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Attractors and chain recurrence for semigroup of continuous maps

Attracteurs et chaîne de récurrence pour le semi-groupe des fonctions continues

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ABSTRACT. In [12], we extended various notions of recurrence for the action of a semigroup analogous to their counterpart in the classical theory of dynamics. In this paper, we shall address the alternative definition of chain recurrent set in terms of attractors, given by Hurley in [10] following Conley's characterization in [5]. We shall also discuss the notion of topological transitivity and chain transitivity in this general setting.
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1. Introduction

A principal aim of this article is to study the notion of attractors and chain recurrence in the context of a semigroup of continuous self maps of a topological space. The concept of chain recurrence in the context of flows, continuous action of a group $(\mathbb{R}, +)$, on a compact metric space was originally introduced by Conley [5]. He described the set of chain recurrent points in terms of attractor and complementary repeller. Later, in [8, 9, 10], Hurley extended this characterization for semiflows, continuous action of a semigroup of nonnegative reals or integers, without the assumption of compactness. Here, we shall follow the treatment of Hurley for compact metric spaces and see how far this characterization applies in this setting of a semigroup.

Most of the questions of dynamics depend on the action of the semigroup. In [3], Barros et al. developed the notion of attractors and chain recurrence for arbitrary semigroup actions. In [16], Souza studied the concept of Poincaré recurrence for semigroup action. They introduced the concept of ω-limit points using a family of subsets of the semigroup, somewhat inspired from the notion of Furstenberg family. In [4], Carvalho et al. studied Poincaré recurrence for the action of finitely generated free semigroup of continuous maps and questions of first return or hitting time maps, the entropy of semigroup actions and the Lyapunov exponents of the generators. We have investigated the dynamics of semigroups of continuous self maps of a topological space. The primary aim is to see, to what extent, the theory of classical dynamics can be applied in this more general setting.

A continuous semigroup is a set of (non-identity) continuous self maps, of a topological space X which is closed under the composition. A semigroup G is said to be generated by a family $\{g_\alpha\}_\alpha$ of continuous self maps of a topological space X if every element of G can be expressed as a composition of finitely many iterates of the elements of the set of generators $\{g_\alpha\}_\alpha$. We denote this by $G = \langle g_\alpha \rangle_\alpha$. The space

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X is assumed to be Hausdorff and first countable throughout.

For any $x \in X$, the subset $\mathcal{O}_G(x) := \{g(x) : g \in G\} \cup \{x\}$ is called the *orbit* of x under G. A subspace $Y \subset X$ is said to be *invariant* under G if $g(y) \in Y$ for all $g \in G$ and $y \in Y$. If $G = \langle g_\alpha \rangle_\alpha$ then for Y to be invariant it is sufficient that $g(y) \in Y$ for all generators $g \in \{g_\alpha\}_\alpha$ and $y \in Y$.

In [11], we have introduced the notion of topological conjugacy of dynamics of two semigroups on two topological spaces as follows: Let $G = \langle g_i \rangle_{i \in \Lambda}$ be a semigroup of continuous self maps of X and $\widetilde{G} = \langle \widetilde{g}_i \rangle_{i \in \Lambda}$ be a semigroup of continuous self maps of Y. Two dynamical systems (X, G) and $(Y, \widetilde{G})$ are said to be topologically conjugate if there exists a homeomorphism $\rho : X \to Y$ such that $\rho \circ g_i = \widetilde{g}_i \circ \rho$ for each $i \in \Lambda$. Note that for $g \in G$, we have, $g = g_{i_1} \circ \cdots \circ g_{i_n}$ for some $g_{i_j} \in \{g_i\}_{i \in \Lambda}$. If $\rho : X \to Y$ is a topological conjugacy, then

$$\begin{aligned} \rho \circ g &= \rho \circ g_{i_1} \circ \cdots \circ g_{i_n} \\ &= \widetilde{g}_{i_1} \circ \cdots \circ \widetilde{g}_{i_n} \circ \rho \\ &= \widetilde{g} \circ \rho \end{aligned}$$

for $\widetilde{g} \in \widetilde{G}$.

In ([3, 16]), the concept of a chain and chain recurrence was generalized for the action of a semigroup on a paracompact Hausdorff space and for choice of a family of subsets of the semigroup. In [12], we have defined chain recurrence for the action of a semigroup on a metric space, which generalizes the classical definition in a canonical way. Let G be a semigroup acting on a metric space (X, d). Let $a, b \in X$, $g \in G$ and $\epsilon > 0$ be given. An (ϵ, g)-chain from a to b means a finite sequence $(a = x_1, \ldots, x_{n+1} = b; g_1, \ldots, g_n)$, where for every i, $x_i \in X$ and $g_i \in \widehat{G}$ such that $d(g_i \circ g(x_i), x_{i+1}) < \epsilon$ for $i = 1, \ldots, n$. A pair of points $a, b \in X$ are called *chain equivalent points* for G if for every $\epsilon > 0$ and every $g \in G$ there exists an (ϵ, g)-chain from a to b and an (ϵ, g)-chain from b to a. The set of all chain equivalent points for G is denoted by $\mathsf{CE}(G)$. A point $x \in X$ is called a *chain recurrent point* for G if for every $\epsilon > 0$ and every $g \in G$ there exists an (ϵ, g)-chain from x to itself. The set of all chain recurrent points for G is denoted by $\mathsf{CR}(G)$.

In [8], Hurley had shown that chain recurrent set for a semiflow is the complement of the union of the set $B(A) \setminus A$, as A varies over the collection of attractors and $B(A)$ denotes the basin of attraction. It says that a chain recurrent point that is in the basin of an attractor must lie in the attractor only. We will see the counterpart of this beautiful result in the context of action of a semigroup in Theorem 2.5. In setting of a semigroup, the situation is fairly complicated. The omega limit set $\omega(x)$ for a point in the phase space need not be same as the omega limit set for any of the point in the orbit of x, classically these sets were same. Since the concept of attractor is based on the notion of limit set, we need to reformulate it in general setting. We have established this in [11].

