) \right),$$

so that

$$(R_k^b f(z))^p R_{k'}^{b,Q} \omega(z) \lesssim C_{a,b,p,q,Q} \frac{\mu_a^{2p}(B_{k'}(z))}{\mu_b^{p+1}(B_{k'}(z)) \mu_{b+\frac{Q-q}{p}}^p (B_{k'}(z))} \int\limits_{B_{k'}(z)} f^p(\zeta) \omega(\zeta) d\mu_q(\zeta),$$

because of Lemma 5.15 since it is possible to dilate the pseudo-balls B_k so that they touch the boundary and the fact that the measures $d\mu_q$ and $d\mu_{q+p'(b-a)}$ are homogeneous. By Lemma 2.3, since $B_{k'}(z) = B(z, k'(1-|z|))$ with $0 < k' < 1$, we have

$$\frac{\mu_a^{2p}(B_{k'}(z))}{\mu_b^{p+1}(B_{k'}(z)) \mu_{b+\frac{Q-q}{p}}^p (B_{k'}(z))} \backsimeq (1-|z|^2)^{2pa-2pb-b-(Q-q)-(N+1)}.$$

Recall that we already have

$$\int\limits_{\mathbb{B}} (m_{a,b} f(z))^p \omega(z) d\mu_Q(z) \leq C_k^p A^p C' \int\limits_{\mathbb{B}} (1-|z|^2)^{c+(b-a)} (R_k^b f(z))^p R_{k'}^{b,Q} \omega(z) d\mu_a(z).$$

Let us set

$$IV = \int\limits_{\mathbb{B}} (1-|z|^2)^{c+(b-a)} (R_k^b f(z))^p R_{k'}^{b,Q} \omega(z) d\mu_a(z).$$

Hence, using the previous control of $(R_k^b f(z))^p R_{k'}^{b,Q} \omega(z)$ and Fubini's theorem, we have

$$IV \lesssim \int\limits_{\mathbb{B}} \left(\int\limits_{B_{k'}(z)} f^p(\zeta) \omega(\zeta) d\mu_q(\zeta) \right) (1-|z|^2)^{c+(b-a)+2pa-2pb-b-Q+q-1-N} d\mu_a(z)$$

$$\lesssim \int\limits_{\mathbb{B}} \left(\int\limits_{\mathbb{B}} \chi_{B_{k'}(z)}(\zeta) (1-|z|^2)^{c+(1-2p)(b-a)-b-Q+q-1-N} d\mu_a(z) \right) f^p(\zeta) \omega(\zeta) d\mu_q(\zeta)$$

$$\lesssim \int\limits_{\mathbb{B}} \left(\int\limits_{\mathbb{B}} \chi_{B_{k''}(\zeta)}(z) (1-|z|^2)^{c+(1-2p)(b-a)-b-Q+q-1-N} d\mu_a(z) \right) f^p(\zeta) \omega(\zeta) d\mu_q(\zeta)$$

$$\lesssim \int_{\mathbb{B}} \left(\int_{\mathbb{B}} \chi_{B_{k''}(\zeta)}(z)(1-|z|^2)^{c-2p(b-a)-Q+q-1-N} d\mu(z) \right) f^p(\zeta)\omega(\zeta)d\mu_q(\zeta)$$
$$\lesssim \int_{\mathbb{B}} (1-|\zeta|^2)^{c-2p(b-a)-(Q-q)} f^p(\zeta)\omega(\zeta)d\mu_q(\zeta).$$

The proof is complete if we take

$$c = 2p(b-a) + (Q-q).$$

□

Remark 5.21. The result in Proposition 5.20 says that if we assume that the Sawyer type condition (5.14) holds, then the necessary condition $w \in (B_p^{a,b,q,Q})$ for the boundedness of $T_{a,b}$ from $L^p(wd\mu_q)$ to $L^p(wd\mu_Q)$ is also sufficient by the good lambda inequality in Theorem 1.13. Since we do not know if $w \in (B_p^{a,b,q,Q})$ implies the Sawyer type condition, we will provide in the sequel a testable sufficient condition for the boundedness of $T_{a,b}$ in this situation.

Lemma 5.22. *Let $k \in (0, \frac{1}{4})$. For $s+t > -1$, or for $-N-1 < s+t < -1$ with $s+t+\frac{Q-q}{p} > -1$, if $\omega \in D_p^{s,t,q,Q}$, we have $\sigma(z) = R_{k'}^{s,Q+pt}\omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}} \in (A_{p,s+t+\frac{Q-q}{p}})$.*

Proof. We have $\sigma(z) = R_{k'}^{s,Q+pt}\omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}} \in (A_{p,s+t+\frac{Q-q}{p}})$ if

$$V := \left(\frac{1}{\mu_{s+t+\frac{Q-q}{p}}(B)} \int_B \sigma(z)d\mu_{s+t+\frac{Q-q}{p}}(z) \right) \left(\frac{1}{\mu_{s+t+\frac{Q-q}{p}}(B)} \int_B \sigma^{\frac{-1}{p-1}}(z)d\mu_{s+t+\frac{Q-q}{p}}(z) \right)^{p-1} \leq C_p(\omega).$$

Note that $\sigma(z) \simeq R_{k'}^{s+t+\frac{Q-q}{p},Q+pt}\omega(z)$ since $d\mu_b(B_{k'}(z)) \simeq (1-|z|^2)^{n+1+b}$. We consider two cases.

First case: $B := B(y,r)$ with $r \ll 1-|y|$.

