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Weighted composition operators between two different weighted sequence spaces

Opérateurs de composition pondérés d'espaces de suites à poids

Carlos Carpintero[1], Julio C. Ramos-Fernández[2] and José E. Sanabria[3]

[1]Facultad de Ciencias Básicas, Corporación Universitaria del Caribe-CECAR, Sincelejo, Colombia
carlos.carpinterof@cecar.edu.co
[2]Facultad Tecnológica, Universidad Distrital Francisco José de Caldas, Bogotá, Colombia
jcramosf@udistrital.edu.co
[3]Facultad de Educación y Ciencias, Universidad de Sucre, Sincelejo, Colombia
jose.sanabria@unisucre.edu.co

ABSTRACT. We give simple criteria which characterize the symbols u, φ defining continuous and compact weighted composition operators $W_{u,\varphi}$ acting between two different weighted sequence spaces. Also we characterize when $W_{u,\varphi}$ is bounded below and when it has closed range.
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1. Introduction

Through this article, we use $\boldsymbol{x} = \{x(k)\}$ for a numerical complex sequence, while $(\boldsymbol{x}_n)$ will denote a sequence of sequences. A sequence $\boldsymbol{r} = \{r(k)\}$ such that $r(k) > 0$ for all $k \in \mathbb{N}$ will be called *a weight*, where $\mathbb{N}$ denotes the set of all positive integers. Given a weight $\boldsymbol{r}$, the weighted sequence space $l^\infty(\boldsymbol{r})$ is defined as the set of all complex sequence $\boldsymbol{x} = \{x(k)\}$ such that

$$\|\boldsymbol{x}\|_r = \sup_{k\in\mathbb{N}} |x(k)|\, r(k) < \infty. \tag{1.1}$$

The pair $(l^\infty(\boldsymbol{r}), \|\cdot\|_r)$ constitute a Banach space. This kind of spaces appear in the literature in a natural way when one studies properties of some operators in certain sequence spaces; for instance, we can see that for $p > 1$ fixed, the Cesàro space ces_p of all complex sequence $\boldsymbol{x} = \{x(k)\}$ such that

$$\|\boldsymbol{x}\|_{ces_p} = \left(\sum_{m=1}^{\infty}\left(\frac{1}{m}\sum_{k=1}^{m}|x(k)|\right)^p\right)^{1/p} < \infty, \tag{1.2}$$

is contained in $l^\infty\left(k^{1-p}\right)$ which tell us that every evaluation functional on ces_p is continuous.

During the past decade, there has been a surge in new results concerning various linear operators $L : X \to Y$, where at least one of the space X and Y is a sequence space satisfying a growth condition or it is defined by a weight. For instance, Williams and Ye in [15] characterize the continuity of all operators defined by infinite matrices acting from $l^1(\boldsymbol{r})$ into $l^1(\boldsymbol{r})$. Also, this kind of sequence spaces have allowed to define Hilbert spaces of analytic functions such as we can see in the work of Shield [13] and more recently, in the article of Luan and Khoi [7].

For a fixed sequence $\boldsymbol{u} = \{u(k)\}$, a function $\varphi : \mathbb{N} \to \mathbb{N}$ and any $\boldsymbol{x} = \{x(k)\} \in l^\infty(\boldsymbol{r})$, we can define the linear weighted composition operator $W_{u,\varphi}$ with symbols $\boldsymbol{u}, \varphi$ by

$$W_{u,\varphi}(\boldsymbol{x}) := \boldsymbol{u} \cdot (\boldsymbol{x} \circ \varphi).$$

Clearly this operator is obtained by composition of two well known operators: multiplication operator M_u and composition operator C_φ. In fact, when φ is the identity on $\mathbb{N}$, $W_{u,\varphi}$ becomes a multiplication operator which is defined pointwise by $M_u(\boldsymbol{x}) = u \cdot \boldsymbol{x}$. We refer to [12] for properties of multiplication operators acting on Banach sequence spaces (see also [5]). If $u(n) = 1$ for all $n \in \mathbb{N}$ then $W_{u,\varphi}$ becomes a linear composition operator defined as $C_\varphi(\boldsymbol{x}) = \boldsymbol{x} \circ \varphi$ which has been widely studied by a big numerous of authors; in particular, acting on some weighted sequence space such as $l^2(\boldsymbol{r})$ were consider early in [14].

In this article, we are interested in to know when $W_{u,\varphi}(\boldsymbol{x}) \in l^\infty(\boldsymbol{s})$ for all $\boldsymbol{x} \in l^\infty(\boldsymbol{r})$ and to establish other topological properties of $W_{u,\varphi}$ such as compactness, closedness of the range, bounded below among others. This kind of problems have been widely studied in the context of holomorphic function spaces (see for instance [8, 4, 11] and the references cited therein); but in the context of Banach sequence spaces is still a subject in development; for instance, weighted composition operators on l^2 (without weight) was studied by Carlson in [3]. Raj, Komal and Khosla in [9] studied this operator acting on Orlicz sequence spaces and more recently, Luan and Khoi [6] have made an exhaustive study of the properties of $W_{u,\varphi}$ acting on the Hilbert space $l^2(\boldsymbol{r})$.