Furthermore, we shall discuss the concepts of transitivity in the context of a semigroup. We define topological transitivity and chain transitivity in this more general setting, analogous to their counterpart in the classical theory. Subsequently, we shall develop a parallel theory of transitivity for semigroups.

Although, this is a self-contained exposition, for general reference to standard terms and basic facts from the classical theory of dynamics, we recommend Alongi and Nelson's book [1].

2. Attractor

This section consists of systematic investigation to reproduce the concept of attractors for a semigroup. The definition of attractor has substantial variation in the literature. Different attempts incorporated to describe the asymptotic limits for orbits in some neighbourhood of a set. A significant exposition of that can be found in [7, 13]. In [5], Conley has introduced the notion of an attractor as an ω-limit set of a neighbourhood of it. We shall define an attractor for a semigroup in analogous way.

In [11], we have introduced the notion of ω-limit point for the semigroup G as follows: A point $z \in X$ is called an *ω-limit point* for a point $x \in X$ if for some unbounded sequence (f_k) in G, $f_k(x) \to z$ as $k \to \infty$. We denote by $\omega(x)$ the set of all ω-limit points for x. A sequence of functions $(f_k)_{k\in\mathbb{N}}$ in $G = \langle g_\alpha \rangle_\alpha$ is said to be *unbounded* if there is

1. a sequence $(n_k)_k$ of natural numbers with $n_k \to \infty$ as $k \to \infty$; and
2. a generator $g_{\alpha_0} \in \{g_\alpha\}_\alpha$ such that each f_k consists of exactly n_k iterates of g_{α_0}, that is, $f_k = h_{n_k+1} \circ g_{\alpha_0} \circ h_{n_k} \circ g_{\alpha_0} \circ h_{n_k-1} \circ \cdots \circ h_2 \circ g_{\alpha_0} \circ h_1$, where each $h_i \in \widehat{G} = G \cup \{\text{identity}\}$ and the functions h_i are independent of g_{α_0}.

In ([3, 16]), the concept of limit sets was introduced for a family of subsets of a semigroup. The ω-limit set of a topological space X for a family $\mathcal{F}$ of subsets of a semigroup S was defined as

$$\omega(X, \mathcal{F}) = \bigcap_{A\in\mathcal{F}} \mathsf{cl}\{sx : s \in A \text{ and } x \in X\}.$$

Consequently, for different families of subsets, one will get different limit sets. However, we can recover the original concept by choice of a particular family, viz. $\{s \in \mathbb{R} : s \geq t,\ t \in \mathbb{R}\}$.

Here, we have given the notion of an unbounded sequence in a semigroup and using that we define the limit points for a semigroup in the above definition. Our definition follows the sequential approach for obtaining the limit points. Indeed, the unbounded sequences establish a direction for limiting behaviour, as in the classical case.

Now, we introduce the notion of an attractor for a semigroup which is related to the concept of limit points and depends on the concept of unbounded sequence. In ([3, 16]), the concept of attractor depends on a family $\mathcal{F}$ of subsets of a semigroup S. An attractor is defined as a set A which admits a neighbourhood V such that $\omega(V, \mathcal{F}) = A$. Under certain hypothesis on the family, they also defined the complementary repeller of an attractor to establish Conley's characterization following his treatment in [5]. Here, we have established this in Theorem 2.5 using the concept of basin of attraction.

Definition 2.1. Let G be a semigroup acting on a metric space (X, d). An open subset U of X is said to be a *trapping region* for G if there exists an $h \in G$ such that for the set

$$\widetilde{U} := \{fh(x) : x \in U \text{ and } f \in \widehat{G}\},$$

we have $\mathsf{cl}(\widetilde{U}) \subset U$.

Definition 2.2. The *attractor* for G determined by a *trapping region* U for G is defined by

$$\begin{aligned} A := \{&x \in X : \text{for every open subset } V \text{ of } X \text{ containing } x,\ V \cap f_k(h(U)) \neq \emptyset \\ &\text{for infinitely many } k \in \mathbb{N}, \text{where } (f_k)_k \text{ is some unbounded sequence in } G\}, \end{aligned}$$

where $h \in G$ corresponds to the trapping region U as in the Definition 2.1.

Definition 2.3. Let G be a semigroup on a compact metric space (X, d). The *basin of attraction* of an attractor A for G is defined by

$$B(A) := \{x \in X : \omega(x) \cap A \neq \emptyset\}.$$

In extending Conley's theorem in the context of a semigroup we have augmented the concept of a basin. For a compact dynamical system (X, f), if $z \in \omega(x)$ and $z \in A \subset U$ by Proposition 2.3, then openness of U implies the orbit of x must meet U. It follows that $\omega(x) \subset A$. Thus, in the classical case of discrete or continuous dynamical systems, this new definition is equivalent to the old one. We notice the following characteristics of attractors and basin of attraction:

Proposition 2.1. *The attractor A determined by a trapping region U is invariant under G.*

Proof. Let $z \in A$ and g_α be a generator of G. Let V be an open subset of X containing $g_\alpha(z)$. Then $g_\alpha^{-1}(V)$ is an open subset of X containing z.

By Definition 2.2, $g_\alpha^{-1}(V) \cap f_k(h(U)) \neq \emptyset$ for infinitely many $k \in \mathbb{N}$, where $(f_k)_k$ is some unbounded sequence in G. This gives $g_\alpha g_\alpha^{-1}(V) \cap g_\alpha f_k(h(U)) \neq \emptyset$ and hence $V \cap g_\alpha f_k(h(U)) \neq \emptyset$ for infinitely many $k \in \mathbb{N}$.