In this case, since $R_{k'}^{a,b}f(x)$ is defined as in (5.13), Corollary 5.19 shows that there are two constants $0 < c < C$ such that

$$cR_{k'}^{s+t+\frac{Q-q}{p},Q+pt}\omega(y) \leq R_{k'}^{s+t+\frac{Q-q}{p},Q+pt}\omega(x) \leq CR_{k'}^{s+t+\frac{Q-q}{p},Q+pt}\omega(y)$$

for all $x \in B$. We then have

$$V \simeq \frac{\mu_{s+t+\frac{Q-q}{p}}(B)}{\mu_{s+t+\frac{Q-q}{p}}(B)} \left(\frac{\mu_{s+t+\frac{Q-q}{p}}(B)}{\mu_{s+t+\frac{Q-q}{p}}(B)} \right)^{p-1} = 1.$$

Second case: $B := B(y,r)$ touches the boundary.

Recall that our measures are homogeneous, and recall that if $z \in B$ and $x \in B_{k'}(z)$, then $z \in B_{k''}(x)$ and $x \in B' := B(y, 2k'Kr + Kr)$, where $k'' = \frac{k'}{1-k'}$. Let

$$VI = \int_B \left(\frac{1}{\mu_s(B_{k'}(z))} \int_{B_{k'}(z)} \omega(x)d\mu_{Q+pt}(x) \right) (1-|z|^2)^{-t-\frac{Q-q}{p}} d\mu_{s+t+\frac{Q-q}{p}}(z).$$

Then by Fubini's theorem we have,

$$
\begin{aligned}
VI &= \int_B \left(\frac{1}{\mu_s(B_{k'}(z))} \int_{B_{k'}(z)} \omega(x) d\mu_{Q+pt}(x) \right) d\mu_s(z) \\
&= \int_B \int_{B_{k'}(z)} \frac{1}{\mu_s(B_{k'}(z))} \omega(x) d\mu_s(z) d\mu_{Q+pt}(x) \\
&= \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{1}{\mu_s(B_{k'}(z))} \chi_B(z) \chi_{B_{k'}(z)}(x) \omega(x) d\mu_s(z) d\mu_{Q+pt}(x) \\
&\lesssim \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{1}{\mu_s(B_{k''}(x))} \chi_{B'}(x) \chi_{B_{k''}(x)}(z) \omega(x) d\mu_s(z) d\mu_{Q+pt}(x) \\
&\lesssim \int_{B'} \omega(x) d\mu_{Q+pt}(x).
\end{aligned}
$$

So we have

$$
VI = \int_B R_{k'}^{s,Q+pt} \omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}} d\mu_{s+t+\frac{Q-q}{p}}(z) \lesssim \int_{B'} \omega(x) d\mu_{Q+pt}(x). \tag{5.15}
$$

Let us now control $(R_{k'}^{s,Q+pt}\omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}})^{1-p'}$. We have

$$
\begin{aligned}
(R_{k'}^{s,Q+pt}\omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}})^{1-p'} &= \left(\frac{1}{\mu_s(B_{k'}(z))} (1-|z|^2)^{-t-\frac{Q-q}{p}} \int_{B_{k'}(z)} \omega(x) d\mu_{Q+pt}(x) \right)^{1-p'} \\
&\simeq \left[\left(\int_{B_{k'}(z)} \omega(x) d\mu_{Q+pt}(x) \right)^{-1} \left(\int_{B_{k'}(z)} d\mu_{s+t+\frac{Q-q}{p}}(x) \right) \right]^{p'-1}.
\end{aligned}
$$

Setting

$$
VII = \left[\left(\int_{B_{k'}(z)} \omega(x) d\mu_{Q+pt}(x) \right)^{-1} \left(\int_{B_{k'}(z)} d\mu_{s+t+\frac{Q-q}{p}}(x) \right) \right]^{p'-1},
$$

we have, by Hölder's inequality

$$
\begin{aligned}
VII &= \left[\left(\int_{B_{k'}(z)} \omega(x) d\mu_{Q+pt}(x) \right)^{-1} \left(\int_{B_{k'}(z)} \omega^{\frac{1}{p}}(x) \omega^{\frac{-1}{p}}(x) (1-|x|^2)^{s-\frac{q}{p}+\frac{Q+pt}{p}} d\mu(x) \right) \right]^{p'-1} \\
&\leq \left[\left(\int_{B_{k'}(z)} \omega(x) d\mu_{Q+pt}(x) \right)^{\frac{1}{p}-1} \left(\int_{B_{k'}(z)} \omega^{\frac{-p'}{p}}(x) d\mu_{q+p'(s-q)}(x) \right)^{\frac{1}{p'}} \right]^{p'-1}
\end{aligned}
$$

$$\leq \left[\left(\int_{B_{k'}(z)} \omega(x) d\mu_{Q+pt}(x) \right)^{-1} \left(\int_{B_{k'}(z)} \omega^{\frac{-p'}{p}}(x) d\mu_{q+p'(s-q)}(x) \right) \right]^{\frac{1}{p}}.$$

