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Limit sets and global dynamic for 2-D divergence-free vector fields

Ensembles limites et dynamique globale pour les 2-D champs de vecteurs sans divergence

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ABSTRACT. The global structure of divergence-free vector fields on closed surfaces is investigated. We prove that if M is a closed surface and $\mathcal{V}$ is a divergence-free C^1-vector field with finitely many singularities on M then every orbit L of $\mathcal{V}$ is one of the following types: (i) a singular point, (ii) a periodic orbit, (iii) a closed (non periodic) orbit in $M^* = M - \mathrm{Sing}(\mathcal{V})$, (iv) a locally dense orbit, where $\mathrm{Sing}(\mathcal{V})$ denotes the set of singular points of $\mathcal{V}$. On the other hand, we show that the complementary in M of periodic components and minimal components is a compact invariant subset consisting of singularities and closed (non compact) orbits in M^*. These results extend those of T. Ma and S. Wang in [Discrete Contin. Dynam. Systems, 7 (2001), 431–445] established when the divergence-free vector field $\mathcal{V}$ is regular that is all its singular points are non-degenerate.
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1. Introduction

In [5], Ma and Wang described the limit set for a *regular* divergence-free vector field on compact surfaces (with or without boundary). They established a limit set Theorem (see [5, Theorem 1.2]); it says that the ω-limit (resp. α-limit) set of any orbit is either a saddle point, or a non-limiting closed orbit, or an ergodic set which is a closed domain with boundaries consisting of saddle connections of finite length. As a consequence, a global structural classification theorem ([5, Theorem 1.3]) was given; it says that the surface can be decomposed into a finite union of invariant sets consisting of circle cells, circle bands, ergodic sets and saddle connections. The main motivation of their study was to develop the topological and geometrical theory of the Lagrange dynamics of 2-D incompressible fluid flows in the physical space.

The aim of this paper is to extend the above results to the general case of divergence-free vector fields regular or not with finitely many singularities on closed surfaces. We describe its global dynamics. In particular, we obtain a simple proof of Ma and Wang previous results.

Our proof is based on the classification in [6] and [7] of the limit sets of orbits and of the global structure Theorem of a general C^1-vector field with finitely many singularities on a closed surface.

Before stating our main results, we firstly need some definitions.

Let $\mathcal{V}$ be a C^1-vector field on a closed surface M. Here a closed surface means a compact surface without boundary. A point $x \in M$ is called a *singular point* (or a *singularity*) of $\mathcal{V}$ if $\mathcal{V}(x) = 0$. We denote by sing($\mathcal{V}$) the set of singularities of $\mathcal{V}$ and $M^* = M - \text{sing}(\mathcal{V})$. Let $\phi : \mathbb{R} \times M \to M$ be the C^1-flow generated by $\mathcal{V}$. Given a point $x \in M$, the subset $L = \{\phi(t,x) : \ t \in \mathbb{R}\}$ is called the orbit of $\mathcal{V}$ through x. For a subset A of M, we denote by $\text{int}(A)$, $\text{Bd}(A) = \overline{A} - \text{int}(A)$ the interior and the boundary of A, respectively. A is called *invariant* by $\mathcal{V}$ if it is a union of orbits. The *ω-limit* set $\omega(x)$ and the *α-limit* set $\alpha(x)$ of x are defined by

$$\omega(x) = \{y \in M : \text{there exists } t_n \mapsto +\infty \text{ such that } \lim_{n\to+\infty} \phi(t_n, x) = y\}.$$

$$\alpha(x) = \{y \in M : \text{there exists } t_n \mapsto -\infty \text{ such that } \lim_{n\to+\infty} \phi(t_n, x) = y\}.$$

Let L be the orbit passing through x. Since $\alpha(x) = \alpha(y)$ (resp. $\omega(x) = \omega(y)$) for $y \in L$, we denote $\Omega_L = \omega(x)$ (resp. $A_L = \alpha(x)$) and called the *ω-limit* set (resp. *α-limit* set) of L. It is known that Ω_L (resp. A_L) is closed, connected, invariant and non empty.

We say that an orbit L of $\mathcal{V}$ is *proper* if $\overline{L} - L$ is closed in M. For example, a closed orbit in M^* is proper. In particular, if L is a periodic orbit, it is proper and called *trivial recurrent*. A non proper orbit L is called *nontrivial recurrent*, that is, either *locally dense* if $\overline{L}$ has non-empty interior, or *exceptional* if L is nowhere dense. For a nontrivial recurrent orbit L, we have one of the following type: (ω-recurrent) if $L \subset \Omega_L$, (α-recurrent) if $L \subset A_L$, or both (ω-recurrent and α-recurrent) if $\Omega_L = A_L = \overline{L}$. Therefore if L is a nontrivial recurrent orbit we have always $\overline{L} = \Omega_L$ or $\overline{L} = A_L$.