Also, for a generator g_α of G, $(g_\alpha f_k)_k$ is an unbounded sequence. If (n_k) is the sequence of naturals corresponding to the sequence (f_k) and let the corresponding generator be $g_{\alpha'}$. Take $c = 1$ if $g_\alpha = g_{\alpha'}$ and $c = 0$ otherwise. Now consider the sequence (m_k) given by $m_k = n_k + c$. Then for each k, $g_\alpha f_k$ consists of exactly m_k iterates of $g_{\alpha'}$. It follows that $(g_\alpha f_k)$ is an unbounded sequence, and hence $g_\alpha(z) \in A$. ☐

Proposition 2.2. *The attractor A determined by a trapping region U is a closed set.*

Proof. Let $y \in \mathsf{cl}(A)$. Then there is a sequence (z_n) in A converging to y. Let V be an open subset of X containing y. Since (z_n) converges to y, there exists $N \in \mathbb{N}$ such that $z_n \in V$ for all $n \geq N$. Also $z_N \in A$ and V is an open set containing z_N, then by Definition 2.2, $V \cap f_k(h(U)) \neq \emptyset$ for infinitely many $k \in \mathbb{N}$, where $(f_k)_k$ is some unbounded sequence in G. Thus, we have, $y \in A$. ☐

Proposition 2.3. *Let A be the attractor determined by a trapping region U. If the phase space X is compact then A is contained in U.*

Proof. For a trapping region U, there will be an $h \in G$ such that $\mathsf{cl}(\widetilde{U}) \subset U$. Also (X, d) is a compact metric space, therefore $\mathsf{cl}(\widetilde{U})$ is compact. Thus, there exists $\eta > 0$ such that if $y \in \mathsf{cl}(\widetilde{U})$, $z \in X$, and $d(y, z) < \eta$, then $z \in U$.

Now, if $x \in A$ and $\epsilon \in (0, \eta]$ then $B_\epsilon(x) \cap f_k(h(U)) \neq \emptyset$ for infinitely many $k \in \mathbb{N}$, where $(f_k)_k$ is some unbounded sequence in G and $B_\epsilon(x)$ is an open ball radius ϵ centred at x. Also,

$$f_k(h(U)) \subset \widetilde{U} \subset \mathsf{cl}(\widetilde{U}).$$

Therefore $B_\epsilon(x) \cap \mathsf{cl}(\widetilde{U}) \neq \emptyset$. Let $y \in B_\epsilon(x) \cap \mathsf{cl}(\widetilde{U})$. Since

$$d(x, y) < \epsilon \leq \eta,$$

we have $x \in U$. ☐

Proposition 2.4. *Let A be the attractor determined by a trapping region U. If the phase space X is compact then $B(A)$ contains U.*

Proof. Let $x \in U$ and (f_k) be an unbounded sequence. Since X is compact there is a $z \in X$ and a subsequence (n_{k_l}) such that $f_{k_l}h(x) \to z$, where $h \in G$ is such that $\mathsf{cl}(\widetilde{U}) \subset U$. Let V be an open subset of X containing z. Then there exists an $N \in \mathbb{N}$ such that

$$V \cap f_{k_l}h(x) \neq \emptyset, \quad \text{for all } l \geq N.$$

Thus,

$$V \cap f_{k_l}h(U) \neq \emptyset, \quad \text{for all } l \geq N.$$

Therefore $z \in A \cap \omega(x)$ and hence $U \subset B(A)$. □

Theorem 2.5. *Let (X, d) be a compact metric space and G be an abelian semigroup. The chain recurrent set of G is the complement of the union of sets $B(A) \setminus A$ as A varies over the collection of attractors of G:*

$$X \setminus \mathsf{CR}(G) = \bigcup_A [B(A) \setminus A].$$

Proof. Let A be an attractor of G and $p \in B(A) \setminus A$. Let U be a trapping region which determines A, and $h \in G$ according to Definition 2.1, is such that $\mathsf{cl}(\widetilde{U}) \subset U$. As in Proposition 2.3, there exists $\eta > 0$ such that if $y \in \mathsf{cl}(\widetilde{U})$ $z \in X$, and $d(y, z) < \eta$ then $z \in U$.

For $p \in B(A)$, we have $\omega(p) \cap A \neq \emptyset$. Let $z \in \omega(p) \cap A \subset U$. Therefore there exists an unbounded sequence (f_k) in G and an $N \in \mathbb{N}$ such that

$$f_k(p) \to z$$

and

$$f_k(p) \in U,$$

for all $k \geq N$.

Now if $p \in \mathsf{CR}(G)$ then for each $k \geq N$ and $\epsilon \in (0, \eta]$ there exists an $(\epsilon, f_k h)$-chain $(p = x_1, \ldots, x_{l_k+1} = p; h_1, \ldots, h_{l_k})$ from p to itself. Since G is abelian and $f_k(p) \in U$, we have

$$h_1 f_k h(x_1) \in h_1 h(U) \subset \widetilde{U} \subset \mathsf{cl}(\widetilde{U}).$$

Since

$$d(h_1 f_k h(x_1), x_2) < \epsilon \leq \eta,$$

we have $x_2 \in U$. Similarly,

$$h_2 f_k h(x_2) \in \widetilde{U} \subset \mathsf{cl}(\widetilde{U})$$

and $d(h_2 f_k h(x_2), x_3) < \epsilon \leq \eta$ gives $x_3 \in U$. Following the previous arguments, we have $x_i \in U$ for every $i = 1, \ldots, l_k$. Since

$$d(h_{l_k} f_k h(x_{l_k}), p) < \epsilon,$$

we have,

$$B_\epsilon(p) \cap h_{l_k} f_k h(U) \neq \emptyset.$$

Now the sequence of functions defined by

$$\bar{f}_k = \begin{cases} h_{l_k} f_k, & k \geq N \\ f_k, & k < N \end{cases}$$

is an unbounded sequence in G. For if (n_k) is the sequence of naturals corresponding to the sequence (f_k) and let the corresponding generator be $g_{\alpha'}$. Taking h_{l_k} = identity for $k < N$ and expressing each h_{l_k} as composition of $g_{\alpha'}$ and functions in $\widehat{G}$, which are independent of $g_{\alpha'}$, yields a sequence (n'_k) of non-negative integers. The combine sequence (m_k), where $m_k = n_k + n'_k$, tends to infinity and for each k, $\bar{f}_k$ consists of exactly m_k iterates of $g_{\alpha'}$.