Let us set

$$VIII = \int_{B} \left(R_{k'}^{s,Q+pt} \omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}} \right)^{\frac{-1}{p-1}} d\mu_{s+t+\frac{Q-q}{p}}(z),$$

then we have

$$VIII \leq \int_{B} \left[\left(\int_{B_{k'}(z)} \omega(x) d\mu_{Q+pt}(x) \right)^{-1} \left(\int_{B_{k'}(z)} \omega^{\frac{-p'}{p}}(x) d\mu_{q+p'(s-q)}(x) \right) \omega(z)(1-|z|^2)^{Q+pt} \right]^{\frac{1}{p}} \omega^{\frac{-1}{p}}(z)(1-|z|^2)^{s-\frac{q}{p}} d\mu(z)$$

so that by Hölder's inequality and Fubini's theorem, we have

$$VIII \lesssim \left(\int_{B} \omega^{\frac{-p'}{p}}(z) d\mu_{q+p'(s-q)}(z) \right)^{\frac{1}{p'}} \left(\int_{B'} \left[\left(\int_{B_{k''}(x)} \omega(\zeta) d\mu_{Q+pt}(\zeta) \right)^{-1} \left(\int_{B_{k''}(x)} \omega(z) d\mu_{Q+pt}(z) \right) \right] \omega^{\frac{-p'}{p}}(x) d\mu_{q+p'(s-q)}(x) \right)^{\frac{1}{p}}.$$

Finally, we obtain

$$VIII \lesssim \int_{B'} \omega^{\frac{-p'}{p}}(z) d\mu_{q+p'(s-q)}(z). \tag{5.16}$$

Where on the last but one inequality we used Fubini's theorem (as in the control of VI) and the fact that for $x \in B_{k'}(z)$ we have $z \in B_{k''}(x)$ and

$$\int_{B_{k''}(x)} \omega(\zeta) d\mu_{Q+pt}(\zeta) \lesssim \int_{B_{k'}(z)} \omega(\zeta) d\mu_{Q+pt}(\zeta).$$

This is a variant of Corollary 5.19 or simply the application of Lemma 5.17 for $f(\zeta) = (1-|\zeta|^2)^{t+\frac{Q-q}{p}} 1_{B_{k'}(z)}(\zeta)$ with $B := B(z, 2Kk'(1+k'')(1-|z|)) \supseteq B_{k''}(x)$ (Lemma 5.9) and the fact that our measures are homogeneous. Since $\omega \in D_p^{s,t,q,Q}$, we use (5.15) and (5.16) to conclude that $\sigma \in (A_{p,s+t+\frac{Q-q}{p}})$. □

Theorem 5.23. *In the case both $Q \geq q$ and $s+t > -1$ hold, and in the case both $s+t+\frac{Q-q}{p} > -1$ and $-1 > s+t > -N-1$ hold, if $\omega \in D_p^{s,t,q,Q}$, there is a constant $C_{s,t,p,q,Q} > 0$ such that $\forall f \in L^p(\omega d\mu_q)$,*

$$\int_{\mathbb{B}} (O_{s,t} f(z))^p \omega(z) d\mu_Q(z) \leq C_{s,t,p,q,Q} \int_{\mathbb{B}} |f(z)|^p \omega(z) d\mu_q(z).$$

Proof. Using in this order Lemma 5.11, Lemma 5.14, Hölder's inequality and Lemma 5.12 we have

$$
\begin{aligned}
\int_{\mathbb{B}} (O_{s,t}f(z))^p \omega(z) d\mu_Q(z) &= \int_{\mathbb{B}} (m_{s+t,s}f(z))^p \omega(z) d\mu_{Q+pt}(z) \\
&\leq C_k^p \int_{\mathbb{B}} (m_{s+t,s} R_k^s f(z))^p \omega(z) d\mu_{Q+pt}(z) \\
&\leq C_k^p A^p \int_{\mathbb{B}} [R_k^s(m_{s+t,s} R_k^s f(z))]^p \omega(z) d\mu_{Q+pt}(z) \\
&\leq C_k^p A^p \int_{\mathbb{B}} R_k^s[(m_{s+t,s} R_k^s f(z))^p] \omega(z) d\mu_{Q+pt}(z) \\
&\leq C_k^p A^p C \int_{\mathbb{B}} (m_{s+t,s} R_k^s f(z))^p R_{k'}^{s,Q+pt} \omega(z) d\mu_s(z).
\end{aligned}
$$

By Lemma 5.22, $R_{k'}^{s,Q+pt}\omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}} \in A_{p,s+t+\frac{Q-q}{p}}$. Using natural domination between our maximal operator defined by equation 5.4 and equation 5.6, we have

$$
\begin{aligned}
\int_{\mathbb{B}} (O_{s,t}f(z))^p \omega(z) d\mu_Q(z) &\leq C_k^p A^p C \int_{\mathbb{B}} (m_{s+t,s} R_k^s f(z))^p R_{k'}^{s,Q+pt} \omega(z) d\mu_s(z) \\
&\leq C_k^p A^p C \int_{\mathbb{B}} (M_{s+t,s} R_k^s f(z))^p R_{k'}^{s,Q+pt} \omega(z) d\mu_s(z) \\
&= C_k^p A^p C \int_{\mathbb{B}} (M_{s+t,s+t}[(1-|z|^2)^{-t} R_k^s f(z)])^p R_{k'}^{s,Q+pt} \omega(z) d\mu_s(z).
\end{aligned}
$$