Definition 1. We call *quasiminimal set* of $\mathcal{V}$ to the closure $\overline{L}$ of a nontrivial recurrent orbit L. If L is locally dense, $\overline{L}$ is called a *quasiminimal set of dense type*. If L is exceptional, $\overline{L}$ is called a *quasiminimal set of exceptional type*.

Let M_1 be the union of all closed orbits in M^* and we let $U_1 = M^* - M_1$. If sing($\mathcal{V}$) is finite, then the set U_1 is open in M by ([3, Theorem p. 386]).

Definition 2. Let L be a locally dense orbit. The connected component of U_1 containing L is called the *minimal component* containing L.

Hence any minimal component is a connected open and invariant set whenever sing($\mathcal{V}$) is finite.

We say that $\mathcal{V}$ is *divergence-free vector field* on M if the divergence operator of $\mathcal{V}$ vanishes (i.e. div($\mathcal{V}(x)) = 0$ for $x \in M$). Let μ be the normalized Lebesgue measure of M. By Liouville's Theorem, $\mathcal{V}$ is a divergence-free vector field on M if μ is ϕ-invariant; that is for any measurable set A on M, we have $\mu(\phi(t, A)) = \mu(A)$ for every $t \in \mathbb{R}$.

In [5], Ma and Wang considered a class of vector fields called *regular*. A vector field $\mathcal{V}$ on M is said to be *regular* if all its singular points are non-degenerate. Recall that a singular point $x \in M$ is called *non-degenerate* if the Jacobian matrix $\mathrm{D}\mathcal{V}(x)$ is invertible. We know that a non degenerate singular point of a C^1-vector field $\mathcal{V}$ must be either a center or a saddle or a node or a focus or a center-focus (cf. [8, Sections 2.8 and 2.10]). Notice that a center-focus cannot occur whenever $\mathcal{V}$ is analytic (see [8, Corollary of Theorem 5, p. 144]). Now if $\mathcal{V}$ is a divergence-free vector field, the recurrent points are dense in M, which implies that a non-degenerate singular point of $\mathcal{V}$ can be either a center or a saddle but not a node nor a focus or a center-focus.

The main results of this paper can be stated as follows:

Theorem 1. *Let M be a closed surface and let $\mathcal{V}$ be a divergence-free C^1-vector field with finitely many singularities on M. Then every orbit L of $\mathcal{V}$ is one of the following types:*

(i) a singular point,

(ii) a periodic orbit,

(iii) a closed (non periodic) orbit in M^*,

(iv) a locally dense orbit.

Corollary 1. ($Limit set Theorem$) *Let M be a closed surface and let $\mathcal{V}$ be a divergence-free C^1-vector field with finitely many singularities on M. For every non compact orbit L, its limit sets Ω_L and A_L, are one of the following types:*

(i) a singular point,

(ii) $\overline{L}$ and is the closure of the minimal component X containing L with $\mathrm{Bd}(X)$ consisting of singularities and closed (non compact) orbits in M^*.

The description of a minimal component in (ii) is given in Proposition 3.

Remark 1. When M is of genus 0, the case (ii) in Corollary 1 cannot occur since we know by the Poincaré-Bendixson Theorem that there are no nontrivial recurrence on surface of genus 0 (closed or not). However, there exist vector fields with a dense orbit on any other closed orientable surface and on any non orientable closed surface of genus greater or equal than 3 (cf. [4]).

As a consequence:

Corollary 2. *[5] Let M be a closed surface and $\mathcal{V}$ be a regular divergence-free C^1-vector field on M. Let V be the union of periodic orbits and centers. Then for every orbit L, its limit sets Ω_L and A_L, are one of the following types:*

(i) a saddle point,

(ii) a periodic orbit but not a limit cycle,

(iii) a connected closed domain $D \subset \overline{M - \overline{V}}$ with boundaries $\mathrm{Bd}(D)$ consisting of saddle connections having finite length.

In Corollary 2:

– a limit cycle is a periodic orbit γ of $\mathcal{V}$ that is either the Ω_L or the A_L of some orbit L of $\mathcal{V}$ other than γ,

– a closed domain is a set D such that $D = \overline{\mathrm{int}(D)}$,

– a saddle connection is an orbit L of $\mathcal{V}$ such that Ω_L (resp. A_L) is a saddle point.