Also for any open subset V of X containing p, there is $\epsilon \in (0, \eta]$ such that $B_\epsilon(p) \subset V$ and an unbounded sequence $\bar{f}_k$ in G corresponding to the ϵ. Thus,

$$V \cap \bar{f}_k(h(U)) \supset B_\epsilon(p) \cap \bar{f}_k(h(U)) \neq \emptyset,$$

for all $k \geq N$. Thus $p \in A$, which is a contradiction. Therefore if $p \in B(A) \setminus A$ then $p \notin \mathsf{CR}(G)$.

Conversely, let $p \notin \mathsf{CR}(G)$. Then there exists an $\epsilon > 0$ and $h_1 \in G$ such that there is no (ϵ, h_1)-chain from p to itself. Consider the set defined by

$$U := \{x \in X : \text{there is an } (\epsilon, h_1)\text{−chain from } p \text{ to } x\}.$$

Then $p \notin U$ and

$$\widetilde{U} := \{f h_1(x) : x \in U \text{ and } f \in \widehat{G}\}.$$

Clearly U is an open subset of X. Indeed, for $x \in U$, there exists an (ϵ, h_1)-chain $(p = x_1, \ldots, x_{n+1} = x; f_1, \ldots, f_n)$ from p to x. Then $x \in B_\epsilon(f_n h_1(x_n))$. If $y \in B_\epsilon(f_n h_1(x_n))$ then $(p = x_1, \ldots, x_{n+1} = y; f_1, \ldots, f_n)$ from p to y is an (ϵ, h_1)-chain and hence $y \in U$. Thus $B_\epsilon(f_n h_1(x_n)) \subset U$ and U is open.

Let $y \in \mathsf{cl}(\widetilde{U})$. Then for some $x \in U$ and $f \in \widehat{G}$, we have $f h_1(x) \in B_\epsilon(y)$. Thus $(x, y; f)$ is an (ϵ, h_1)-chain from x to y. By transitivity, we can produce an (ϵ, h_1)-chain from p to y. Hence $\mathsf{cl}(\widetilde{U})) \subset U$. Therefore U is a trapping region for G.

Let A be an attractor determined by U. Since $p \notin U$ and $A \subset U$, it follows that $p \notin A$. Moreover $(p, h_1(p); \text{identity})$ is an (ϵ, h_1)-chain from p to $h_1(p)$. Hence $h_1(p) \in U$. By Proposition 2.4, we have $\omega(h_1(p)) \cap A \neq \emptyset$. As $\omega(h_1(p)) \subset \omega(p)$ thus $p \in B(A)$. Therefore $p \in B(A) \setminus A$. □

Now Theorem 2.5 generalizes the result obtained by Hurley in [8] following Conley's characterization in [5].

The following example will illustrate an application of the above theorem.

Example 2.1. Consider the family of polynomial mappings on closed unit disc $X = \{z \in \mathbb{C} : |z| \leq 1\}$ in complex plane,

$$g_n : X \to X$$

$$g_n(z) = z^n.$$

The semigroup $G = \{g_n : n \in \mathbb{N} \setminus \{1\}\}$, which is closed under composition, consists of non-identity continuous self maps of the closed unit disc in complex plane $\mathbb{C}$. By Fundamental Theorem of Arithmetic, we can see that the semigroup G has set of generators given by $\{g_p : p \text{ is a prime number}\}$.

We shall identify all the trapping regions and attractors for the dynamics of G on the phase space X. Evidently, the empty set and the phase space itself are trapping regions for G. Moreover the corresponding attractor and basin of attraction coincide with the trapping regions respectively. Thus, we have $B(A) \setminus A = \emptyset$ for each attractor.

For each $r \in (0, 1]$, the set $U_r = \{z \in \mathbb{C} : |z| < r\}$ is a trapping region for G. Since $z \in U_r$ implies $|g(z)| \leq |z|^2 < r$ for every $g \in G$. In particular, for $\widetilde{U}_r = \{g \circ g_2(z) : z \in U_r \text{ and } g \in \widehat{G}\}$, we have $\mathsf{cl}(\widetilde{U}_r) \subset U_r$. Here for each $r \in (0, 1)$, the attractor A_r is $\{0\}$. If $z \in X \setminus \{0\}$ then the open balls $B(0, \frac{|z|}{3})$ and $B(z, \frac{|z|}{3})$ of radius $\frac{|z|}{3}$ centred at 0 and z respectively are disjoint. Now, for all k sufficiently large and $w \in U_r$, we get $g_k \circ g_2(w) \in B(0, \frac{|z|}{3})$. Hence $A_r = \{0\}$.

Also, for $|z| = 1$, we have $\omega(z)$ is contained in unit circle and hence disjoint from A_r. If $|z| < 1$ then $\omega(z) = \{0\} = A_r$. Thus, the basin of attraction $B(A_r) = \{z \in \mathbb{C} : |z| < 1\}$ for each $r \in (0, 1]$. Therefore $B(A_r) \setminus A_r = \{z \in \mathbb{C} : 0 < |z| < 1\}$. By Theorem 2.5, we have

$$\begin{aligned} X \setminus \mathsf{CR}(G) &= \bigcup_A [B(A) \setminus A] \\ &= \emptyset \cup \{z \in X : 0 < |z| < 1\} \\ &= \{z \in X : |z| \neq 0, |z| \neq 1\}. \end{aligned}$$

Consequently, we have $\mathsf{CR}(G) = \{z \in X : |z| = 0 \text{ or } |z| = 1\}$.