Because in each case we have $\frac{Q-q}{p} \geq 0$, using Theorem 2.13, we have that

$$
\begin{aligned}
\int_{\mathbb{B}} (O_{s,t}f(z))^p \omega(z) d\mu_Q(z) &\lesssim \int_{\mathbb{B}} (M_{s+t+\frac{Q-q}{p},s+t}[(1-|z|^2)^{-t} R_k^s f(z)])^p R_{k'}^{s,Q+pt} \omega(z) d\mu_s(z) \\
&\lesssim \int_{\mathbb{B}} (M_{s+t+\frac{Q-q}{p},s+t+\frac{Q-q}{p}}[(1-|z|^2)^{-t-\frac{Q-q}{p}} R_k^s f(z)])^p R_{k'}^{s,Q+pt} \omega(z) d\mu_s(z) \\
&\lesssim \int_{\mathbb{B}} (1-|z|^2)^{-pt-(Q-q)} (R_k^s f(z))^p R_{k'}^{s,Q+pt} \omega(z) d\mu_s(z). \qquad (5.17)
\end{aligned}
$$

Now let us control $IX = (R_k^s f(z))^p R_{k'}^{s,Q+pt}\omega(z)$. We have

$$
\begin{aligned}
IX &= \left(\frac{1}{\mu_s(B_k(z))} \int_{B_k(z)} f(\zeta) d\mu_s(\zeta) \right)^p \left(\frac{1}{\mu_s(B_{k'}(z))} \int_{B_{k'}(z)} \omega(\zeta) d\mu_{Q+pt}(\zeta) \right) \\
&\lesssim \left(\frac{1}{\mu_s(B_{k'}(z))} \int_{B_{k'}(z)} f(\zeta) d\mu_s(\zeta) \right)^p \left(\frac{1}{\mu_s(B_{k'}(z))} \int_{B_{k'}(z)} \omega(\zeta) d\mu_{Q+pt}(\zeta) \right)
\end{aligned}
$$

$$\lesssim \frac{\mu^p_{s+t+\frac{Q-q}{p}}(B_{k'}(z))}{\mu_s^{p+1}(B_{k'}(z))}\left(\frac{1}{\mu_{s+t+\frac{Q-q}{p}}(B_{k'}(z))}\int\limits_{B_{k'}(z)} f(\zeta)d\mu_s(\zeta)\right)^p\left(\int\limits_{B_{k'}(z)}\omega(\zeta)d\mu_{Q+pt}(\zeta)\right)$$

$$\lesssim C_{s,t,p,q,Q}\frac{\mu^p_{s+t+\frac{Q-q}{p}}(B_{k'}(z))}{\mu_s^{p+1}(B_{k'}(z))}\int\limits_{B_{k'}(z)} f^p(\zeta)\omega(\zeta)d\mu_q(\zeta)$$

$$\lesssim C_{s,t,p,q}(1-|z|^2)^{pt+(Q-q)-s-N-1}\int\limits_{B_{k'}(z)} f^p(\zeta)\omega(\zeta)d\mu_q(\zeta),$$

where for the last but one inequality we used Lemma 5.17. Hence using (5.17) and Fubini's theorem we have

$$\int\limits_{\mathbb{B}}(O_{s,t}f(z))^p\omega(z)d\mu_Q(z)\lesssim\int\limits_{\mathbb{B}}\left(\int\limits_{B_{k'}(z)} f^p(\zeta)\omega(\zeta)d\mu_q(\zeta)\right)(1-|z|^2)^{-1-N}d\mu(z)$$

$$\lesssim\int\limits_{\mathbb{B}}\left(\int\limits_{\mathbb{B}}\chi_{B_{k''}(\zeta)}(z)(1-|z|^2)^{-1-N}d\mu(z)\right)f^p(\zeta)\omega(\zeta)d\mu_q(\zeta)$$

$$\lesssim\int\limits_{\mathbb{B}} f^p(\zeta)\omega(\zeta)d\mu_q(\zeta).$$

Here we used Lemma 5.8 with $k'' := (k')'$. The proof is complete. ☐

6. Good lambda inequality and sufficient conditions

In this section we will establish the good lambda inequality that allow us to provide sufficient conditions for the boundedness of our operators. We first need some preliminary results. The first result extends to the maximal function $m'_{s+t,s}$ the analogue result in [3]. For the sake of completeness, we include the proof.

Proposition 6.1. *Let $\beta > 0$ and $s+t > -1-N$ there is a constant $A > 0$ such that for all $\xi \in \mathbb{B}$ $R > 1-|z_0|$ and a positive locally integrable function f and for all $z \in B(z_0, R)$ if $s+t > -1$, then*

$$R^\beta\int\limits_{d(z_0,\xi)\geq R}\frac{f(\xi)}{d(z_0,\xi)^{N+1+s+t+\beta}}\mathrm{d}\mu_s(\xi)\leq A m_{s+t,s}f(z).$$

More generally if $-1-N < s+t$ then

$$R^\beta\int\limits_{d(z_0,\xi)\geq R}\frac{f(\xi)}{d(z_0,\xi)^{N+1+s+t+\beta}}\mathrm{d}\mu_s(\xi)\leq A m'_{s+t,s}f(z).$$