Let $\mathrm{Per}(\mathcal{V})$ be the set of periodic orbits and suppose that $\mathrm{int}(\mathrm{Per}(\mathcal{V}))$ is non empty. We call a *periodic component* a connected component of $\mathrm{int}(\mathrm{Per}(\mathcal{V}))$.

Let $\overline{L_1},\ \overline{L_2}, \ldots,\ \overline{L_p}$ be the quasiminimal sets of dense type of $\mathcal{V}$ since we know by ([1]), that the quasiminimal sets of $\mathcal{V}$ are of finite number $\leq g$ (if M is orientable), and $\leq [\frac{g-1}{2}]$ (if M is non orientable), where g is the genus of M and $[\ ]$ is the integer part. Therefore there are a finite number of minimal components which correspond to L_1, L_2, ..., L_p.

With the above notations, we obtain immediately the following structure Theorem:

Theorem 2. (*Structure Theorem*) *Let $(M;\mathcal{V})$ be as in Theorem 1. Then, the complementary in M of periodic components and minimal components is a compact invariant subset consisting of singularities and closed (non periodic) orbits in M^*.*

We mention that in ([5, Section 3]) circle bands and circle cells correspond here to periodic components, and ergodic set corresponds to quasiminimal set of dense type. As a consequence, the structural classification Theorem ([5, Theorem 3.1]) follows from Theorem 2.

Throughout this paper, M is a closed surface and $\mathcal{V}$ be a C^1-vector field with finitely many singularities on M.

The outline of this paper is as follows. In Section 2 we recall some results for general vector fields with finitely many singularities on closed surfaces and some properties of nontrivial recurrent orbits. The description of minimal components is given in Section 3. In Section 4, we prove the results stated in Section 1.

2. Recurrence and general results

In this section, we start first with some general results valid for any C^1-vector fields with finitely many singularities on closed surfaces, and second we state some known properties of the dynamic of nontrivial recurrent orbits. The following theorems was proven for M, closed orientable surface, and remain true if M is non orientable by passing to the orientation covering of M.

Theorem 3. (*Limit set Theorem*) *[6] Let M be a closed surface and let $\mathcal{V}$ be a C^1-vector field with finitely many singularities on M. For every orbit L of $\mathcal{V}$, each of its limit sets Ω_L (resp. A_L) is one of the following types:*

(i) a singular point,

(ii) a periodic orbit,

(iii) a union of singularities and closed (non periodic) orbits in M^*,

(iv) a quasiminimal set.

This theorem can be considered as a generalization of the Poincaré-Bendixson Theorem on non-zero genus surfaces. As we know, when M is of genus 0, the Poincaré-Bendixson Theorem asserts that only first three could happen since there are no nontrivial recurrence.

The following theorems characterize the dynamic of $\mathcal{V}$ near a proper non closed orbit in M^* and near an exceptional orbit:

Theorem 4. *([7, Theorem 2.1']). Let $(M;\mathcal{V})$ be as in Theorem 3. Let G be a proper orbit and $O \subset \Omega_G$ (resp. A_G) be an orbit. Then there exists a connected, open, invariant neighborhood W of G such that: Every orbit γ of $\mathcal{V}_{|W}$, is proper and $O \subset \Omega_\gamma$ (resp. $O \subset A_\gamma$).*

Theorem 5. *([7, Theorem 2.2]) Let $(M;\mathcal{V})$ be as in Theorem 3. Let L be an exceptional orbit of $\mathcal{V}$. Then there exists a connected, open, invariant neighborhood W of L such that: For every orbit G of $\mathcal{V}_{|W}$, $L \subset \overline{G}$. Moreover $\Omega_G = \overline{L}$ or $A_G = \overline{L}$.*

We give below some properties concerning nontrivial recurrent orbits.

Definition 3. We call *class of an orbit* L of $\mathcal{V}$ the union $\mathrm{cl}(L)$ of orbits G of $\mathcal{V}$ such that $\overline{G} = \overline{L}$.

Notice that orbits which are in the same class are either all proper or locally dense or exceptional. In particular, if L is proper then $\mathrm{cl}(L) = L$.

We call *lower structure* of an orbit L of $\mathcal{V}$ the subset $\mathrm{SI}(L) = \overline{L} - \mathrm{cl}(L)$. In the case where L is proper, $\mathrm{SI}(L) = \overline{L} - L$ is always closed in M.