Remark 2.6. Herewith a direct proof of $\mathsf{CR}(G) = \{z \in X : |z| = 0 \text{ or } |z| = 1\}$ in Example 2.1. As 0 is a fixed point of G, hence a chain recurrent point for G. Let $z \in \{z \in X : 0 < |z| < 1\}$, we shall show that z is not a chain recurrent point for G. Let $m, M \in \mathbb{N}$ be sufficiently large, so that $1/m$ is small enough such that $\big||g_{_M}(z)| - |z|\big| > \frac{1}{m}$. Then there is no $(\frac{1}{m}, g_{_M})$-chain from z to itself. If $(z = z_1, \ldots, z_{n+1}; h_1, \ldots, h_n)$ is an $(\frac{1}{m}, g_{_M})$-chain from z then $B(h_i \circ g_{_M}(z_i), \frac{1}{m}) \subset B(0, |z|)$ for all i. In particular, $z_{n+1} \in B(h_n \circ g_{_M}(z_n), \frac{1}{m}) \subset B(0, |z|)$ and hence $z \neq z_{n+1}$. Therefore z is not a chain recurrent point for G.

Let $z_0 \in \mathbb{C}$ be such that $|z_0| = 1$, that is, $z_0 = \exp(i\pi\theta_0)$, where $\exp(\cdot)$ is the exponential function. Let $\epsilon > 0$ and $n_0 \in \mathbb{N} \setminus \{1\}$. We shall construct an (ϵ, g_{n_0})-chain from z_0 to itself.

For rationals are dense in reals, we can pick a $z_1 = \exp(i\pi p_1/q_1) \in B_\epsilon(g_{n_0}(z_0)) = \{z : |g_{n_0}(z_0) - z| < \epsilon\}$, for some $p_1, q_1 \in \mathbb{N}$. Then,

$$\begin{aligned} g_{2q_1} g_{n_0}(z_1) &= \exp(i\pi p_1 n_0 2q_1/q_1) \\ &= 1. \end{aligned}$$

Let $p, q \in \mathbb{N}$ be such that $w = \exp(i\pi p/q) \in B_\epsilon(z_0)$. There is an $N \in \mathbb{N}$ sufficiently large such that $z_2 = \exp(i\pi p/(qn_0N)) \in B_\epsilon(1)$. Then $g_{_N} \circ g_{n_0}(z_2) = w$ and $|w - z_0| < \epsilon$.

Therefore, $(z_0, z_1, z_2, z_0; \text{identity}, g_{2q_1}, g_{_N})$ is an (ϵ, g_{n_0})-chain from z_0 to itself. Thus z_0 is a chain recurrent point for G and we have $\mathsf{CR}(G) = \{z \in X : |z| = 0 \text{ or } |z| = 1\}$.

3. Transitivity

In this section, we discuss the notion of topological transitivity and chain transitivity for the action of a semigroup. The topological transitivity is a global attribute of a dynamical system. A topological transitive system has points which move from one neighbourhood to any other, under the dynamics of some map. Thus the dynamical system can not be decomposed into two invariant subsystems. For a general semigroup, the topological transitivity is defined as follows:

Definition 3.1. The dynamical system (X, G) is *topologically transitive* if for any two nonempty open subsets U and V of X, there exists a $g \in \widehat{G} = G \cup \{\text{identity}\}$ such that $g(U) \cap V \neq \emptyset$. A nonempty invariant subset A of X is said to be *topologically transitive* if for any two nonempty open subsets U and V of A, there exists a $g \in \widehat{G}$ such that $g(U) \cap V \neq \emptyset$.

Remark 3.1. Equivalently, the definition for topological transitivity can also be formulated as: if for any two nonempty open subsets U and V of X, there exists a $g \in \widehat{G} = G \cup \{\text{identity}\}$ such that $U \cap g^{-1}(V) \neq \emptyset$. Furthermore, since each $g \in G$ is a continuous self map of X, the set $g^{-1}(V)$ is also open in X. Therefore by definition of topological transitivity, given two nonempty open subsets U and V and a $g \in G$ with $g^{-1}(V) \neq \emptyset$, there exists an $h \in \widehat{G}$ such that $U \cap h^{-1} \circ g^{-1}(V) \neq \emptyset$, or equivalently, $g \circ h(U) \cap V \neq \emptyset$.

Some authors also define the concept of topological transitive system in terms of existence of a dense orbit. We show that if the phase space is second countable Baire space then the topological transitive system has a dense orbit.

Definition 3.2. A subset A of a topological space X is said to be residual in X if A is countable intersection of a collection of open subsets of X each of which is dense in X.

Definition 3.3. A topological space X is said to be a Baire space if given any countable collection of closed subsets of X, each of which has empty interior, their union also has empty interior in X.

Lemma 3.2. *[14] A topological space X is a Baire space if and only if given any countable collection of open subsets of X, each of which is dense in X, their intersection is also dense in X.*

Lemma 3.3. *Let G be a semigroup of continuous self maps of a second countable Baire space X. If $\bigcup_{g \in \widehat{G}} g^{-1}(U)$ is dense in X for every nonempty open subset U of X, then there exists $D \subset X$, such that D is residual in X and $\mathcal{O}_G(x)$ is dense in X for all $x \in D$.*

Proof. Let $\{U_i\}_{i \in \mathbb{N}}$ be a countable base for X. Then for each $i \in \mathbb{N}$, $\bigcup_{g \in \widehat{G}} g^{-1}(U_i)$ is dense in X. Since each $g \in G$ is continuous and U_i is open, hence $g^{-1}(U_i)$ is open. Thus $\bigcup_{g \in \widehat{G}} g^{-1}(U_i)$ is open in X for each $i \in \mathbb{N}$. Therefore, the set

$$D = \bigcap_{i \in \mathbb{N}} \bigcup_{g \in \widehat{G}} g^{-1}(U_i)$$

is residual in X and hence dense in X.