Proof. Recall that if $s+t > -1$, by Lemma 2.3, there is a constant $a > 0$ such that for all $k \in \mathbb{N}$, we have $\mu_{s+t}(B(z_0, 2^{k+1}R)) \leq a(2^{k+1}R)^{N+1+s+t}$, so that setting

$$X = R^\beta\int\limits_{d(z_0,\xi)\geq R}\frac{f(\xi)}{d(z_0,\xi)^{N+1+s+t+\beta}}\mathrm{d}\mu_s(\xi)$$

we have

$$\begin{aligned} X &\leq \sum_{k=0}^{+\infty} \frac{1}{2^{k(N+1+s+t+\beta)} R^{N+1+s+t}} \int\limits_{d(z_0,\xi)<2^{k+1}R} f(\xi)\mathrm{d}\mu_s(\xi) \\ &\leq a2^{N+1+s+t} \sum_{k=0}^{+\infty} 2^{-k\beta} \frac{1}{\mu_{s+t}(B(z_0, 2^{k+1}R))} \int\limits_{B(z_0,2^{n+1}R)} f(\xi)\mathrm{d}\mu_s(\xi) \\ &\leq a2^{N+1+s+t} m_{s+t,s} f(z) \sum_{n=0}^{+\infty} 2^{-k\beta} = \frac{a2^{N+1+s+t}}{1-2^{-\beta}} m_{s+t,s} f(z). \end{aligned}$$

We can take $A = \dfrac{a2^{N+1+s+t}}{1-2^{-\beta}}$ to conclude. □

Proposition 6.2. *Let $\omega \in (D_p^{s,t,q,Q})$, we set again $\sigma(z) = R_k^{s,Q+pt}\omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}}$ with $k \in (0,1/2)$. Set $B = B(z',r)$ with $1-|z'| < cr$ and $L = \left\{z \in B : 1-|z| < C_0'\gamma^{\frac{1}{N+1+s+t}} r\right\}$ where $C_0' > 0$, $0<\gamma<1$, $r>0$ and $c>0$ are constants. Then if we set $L' = \left\{z \in \bar{B} : 1-|z| < 2C_0'\gamma^{\frac{1}{N+1+s+t}} r\right\}$ and $\bar{B} = B(z', ar)$ with $a = K(C_0'+1)$, there are two constants C_1 and $C_2 > 0$ independent of γ such that if $s+t+\frac{Q-q}{p} > -1$ then*

$$\omega d\mu_{Q+pt}(L) \leq C_1 \sigma d\mu_{s+t+\frac{Q-q}{p}}(L') \quad \text{and} \quad \mu_{s+t+\frac{Q-q}{p}}(L') \leq C_2 \gamma^{\frac{s+t+\frac{Q-q}{p}+1}{N+1+s+t}} \mu_{s+t+\frac{Q-q}{p}}(\bar{B}). \tag{6.1}$$

Proof. Let $k_1 = \frac{k}{1+k}$, then for $z \in L$ and $\xi \in B_{k_1}(z)$ we have $z \in B = B(z',r)$, $1-|z| < C_0'\gamma^{\frac{1}{N+1+s+t}} r$ and $z \in B_k(\xi)$ because $k_1' = k$. Then

$$d(z',\xi) \leq K(d(z',z) + d(z,\xi)) < K[r + k_1(1-|z|)] < K[r + k_1 C_0'\gamma^{\frac{1}{N+1+s+t}} r] < (C_0'+1)rK$$

because $0 < k_1, \gamma < 1$. Then $\xi \in \overline{B} = B(z', ar)$ with $a = K(C_0'+1)$. Moreover,

$$1-|\xi| < 1-|z| + d(z,\xi) < (k_1+1)(1-|z|) < 2C_0'\gamma^{\frac{1}{N+1+s+t}} r$$

because $0<k_1<1$, so that $\xi \in L' = \left\{z \in \overline{B} : 1-|z| < 2C_0'\gamma^{\frac{1}{N+1+s+t}} r\right\}$. Then we have

$$\chi_L(z)\chi_{B_{k_1}(z)}(\xi) \leq \chi_{L'}(\xi)\chi_{B_k(\xi)}(z).$$

Remember that

$$\mu_{s+t+\frac{Q-q}{p}}(B_k(\xi)) \simeq \mu_{s+t+\frac{Q-q}{p}}(B_{k_1}(z)).$$

Hence,

$$\begin{aligned} \omega d\mu_{Q+pt}(L) &= \int\limits_L \omega(z) d\mu_{Q+pt}(z) \\ &= \int\limits_L \left(\frac{\omega(z)}{\mu_s(B_{k_1}(z))} \int\limits_{B_{k_1}(z)} d\mu_s(\xi) \right) d\mu_{Q+pt}(z) \end{aligned}$$

$$
\begin{aligned}
&= \int\limits_{\mathbb{B}} \left(\int\limits_{\mathbb{B}} \frac{\chi_L(z)\chi_{B_{k_1}(z)}(\xi)\omega(z)}{\mu_s(B_{k_1}(z))} d\mu_{Q+pt}(z) \right) d\mu_s(\xi) \\
&\lesssim \int\limits_{\mathbb{B}} \left(\int\limits_{\mathbb{B}} \frac{\chi_{L'}(\xi)\chi_{B_k(\xi)}(z)\omega(z)}{\mu_s(B_k(\xi))} d\mu_{Q+pt}(z) \right) d\mu_s(\xi) \\
&= \int\limits_{\mathbb{B}} \left(\frac{\chi_{L'}(\xi)}{\mu_s(B_k(\xi))} \int\limits_{B_k(\xi)} \omega(z) d\mu_{Q+pt}(z) \right) (1-|\xi|^2)^{-t-\frac{Q-q}{p}} d\mu_{s+t+\frac{Q-q}{p}}(\xi) \\
&= \int\limits_{L'} \left(R_k^{s,Q+pt} w(\xi) \right) (1-|\xi|^2)^{-t-\frac{Q-q}{p}} d\mu_{s+t+\frac{Q-q}{p}}(\xi) \\
&= \sigma d\mu_{s+t+\frac{Q-q}{p}}(L').
\end{aligned}
$$