Proposition 1. *([6, Proposition 2.1]). If L is a nontrivial recurrent orbit of $\mathcal{V}$ then every orbit contained in $\mathrm{SI}(L)$ is closed in M^*.*

Proposition 2. *Let L be an orbit of $\mathcal{V}$. If $\overline{L}$ contains a periodic orbit γ then L is proper.*

Proof. Suppose that L is non proper. Then $\overline{L}$ is one of its limit set, say $\overline{L} = \Omega_L$. By ([2, Proposition 7.11]), we will have $\Omega_L = \gamma$ and thus $\gamma = \overline{L}$, which is impossible. □

Corollary 3. *If L is a nontrivial recurrent orbit of $\mathcal{V}$ then $\mathrm{cl}(L) = \overline{L} \cap U_1$.*

Proof. Let L be a nontrivial recurrent orbit. If $G \subset \overline{L} \cap U_1$ is an orbit of $\mathcal{V}$ then G is non closed in M^*. From Proposition 1, we have $\overline{G} = \overline{L}$. So, $G \subset \mathrm{cl}(L)$ and therefore $\mathrm{cl}(L) = \overline{L} \cap U_1$. □

3. Structure of minimal components

In this section, we give a description of the dynamic and topological structure of the minimal components.

Proposition 3. *Let L be a locally dense orbit of $\mathcal{V}$ and X be the minimal component containing L. Then:*

(i) $\mathrm{cl}(L) = X$,

(ii) *every orbit of $\mathcal{V}_{|X}$ is dense in X,*

(iii) *$\mathrm{Bd}(X)$ is a union of singularities and closed (non periodic) orbits in M^*.*

Proof. Assertion (i). Since L is locally dense, $L \subset \text{int}(\overline{L})$ and therefore $\text{cl}(L) \subset \text{int}(\overline{L})$. By Corollary 3, we have also $\overline{L} \cap X = \text{cl}(L)$. It follows that $\text{cl}(L) = \overline{L} \cap X = \text{int}(\overline{L}) \cap X$. Thus, $\text{cl}(L)$ is open and closed in X. As X is connected, we have $\text{cl}(L) = X$.

Assertion (ii) is clear since for every orbit G of $\mathcal{V}_{|X}$, we have $\overline{G} = \overline{L} = \overline{X}$.

Assertion (iii): Since $\text{cl}(L) = X$, $\text{Bd}(X) = \overline{X} - X = \text{SI}(L)$. Hence, Assertion (iii) follows from Propositions 1 and 2. ☐

We deduce that if L is a locally dense orbit then $\overline{L}$ is the closure of the minimal component X containing L and the restriction $\mathcal{V}_{|X}$ of $\mathcal{V}$ to X is minimal. This later justifies the terminology of minimal component.

To describe the topological structure of a minimal component, notice that a minimal component X is a connected open subset of M. Hence, viewed as an open surface, we can define its genus $g(X) = k$ (cf. [9]). Let g be the genus of M. We know by the Poincaré-Bendixson Theorem that there is no locally dense orbit on orientable surface of genus 0. It follows that we have $1 \leq k \leq g$ if M is orientable. In particular, when M is the torus then $g(X) = 1$, so X is an open torus. Also, there is (cf. [1]) no a locally dense orbit on non orientable surfaces with genus $g \leq 2$. Therefore, we have $3 \leq k \leq g$ if M is non orientable.

In the case where $\mathcal{V}$ has a dense orbit, we obtain the following Corollary:

Corollary 4. *If L is a dense orbit in M then:*

(i) U_1 *is the minimal component containing* L*,*

(ii) every orbit of $\mathcal{V}_{|U_1}$ *is dense in* U_1*,*

(iii) $M - U_1$ *consists of sing($\mathcal{V}$) and some closed (non periodic) orbits in* M^**.*

Proof. By Corollary 3, we have $\text{cl}(L) = \overline{L} \cap U_1 = U_1$ and then $\text{Bd}(U_1) = M - U_1$. Hence, the Corollary is a consequence of Proposition 3. ☐

4. Proofs

Since $\mathcal{V}$ is a divergence free-vector field, one can associate to $\mathcal{V}$ a transverse invariant measure ν for $\mathcal{V}$ as follows:

ν is a positive and finite Borel measure ν_J on each transverse segment J to $\mathcal{V}$ (the extremities of J may be singularities of $\mathcal{V}$). The collection of (ν_J) must have the following properties:

(i) If $J \subset J'$, the induced measure $(\nu_{J'})_{|J}$ on J is ν_J.