For $x \in D$, we have $x \in \bigcup_{g \in \widehat{G}} g^{-1}(U_i)$ for each $i \in \mathbb{N}$. Then for each $i = 1, 2, 3, \ldots$ there exists a $g_i \in \widehat{G}$ such that $x \in g_i^{-1}(U_i)$, that is, $g_i(x) \in U_i$. Thus, $\mathcal{O}_G(x) \cap U_i \neq \emptyset$ for each $i = 1, 2, 3, \ldots$. Since $\{U_i\}_{i \in \mathbb{N}}$ is a base for the topology of X, we have orbit of x is dense in X. □

Theorem 3.4. *Let G be a semigroup of continuous self maps of a second countable Baire space X. If X is topologically transitive then there exists $x \in X$ such that orbit of x is dense in X.*

Proof. Since X is topologically transitive, we have for any two nonempty open subsets U and V of X, there exists a $g \in \widehat{G}$ such that $U \cap g^{-1}(V) \neq \emptyset$. Therefore the set $\bigcup_{g \in \widehat{G}} g^{-1}(V)$ is dense in X. By Lemma 3.3, there exists $D \subset X$ such that D is residual in X and $\mathcal{O}_G(x)$ is dense in X for all $x \in D$. Since X is a Baire space and D is residual in X and hence D is nonempty. Thus there exists $x \in D$ such that orbit of x is dense in X. □

However, in case of a flow on a perfect set, existence of a point with dense orbit also implies topologically transitive, see [6, 15]. But this fails to hold in case of dynamics of action of a semigroup.

Example 3.1. [2] Let $X = [0, 1]$ be the unit interval with the usual topology and for $0 < \alpha < 1$, define $f_\alpha : X \to X$ as

$$f_\alpha(x) = \alpha x.$$

Now $G = \{f_\alpha : 0 < \alpha < 1\}$ is a semigroup of continuous self maps of X.

The orbit $\mathcal{O}(1) = \{\alpha : 0 < \alpha < 1\} \cup \{1\} = (0, 1]$ is dense in X. Also $U = (0, 1/2)$ is an invariant subset of X. Therefore, we have, for $V = (1/2, 1)$, $g(U) \cap V = \emptyset$ for $g \in \widehat{G}$. Thus (X, G) is not topologically transitive.

The following proposition shows that topological transitivity is preserved under conjugacy. More precisely,

Proposition 3.5. *Let (X, G) and $(Y, \widetilde{G})$ be two dynamical systems. If $\rho : X \to Y$ is a topological conjugacy and $A \subset X$ is topologically transitive then so is $\rho(A)$.*

Proof. Let $\widetilde{U}$ and $\widetilde{V}$ be two nonempty open subsets of Y whose intersection with $\rho(A)$ is also nonempty. Since ρ is a homeomorphism, the sets $U = \rho^{-1}(\widetilde{U})$ and $V = \rho^{-1}(\widetilde{V})$ are open in X and has nonempty intersection with A. Since A is topologically transitive, there exists a $g \in \widehat{G}$ such that $g(U \cap A) \cap (V \cap A) \neq \emptyset$.

Therefore, we have $\rho[g(U \cap A) \cap (V \cap A)] \neq \emptyset$. Since ρ is a topological conjugacy, $\widetilde{g}\rho(U \cap A) \cap \rho(V \cap A) \neq \emptyset$, for $\widetilde{g} \in \widetilde{G}$. Further, $\widetilde{g}(\widetilde{U} \cap \rho(A)) \cap (\widetilde{V} \cap \rho(A)) \neq \emptyset$. Thus $\rho(A)$ is topologically transitive. □

Now we proceed towards the concept of chain transitive system – a more general type of irreducibility that extends topological transitivity.

Definition 3.4. Let G be a semigroup on a metric space (X, d). A nonempty subset A of X is said to be *chain transitive* if for each $a, b \in A$, $\epsilon > 0$ and $g \in G$, there exists an (ϵ, g)-chain from a to b.

Remark 3.6. In the above definition, one can see, by interchanging the roles of a and b, any two points in a chain transitive set are chain equivalent. Also letting $a = b$, we have every element of a chain transitive set is chain recurrent.

The following theorem shows that any topologically transitive subset of a surjective dynamical system is chain transitive.

Theorem 3.7. *If G is an abelian semigroup on a metric space (X,d) such that each $g \in G$ is surjective, then any topologically transitive subset of X with respect to G is chain transitive.*

Proof. Let A be a topologically transitive subset of X and $a, b \in A$. Let $g \in G$ and $\epsilon > 0$ be given. Since g is continuous there exists a $\delta > 0$ such that $d(a,y) < \delta$ implies $d(g(a), g(y)) < \epsilon$, $y \in X$.

Also A is topologically transitive and each $g \in G$ is surjective. By Remark 3.1, for open sets $B(a,\delta)$ and $B(b,\epsilon)$ there exists $h \in \widehat{G}$ such that

$$g^2 \circ h(B(a,\delta) \cap A) \cap (B(b,\epsilon) \cap A) \neq \emptyset.$$

For G is abelian,

$$h \circ g^2(B(a,\delta) \cap A) \cap (B(b,\epsilon) \cap A) \neq \emptyset.$$

Thus there is a $c \in B(a,\delta) \cap A$ such that $hg^2(c) \in B(b,\epsilon) \cap A$. That is, $d(hg^2(c), b) < \epsilon$ and $d(a,c) < \delta$ hence $d(g(a), g(c)) < \epsilon$. So that

$$(a, g(c), b; \text{identity}, h)$$

is an (ϵ, g)-chain from a to b. Therefore A is chain transitive. □

Theorem 3.8. *Let (X,G) and $(Y,\widetilde{G})$ be two dynamical systems on the metric spaces (X,d_X) and (Y,d_Y). If a uniformly continuous homeomorphism $\rho : X \to Y$ is a topological conjugacy and $A \subset X$ is chain transitive then so is $\rho(A)$.*

Proof. Let $x, \bar{x} \in A$. Let $\epsilon > 0$ be given and some $\widetilde{g} \in \widetilde{G}$. We shall construct an $(\epsilon, \widetilde{g})$-chain from $y = \rho(x)$ to $\bar{y} = \rho(\bar{x})$. Let $g \in G$ be such that $\rho \circ g = \widetilde{g} \circ \rho$.