Let us show the second inequality. Let $z \in \overline{B}$, then, $d(z', z) = ||z'| - |z|| + \left|1 - \frac{\langle z', z\rangle}{|z'||z|}\right| < ar$. Then $||z'|-|z|| < ar$ and $\left|1 - \frac{\langle z', z\rangle}{|z'||z|}\right| < ar$. Moreover, as $1-|z'| < cr$ then $1-|z| = 1-|z'|+|z'|-|z| < cr+ar$. Then for $z \in L'$, setting $\beta_1 = 2C_0'\gamma^{\frac{1}{N+1+s+t}} r$ and $\beta_2 = (c+a)r$, we get $1 - |z| < \beta_1$, $1 - |z| < \beta_2$ and $\left|1 - \frac{\langle z', z\rangle}{|z'||z|}\right| < ar$. Then, $L' \subset \{z \in \mathbb{B} : 1 - |z| < \min(\beta_1, \beta_2),\ \left|1 - \frac{\langle z', z\rangle}{|z'||z|}\right| < ar\}$. In spherical coordinates we have, for $s + t + \frac{Q-q}{p} > -1$

$$
\begin{aligned}
\mu_{s+t+\frac{Q-q}{p}}(L') &\lesssim (s+t+\frac{Q-q}{p}+1) \int\limits_{1-\rho<\min(\beta_1,\beta_2)} (1-\rho^2)^{s+t+\frac{Q-q}{p}} \rho \mathrm{d}\rho \int\limits_{|1-\frac{\langle z', z\rangle}{|z'||z|}|<ar} \mathrm{d}\sigma(\xi) \\
&\leq (s+t+\frac{Q-q}{p}+1) \int\limits_{1-\min(\beta_1,\beta_2)<\rho<1} (1-\rho)^{s+t+\frac{Q-q}{p}} \mathrm{d}\rho \int\limits_{|1-\frac{\langle z', z\rangle}{|z'||z|}|<ar} \mathrm{d}\sigma(\xi) \\
&\lesssim r^N[-(1-\rho)^{s+t+\frac{Q-q}{p}+1}]^1_{1-\min(\beta_1,\beta_2)} = r^N(\min(\beta_1,\beta_2))^{s+t+\frac{Q-q}{p}+1}.
\end{aligned}
$$

Then,

$$
\mu_{s+t+\frac{Q-q}{p}}(L') \lesssim r^N \beta_1^{s+t+\frac{Q-q}{p}+1} \backsimeq r^{N+s+t+\frac{Q-q}{p}+1} \gamma^{\frac{s+t+\frac{Q-q}{p}+1}{N+1+s+t}}. \tag{6.2}
$$

On the other hand, as $1 - |z'| < cr$, we have by Lemma 2.3

$$
\mu_{s+t+\frac{Q-q}{p}}(\bar{B}) \simeq (ar)^{N+1}(\max(1-|z'|, ar))^{s+t+\frac{Q-q}{p}} \lesssim r^{N+1+s+t+\frac{Q-q}{p}}.
$$

□

We are now ready to prove our good lambda inequality (Theorem 1.13) that we recall here for the reader convenience. It is used to show that $S_{s+t,s}f \in L^p(\omega d\mu_{Q+pt})$ when $m'_{s+t,s}f \in L^p(\omega d\mu_{Q+pt})$.

Theorem 6.3 (Good lambda inequality)**.** *Let* $\omega \in (D_p^{s,t,q,Q})$ $(1 < p < +\infty)$ *in the case both* $s+t+\frac{Q-q}{p} > -1$ *and* $-1 > s+t > -N-1$ *hold, or both* $s+t > -1$ *and* $Q \geq q$ *hold. There are two positive constants*

C and β such that for all γ sufficiently small, $\lambda > 0$ and for all positive locally integrable functions f, we have

$$\omega d\mu_{Q+pt}(\{z \in \mathbb{B} : S_{s+t,s}f(z) > 2\lambda, m'_{s+t,s}f(z) \leq \gamma\lambda\}) \leq \\ CD_p^{s,t,q,Q}(\omega)\gamma^\beta \omega d\mu_{Q+pt}(\{z \in \mathbb{B} : S_{s+t,s}f(z) > \lambda\}). \quad (6.3)$$

Proof. Let $\lambda > 0$, $0 < \gamma < 1$ and f a positive locally integrable function. Let $E_\lambda = \{z \in \mathbb{B} : S_{s+t,s}f(z) > \lambda\}$. By the Whitney decomposition Lemma (see [8]), there are a positive integer J, $\delta > 1$ and a sequence of pseudo-balls $\{B_j\}_{j=1}^{\infty}$, with $B_j = B(z_j, r_j)$, such that

- $E_\lambda = \bigcup_{j=1}^{\infty} B_j$;
- every point of E_λ is at most in J balls B_j;
- the balls $B'_j = B(z_j, \delta r_j)$ touch the complement of E_λ in $\mathbb{B}$.