(ii) The measure ν_J is kept invariant if we move J such that its extremities do not change orbits.

Proof of Theorem 1. Suppose that L is non closed in M^*; that is $L \subset U_1$. Then, Ω_L or A_L is not a singular point, say Ω_L. Then there exists an orbit $O \subset \Omega_L$. We will show that L is locally dense. For this, we divide the proof into two cases as follows:

Case 1. *L is a proper orbit.* Then, by Theorem 4, there exists an open, connected, invariant neighborhood W of L such that: every orbit γ of $\mathcal{V}_{|W}$ is proper and $O \subset \Omega_\gamma$. Let T be an open transverse arc such that $O \cap T \neq \emptyset$, and let $x \in L \cap T$. Since L is proper and non-closed in M^*, there exists ([10, Theorem 6.2, p. 264]) an arc $]a_x, b_x[\subset W \cap T$ containing x such that for every orbit G through $z \in]a_x, b_x[$, we have $L^+(z) \cap]a_x, b_x[= \{z\}$, where $L^+(z) = \{\phi(t, z) : t \in \mathbb{R}_+\}$. We deduce that the positive semi-orbit $L^+(x)$ will be contained in the image of an injective immersion φ of $]-1, 1[\times[0, +\infty[$ in W which sends each semi-straight line $\{t\} \times [0, +\infty[$ on a semi-orbit of $\mathcal{V}_{|W}$. The existence of this immersion implies that $\nu(T)$ is infinite, this contradicts the definition of ν.

Case 2. *L is non proper.* Suppose that L is not locally dense; that is L is an exceptional orbit. Then, by Theorem 5, $W - \text{cl}(L)$ is non empty (because on the contrary, $\text{cl}(L)$ will be open and then L will be locally dense, a contradiction). Therefore, there exists a proper orbit $G \subset W$ such that $L \subset \overline{G}$ and $\Omega_G = \overline{L}$ or $A_G = \overline{L}$, say for example, $\Omega_G = \overline{L}$. As in the proof of the case 1, the measure $\nu(T)$ will be infinite, a contradiction. □

Proof of Corollary 1. The proof will be a consequence of Theorem 1, the Proposition 4 below and Proposition 3, respectively.

Proposition 4. *If L is a locally dense orbit then Ω_L (resp. A_L) is one of the following types:*

(i) a singular point,

(ii) $\overline{L}$.

Proof. It suffices to prove the proposition for Ω_L, the same holds for A_L. If L is ω-recurrent that is $L \subset \Omega_L$ then $\Omega_L = \overline{L}$. Now suppose that $L \not\subset \Omega_L$. Then $\Omega_L \subset \overline{L} - \text{cl}(L) = \text{SI}(L)$. To prove that Ω_L is a singular point, suppose the contrary: there exists an orbit $O \subset \Omega_L$. Take a point $x \in L$. Since $\Omega_L \subset \text{SI}(L)$, so the semi-orbit $L^+(x) = \{\phi(t, x) : t \in \mathbb{R}_+\}$ is proper and we have $O \subset \Omega_L^+(x) = \Omega_L$. By the same proof as in the Case 1 of Theorem 1 before, we get a contradiction. Therefore, Ω_L is reduced to a singular point. □

Proof of Corollary 2. Let L be an orbit of $\mathcal{V}$. If L is a periodic orbit then it is not a limit cycle by Theorem 1. Assume that L is neither a saddle point, nor a periodic orbit. By Corollary 1, Ω_L (resp. A_L) is a saddle point or $\overline{L}$, where L is locally dense. We let $D = \overline{L}$. Then $\text{int}(D)$ is non empty and we have $D = \overline{\text{int}(D)}$. So D is a closed domain. Moreover, if V is the union of periodic orbits and centers then $D \subset \overline{M - \overline{V}}$ (otherwise, $\overline{L}$ will contains a periodic orbit, a contradiction by Proposition 2). Now, since $\text{cl}(L) \subset \text{int}(D)$, we have $\text{Bd}(D) = D - \text{int}(D) \subset \overline{L} - \text{cl}(L) = \text{SI}(L)$. Hence, by Proposition 1, $\text{Bd}(D)$ is a finite union of saddle points and saddle connections. □

Proof of Theorem 2. Since U_1 is open, so for each periodic component C, $\text{Bd}(C)$ consists of singularities and closed (non periodic) orbit in M^*. Also, from Proposition 3, for each minimal component X, $\text{Bd}(X)$ consists of singularities and closed (non periodic) orbits in M^*. Therefore Theorem 2 follows from Theorem 1. □

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