Since $\rho : X \to Y$ is uniformly continuous, there exists a $\delta > 0$ such that if $d_X(x_1, x_2) < \delta$ then $d_Y(\rho(x_1), \rho(x_2)) < \epsilon$, for $x_1, x_2 \in X$. Also $x, \bar{x} \in A$ implies there is a (δ, g)-chain $(x = x_1, \ldots, x_{n+1} = \bar{x}; g_1, \ldots, g_n)$ from x to $\bar{x}$. That is, for each $i = 1, \ldots, n$, we have

$$d_X(g_i \circ g(x_i), x_{i+1}) < \delta.$$

For each $i = 1, \ldots, n+1$, let $y_i = \rho(x_i)$. Since $d_X(g_i \circ g(x_i), x_{i+1}) < \delta$, we have

$$d_Y(\widetilde{g}_i \circ \widetilde{g}(y_i), y_{i+1}) = d_Y(\rho \circ g_i \circ g(x_i), \rho(x_{i+1})) < \epsilon.$$

Thus $(\rho(x) = y_1, \ldots, y_{n+1} = \rho(\bar{x}); \widetilde{g}_1, \ldots, \widetilde{g}_n)$ is an $(\epsilon, \widetilde{g})$-chain from $y = \rho(x)$ to $\bar{y} = \rho(\bar{x})$. Therefore $\rho(A) \subset Y$ is chain transitive. □

Proposition 3.9. *The relation R defined by: aRb if and only if $(a,b) \in \mathsf{CE}(G)$, is an equivalence relation on $\mathsf{CR}(G)$.*

Proof. Reflexive: Let $a \in \mathsf{CR}(G)$. Then for every $\epsilon > 0$ and $g \in G$ there exists an (ϵ, g)-chain from a to itself. Hence aRa.

Symmetric: Let aRb, that is, $(a, b) \in \mathsf{CE}(G)$. Then for every $\epsilon > 0$ and $g \in G$ there exists an (ϵ, g)-chain from a to b and an (ϵ, g)-chain from b to a. Thus bRa.

Transitive: Let aRb and bRc. Then for every $\epsilon > 0$ and $g \in G$ there exists an (ϵ, g)-chain from a to b and an (ϵ, g)-chain from b to a. Also bRc implies for every $\epsilon > 0$ and $g \in G$ there exists an (ϵ, g)-chain from c to b and an (ϵ, g)-chain from b to c. By concatenating the two (ϵ, g)-chains from a to b and b to c, we have an (ϵ, g)-chain from a to c. Again, by concatenating the two (ϵ, g)-chains from c to b and b to a, we have an (ϵ, g)-chain from c to a. Thus, we have aRc. □

Definition 3.5. The equivalence relation R defined by: aRb if and only if $(a, b) \in \mathsf{CE}(G)$, is the chain equivalence relation for G on X. An equivalence class of the chain equivalence relation for G is called a *chain component* of G.

Proposition 3.10. *If G is abelian then every chain component of G is invariant under G.*

Proof. Let C be a chain component of G on a metric space (X, d). Let $x \in C$ and g be any generator of G. Let $\epsilon > 0$ be given and some $h \in G$. We shall construct (ϵ, h)-chains from $g(x)$ to x and x to $g(x)$. As C is a chain component, it will follow that $g(x) \in C$.

For $x \in \mathsf{CR}(G)$, there is an $(\epsilon, g \circ h)$-chain $(x = x_1, \ldots, x_{n+1} = x; h_1, \ldots, h_n)$ from x to itself. That is, for each $i = 1, \ldots, n$, we have

$$d(h_i \circ g \circ h(x_i), x_{i+1}) < \epsilon.$$

In particular, since G is abelian, we have

$$\begin{aligned} d(h_1 \circ h \circ g(x_1), x_2) &= d(h_1 \circ g \circ h(x_1), x_2) \\ &< \epsilon. \end{aligned}$$

Hence $(g(x), x_2, \ldots, x_n, x; h_1, h_2 \circ g, \ldots, h_{n-1} \circ g, h_n \circ g)$ is an (ϵ, h)-chain from $g(x)$ to x.

Further, since g is continuous, there exists a $\delta \in (0, \epsilon]$ such that if $d(x, y) < \delta$ then $d(g(x), g(y)) < \epsilon$. Also $x \in \mathsf{CR}(G)$ implies there is a $(\delta, g \circ h)$-chain $(x = x_1, \ldots, x_{n+1} = x; h_1, \ldots, h_n)$ from x to itself. That is, for each $i = 1, \ldots, n$, we have

$$d(h_i \circ g \circ h(x_i), x_{i+1}) < \delta \leq \epsilon.$$

Also from the continuity of g, we have

$$d(h_n \circ g \circ h(x_n), x) < \delta \qquad \text{implies} \qquad d(g \circ h_n \circ g \circ h(x_n), g(x)) < \epsilon.$$

Hence $(x, x_2, \ldots, x_n, g(x); h_1 \circ g, h_2 \circ g, \ldots, h_{n-1} \circ g, g \circ h_n \circ g)$ is an (ϵ, h)-chain from x to $g(x)$.