To obtain (6.3), it is then sufficient to show that

$$\omega\mu_{Q+pt}(\{z \in B : S_{s+t,s}f(z) > 2\lambda, m'_{s+t,s}f(z) \leq \gamma\lambda\}) \leq CD_p^{s,t,q,Q}(\omega)\gamma^\beta \omega\mu_{Q+pt}(B), \quad (6.4)$$

where $B = B(z', r)$ is a ball in the Whitney decomposition of E_λ. From the third property of the Whitney decomposition, there is $z_0 \in B' = B(z', \delta r)$ such that $S_{s+t,s}f(z_0) \leq \lambda$. Without loss of generality, assume that there is $\xi_0 \in B$ such that $m'_{s+t,s}f(\xi_0) \leq \gamma\lambda$. Let $\widetilde{B} = B(z_0, R)$ with $R = \max(1 - |z_0|, C_0 r)$ where we choose $C_0 \geq \max(c_1 K(1+\delta), \delta)$ where c_1 is the constant C_1 in Lemma 2.15.

We set $f_1 = 1_{\widetilde{B}} f$ and $f_2 = 1_{\mathbb{B}\setminus\widetilde{B}} f$, then $f = f_1 + f_2$ and we have

$$S_{s+t,s}f_2(z) \leq \int_{\mathbb{B}\setminus\widetilde{B}} \left| \frac{f(\xi)}{|1 - \langle z_0, \xi\rangle|^{N+1+s+t}} \right| d\mu_s(\xi) + \\ \int_{\mathbb{B}\setminus\widetilde{B}} \left| \frac{1}{|1 - \langle z, \xi\rangle|^{N+1+s+t}} - \frac{1}{|1 - \langle z_0, \xi\rangle|^{N+1+s+t}} \right| f(\xi) d\mu_s(\xi).$$

So that, by Lemma 2.15 and Proposition 6.1, we finally have, for $z \in B$

$$S_{s+t,s}f_2(z) \;\leq\; S_{s+t,s}f(z_0) + A' m'_{s+t,s}f(\xi_0) \leq \lambda + A'\gamma\lambda.$$

So, since

$$S_{s+t,s}f(z) \leq S_{s+t,s}f_1(z) + S_{s+t,s}f_2(z) \leq S_{s+t,s}f_1(z) + \lambda + A'\gamma\lambda,$$

we have that $S_{s+t,s}f(z) > 2\lambda$ implies that $S_{s+t,s}f_1(z) > (1 - A'\gamma)\lambda$. Therefore, to prove (6.4), it will be enough to show that

$$\omega d\mu_{Q+pt}(\{z \in B : S_{s+t,s}f_1(z) > b\lambda\}) \leq CD_p^{s,t,q,Q}(\omega)\gamma^\beta \omega d\mu_{Q+pt}(B), \quad (6.5)$$

where $b = 1 - A'\gamma$ with an appropriate choice of γ. We are going to discuss according to the values of the radius $R = \max(1 - |z_0|, C_0 r)$ of $\widetilde{B} = B(z_0, R)$. Let $E'_\lambda = \{S_{s+t,s}f_1 \geq b\lambda\} \cap B$.

First case: $C_0 r \leq 1 - |z_0|$.

Then, $\widetilde{B} = B(z_0, 1 - |z_0|)$. Therefore for all $z \in B$ and $\xi \in \widetilde{B}$, $|1 - \langle z, \xi\rangle| \geq 1 - |z| > C'(1 - |z_0|)$, so that for all $z \in B$,

$$S_{s+t,s} f_1(z) = \int_{\widetilde{B}} \frac{f(\xi) d\mu_s(\xi)}{|1 - \langle z, \xi\rangle|^{N+1+s+t}} \leq \frac{1}{(C'(1 - |z_0|))^{N+1+s+t}} \int_{\widetilde{B}} f(\xi) d\mu_s(\xi).$$

Then

$$S_{s+t,s} f_1(z) < C'' m'_{s+t,s} f(\xi_0) \leq C'' \gamma \lambda.$$

Hence, if we take $0 < \gamma < \gamma_0 = \min(\frac{1}{A'}, \frac{b}{C''})$ then it remains only to prove the following case.

Second case: $1 - |z_0| < C_0 r$.

Then $\widetilde{B} = B(z_0, C_0 r)$ and $E'_\lambda \subseteq L$ for L defined in Proposition 6.2. In fact, if $z \in E'_\lambda$, then $z \in B$ and

$$\begin{aligned} b\lambda \leq S_{s+t,s} f_1(z) &= \int_{\widetilde{B}} \frac{f(\xi) d\mu_s(\xi)}{|1 - \langle z, \xi\rangle|^{N+1+s+t}} \leq \frac{1}{(1 - |z|)^{N+1+s+t}} \int_{\widetilde{B}} f(\xi) d\mu_s(\xi) \\ &\leq \frac{(C_0 r)^{N+1+s+t}}{(1 - |z|)^{N+1+s+t}} m'_{s+t,s} f(\xi_0) \\ &\leq \frac{(C_0 r)^{N+1+s+t}}{(1 - |z|)^{N+1+s+t}} \gamma \lambda. \end{aligned}$$

For $\sigma(z) = R_{k'}^{s,Q+pt} \omega(z)(1 - |z|^2)^{-t-\frac{Q-q}{p}}$, with $k' \in (0, \frac{1}{2})$, by Lemma 5.22 we have $\sigma \in (A_{p,s+t+\frac{Q-q}{p}})$ so that $\sigma \in (A_{\infty,s+t+\frac{Q-q}{p}})$ because $(A_{p,s+t+\frac{Q-q}{p}}) \subseteq (A_{\infty,s+t+\frac{Q-q}{p}})$.