Thus $(x, g(x)) \in \mathsf{CE}(G)$. Also, by concatenating the two (ϵ, g)-chains from $g(x)$ to x and x to $g(x)$, we obtain an (ϵ, g)-chain from $g(x)$ to itself. Hence $g(x)$ is a chain recurrent point and an element of C. Therefore every chain component of G is invariant under G, provided G is abelian. □

Next corollary provides another proof of Theorem 5.3 in [12].

Corollary 3.11. *If G is abelian then the chain recurrent set $\mathsf{CR}(G)$ is invariant under G.*

Proof. By Proposition 3.10, every chain component of G is invariant, provided G is abelian. Also the collection of chain components partitions the chain recurrent set. Therefore the chain recurrent set is invariant under G. □

Also, the following theorem presents an alternative proof of Theorem 5.1 in [12].

Theorem 3.12. *Every chain component of G is a closed subset of X.*

Proof. Let C be a chain component of G on a metric space (X, d). Suppose that a is a limit point of C and $y \in C$. Let $\epsilon > 0$ be given and some $g \in G$. First we shall construct an (ϵ, g)-chain from a to y.

Since g is continuous, there exists a $\delta > 0$ such that

$$d(x,a) < \delta \qquad \text{implies} \qquad d(g(x), g(a)) < \epsilon.$$

Take $\delta' = \min\{\delta, \epsilon/2\}$. As a is a limit point of C, there exists an $x \in C$ such that $d(x,a) < \delta'$. Then $d(g(x), g(a)) < \epsilon$.

For $x, y \in C$, there exists a (δ', g^2)-chain $(x = x_1, \ldots, x_{n+1} = y; g_1, \ldots, g_n)$ from x to y. So that $(a, g(x), x_2, \ldots, x_{n+1} = y; \text{identity}, g_1, g_2 g, \ldots, g_n g)$ is an (ϵ, g)-chain from a to y.

Also, $x, y \in C$ implies there exists an $(\epsilon/2, g)$-chain $(y = y_1, \ldots, y_{n+1} = x; h_1, \ldots, h_n)$ from y to x. Further,

$$\begin{aligned} d(h_n g(y_n), a) &\leq d(h_n g(y_n), x) + d(x, a) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon, \end{aligned}$$

implies $(y = y_1, \ldots, y_n, a; h_1, \ldots, h_n)$ is an (ϵ, g)-chain from y to a.

By transitivity via concatenating these (ϵ, g)-chains, there is an (ϵ, g)-chain from a to itself. Hence a is a chain recurrent point and an element of C. Therefore every chain component of G is closed. □

Next we show that a chain component is a maximal chain transitive set.

Proposition 3.13. *A chain component of G is a maximal chain transitive set.*

Proof. By definition of a chain component, any two points in a chain component are chain equivalent. Hence a chain component is a chain transitive set.

Furthermore, by Remark 3.6, every chain transitive set is contained in the chain recurrent set. Since the chain recurrent set has been partitioned by disjoint equivalence classes of chain components, therefore any chain transitive set is contained in a unique chain component. If A and B are two chain transitive sets with $A \subset B$ and C is the unique chain component containing A, then, $B \cap C \neq \emptyset$ and hence C is the unique chain component containing B. Thus a chain component of G is a maximal chain transitive set. □

Theorem 3.14. *Let (X, d) be a compact metric space and G be an abelian semigroup. For $a, b \in \mathsf{CR}(G)$, a and b are chain equivalent if and only if $a, b \in A$ or $a, b \notin B(A)$ for each attractor A for G.*

Proof. Assume that $(a, b) \in \mathsf{CE}(G)$ and A is an attractor for G. Let U be a trapping region which determines A, and following Definition 2.1, $h \in G$ be such that $\mathsf{cl}(\widetilde{U}) \subset U$.

Since a and b are chain equivalent, for $0 < \epsilon < d(\mathsf{cl}(\widetilde{U}), X \setminus U)$ there is an (ϵ, h)-chain

$$(a = x_1, \ldots, x_{n+1} = b; g_1, \ldots, g_n)$$

from a to b. If $x_i \in U$ for some $i = 1, \ldots, n+1$, then

$$g_i \circ h(x_i) \in \widetilde{U} \subset \mathsf{cl}(\widetilde{U}) \subset U.$$

Also, $d(g_i \circ h(x_i), x_{i+1}) < \epsilon < d(\mathsf{cl}(\widetilde{U}), X \setminus U)$ hence $x_{i+1} \in U$.

Thus $a \in A \subset U$ implies $b \in U$. Therefore, by Theorem 2.5, we have $b \in A$.

Further $a \notin B(A)$ implies $b \notin B(A)$. Indeed if $b \in B(A)$ then, by Theorem 2.5, we have $b \in A$. Applying the previous argument to b, we have $a \in A \subset B(A)$, a contradiction.

Conversely, assume that a and b are not chain equivalent. Then there exists an $\epsilon > 0$ and $h_1 \in G$ such that there is no (ϵ, h_1)-chain from a to b or there is no (ϵ, h_1)-chain from b to a. With no loss of generality, assume that there is no (ϵ, h_1)-chain from a to b. Consider the set defined by

$$U := \{x \in X : \text{there is an } (\epsilon, h_1)\text{−chain from } a \text{ to } x\}.$$

Then $a \in U$ and

$$\widetilde{U} := \{f h_1(x) : x \in U \text{ and } f \in \widehat{G}\}.$$

As in the proof of Theorem 2.5, U is a trapping region for G. Let A be an attractor determined by U. Since a is chain recurrent and $a \in U$, we have $a \in A$. But b is also chain recurrent and $b \notin U$, it follows that, $b \notin A$. □

We note here that, for an attractor A for the action of an abelian semigroup on a compact metric space, a chain component lies completely either inside A or in the complement of $B(A)$, the basin of attraction of A.

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