Given the fact that L' is a measurable subset of $\bar{B} = B(z', ar)$, we have by Proposition 6.2 and Lemma 2.12

$$\begin{aligned} \omega d\mu_{Q+pt}(L) &\leq C \sigma d\mu_{s+t+\frac{Q-q}{p}}(L') \\ &\leq C \left(\frac{\mu_{s+t+\frac{Q-q}{p}}(L')}{\mu_{s+t+\frac{Q-q}{p}}(\bar{B})}\right)^{\beta_0} \sigma d\mu_{s+t+\frac{Q-q}{p}}(\bar{B}) \\ &\leq C \gamma^{\frac{s+t+\frac{Q-q}{p}+1}{N+1+s+t}\beta_0} \sigma d\mu_{s+t+\frac{Q-q}{p}}(\bar{B}). \end{aligned}$$

As E'_λ is a subset of $L = \{z \in \bar{B} : 1 - |z| < C'_0 \gamma^{\frac{1}{N+1+s+t}} r\}$, it follows that for $\beta = \frac{s+t+\frac{Q-q}{p}+1}{N+1+s+t}\beta_0$ we have

$$\omega d\mu_{Q+pt}(E'_\lambda) \leq C \gamma^\beta \sigma d\mu_{s+t+\frac{Q-q}{p}}(\bar{B}). \tag{6.6}$$

One shows by Fubini's theorem that $\sigma d\mu_{s+t+\frac{Q-q}{p}}(\bar{B}) \leq C \omega d\mu_{Q+pt}(\bar{\bar{B}})$ with $\bar{\bar{B}} = B(z', (2k+1)arK)$. And by Corollary 5.19 we get $\omega d\mu_{Q+pt}(\bar{\bar{B}}) \leq C D_p^{s,t,q,Q}(\omega) \omega d\mu_{Q+pt}(B)$. Then

$$\omega d\mu_{Q+pt}(E'_\lambda) = \omega d\mu_{Q+pt}(\{z \in B : S_{s+t,s} f_1(z) > b\lambda\}) \leq C D_p^{s,t,q,Q}(\omega) \gamma^\beta \omega d\mu_{Q+pt}(B).$$

This ends the proof. □

The following results appear as consequences of Theorem 6.3 and Lemma 2.14.

Theorem 6.4. *Let $p > 1$. For $Q \geq q$ and $s+t > -1$, if $\omega \in D_p^{s,t,q,Q}$ there is a constant $C_{s,t,q,Q} > 0$ such that*

$$\int_{\mathbb{B}} (S_{s+t,s}f(z))^p d\mu_{Q+pt}(z) \leq C_{s,t,q,Q} \int_{\mathbb{B}} (m_{s+t,s}f(z))^p d\mu_{Q+pt}(z).$$

Proof. It is enough to apply Lemma 2.14 to $S_{s+t,s}f$ and $m_{s+t,s}f$ with $t = 2\lambda$, $c = \frac{\gamma}{2}$, $b = \frac{1}{2}$ and $a = CD_p^{s,t,q,Q}(\omega)\gamma^\beta$. We just need to take γ small enough so that $a < b^p$. □

Theorem 6.5. *Let $p > 1$. For $s+t+\frac{Q-q}{p} > -1$ and $-N-1 < s+t < -1$, if $\omega \in D_p^{s,t,q,Q}$ there is a constant $C_{s,t,q,Q} > 0$ such that*

$$\int_{\mathbb{B}} (S_{s+t,s}f(z))^p d\mu_{Q+pt}(z) \leq C_{s,t,q,Q} \int_{\mathbb{B}} (m'_{s+t,s}f(z))^p d\mu_{Q+pt}(z).$$

7. Final remark and open question

This part is simply a direct application of the two preceding sections and Remark 1.12. Therefore for $1 < p < +\infty$ we have the following two corollaries.

Corollary 7.1. *Let ω be a weight on $\mathbb{B}$. Then for $s+t > -1$ and $q = Q$, the following assertions are equivalent.*

1. *$P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_q)$;*
2. *$T_{s+t,s}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_{q+pt})$;*
3. *$S_{s+t,s}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_{q+pt})$;*
4. *$\omega \in (K_p^{s,t,q,q})$.*

Proof. By Remark 1.12, we have that $(K_p^{s,t,q,q}) = (D_p^{s,t,q,q})$, so that by Theorem 6.4, since $(1-|z|^2)^t S_{s+t,s}f(z) = P_{s,t}f(z)$ and $O_{s,t}f(z) = (1-|z|^2)^t m_{s+t,s}f(z)$ and Theorem 5.23, we have (4) implies (1). □

Corollary 7.2. *Let ω be a weight on $\mathbb{B}$. In the case both $s+t+\frac{Q-q}{p} > -1$ and $-1 > s+t > -N-1$ hold, and in the case both $s+t > -1$ and $Q \geq q$ hold, if $\omega \in (D_p^{s,t,q,Q})$ then $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$, so that $S_{s+t,s}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_{Q+pt})$.*

In Theorem 5.5 and in Theorem 5.6 we show that being in $(K_p^{s,t,q,Q})$ is a necessary condition for the continuity of $P_{s,t}$ from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$, while in Corollary 7.2, we have that being in $D_p^{s,t,q,Q}$ is a sufficient one. When $Q = q$, we find out that $(K_p^{s,t,q,q}) = (D_p^{s,t,q,q})$, so that we have a necessary and sufficient condition. But when $Q > q$ we have $(D_p^{s,t,q,Q}) \subseteq (K_p^{s,t,q,Q})$. It is an open question to find in this case a necessary and sufficient condition for the boundedness of $P_{s,t}$ from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$.

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