Motivated by the results due to Montes-Rodríguez [8] and Contreras and Hernández-Díaz [4] (see also [11, 2]), about the properties of weighted composition operators acting on weighted Banach spaces of analytic functions, where the characterizations of the properties of these operators are obtained by composition with certain special functions in this type of spaces, through this article we are going to prove the following results:

1. The operator $W_{u,\varphi}$ is continuous from $l^\infty(\boldsymbol{r})$ into $l^\infty(\boldsymbol{s})$ if and only if
$$L_{u,\varphi} = \sup_{n\in\mathbb{N}} \frac{\|W_{u,\varphi}(\boldsymbol{e}_n)\|_s}{\|\boldsymbol{e}_n\|_r} < \infty.$$
In this case $\|W_{u,\varphi}\| = L_{u,\varphi}$.
2. $W_{u,\varphi} : l^\infty(\boldsymbol{r}) \to l^\infty(\boldsymbol{s})$ is compact if and only if
$$\lim_{n\to\infty} \frac{\|W_{u,\varphi}(\boldsymbol{e}_n)\|_s}{\|\boldsymbol{e}_n\|_r} = 0.$$
3. $W_{u,\varphi} : l^\infty(\boldsymbol{r}) \to l^\infty(\boldsymbol{s})$ is bounded below if and only if
$$\inf_{n\in\mathbb{N}} \frac{\|W_{u,\varphi}(\boldsymbol{e}_n)\|_s}{\|\boldsymbol{e}_n\|_r} > 0.$$
4. $W_{u,\varphi} : l^\infty(\boldsymbol{r}) \to l^\infty(\boldsymbol{s})$ is injective if and only if
$$\frac{\|W_{u,\varphi}(\boldsymbol{e}_n)\|_s}{\|\boldsymbol{e}_n\|_r} > 0$$
for all $n \in \mathbb{N}$.
5. $W_{u,\varphi} : l^\infty(\boldsymbol{r}) \to l^\infty(\boldsymbol{s})$ has closed range if and only if
$$\inf_{n\in S} \frac{\|W_{u,\varphi}(\boldsymbol{e}_n)\|_s}{\|\boldsymbol{e}_n\|_r} > 0,$$
being $S = \left\{n \in \mathbb{N} : \|W_{u,\varphi}(\boldsymbol{e}_n)\|_s \neq 0\right\}$.

Where $(\boldsymbol{e}_n)$ is the so called canonical basis, that is, the sequence $\boldsymbol{e}_n = \{e_n(k)\}$ is defined, as is usual, by $e_n(n) = 1$ and $e_n(k) = 0$ for all $k \neq n$. In particular, $\boldsymbol{e}_n \in l^\infty(\boldsymbol{r})$ for all $n \in \mathbb{N}$ and $\|\boldsymbol{e}_n\|_r = r(n)$. As important consequences of the above results, also we characterize when this operator has finite dimensional range and when it is upper semi-Fredholm.

2. Continuity of the weighted composition operators on $l^\infty(\mathrm{r})$

In this section we characterize all continuous weighted composition operators between weighted sequence spaces in terms of the norm of the images of the normalized canonical basis. From now, it is convenient to define

$$\mathrm{Ran}(\varphi) = \{n \in \mathbb{N} : n = \varphi(k) \text{ for some } k \in \mathbb{N}\}.$$

We can see that if $n \notin \mathrm{Ran}(\varphi)$, then $\|W_{u,\varphi}(\boldsymbol{e}_n)\|_s = 0$. Thus, we have the following result:

Theorem 2.1. *Let $\boldsymbol{r}, \boldsymbol{s}$ be two weights, suppose that $\boldsymbol{u} = \{u(k)\}$ is a complex sequence and that $\varphi : \mathbb{N} \to \mathbb{N}$ is a function. The operator $W_{u,\varphi}$ is continuous from $l^\infty(\boldsymbol{r})$ into $l^\infty(\boldsymbol{s})$ if and only if*

$$L_{u,\varphi} = \sup_{n\in\mathbb{N}} \frac{\|W_{u,\varphi}(\boldsymbol{e}_n)\|_s}{\|\boldsymbol{e}_n\|_r} < \infty. \tag{2.1}$$

In this case $\|W_{u,\varphi}\| = L_{u,\varphi}$.

Proof. Let us suppose first that $W_{u,\varphi} : l^\infty(\boldsymbol{r}) \to l^\infty(\boldsymbol{s})$ is continuous, then for each $n \in \mathbb{N}$, the sequence $\boldsymbol{e}_n \in l^\infty(\boldsymbol{r})$ and hence

$$\|W_{u,\varphi}(\boldsymbol{e}_n)\|_s \leq \|W_{u,\varphi}\| \, \|\boldsymbol{e}_n\|_r.$$

This prove that

$$L_{u,\varphi} = \sup_{n\in\mathbb{N}} \frac{\|W_{u,\varphi}(\boldsymbol{e}_n)\|_s}{\|\boldsymbol{e}_n\|_r} \leq \|W_{u,\varphi}\|.$$

Conversely, if the relation (2.1) holds, then for any $\boldsymbol{x} = \{x(k)\} \in l^\infty(\boldsymbol{r})$ and for any $k \in \mathbb{N}$, we have

$$\begin{aligned}
|u(k)|\,|x(\varphi(k))|\,s(k) &\leq \|\boldsymbol{x}\|_r\,|u(k)|\,\frac{s(k)}{r(\varphi(k))} \\
&\leq \|\boldsymbol{x}\|_r \sup_{n\in\mathrm{Ran}(\varphi)} |u(k)|\,s(k)\,\frac{e_n(\varphi(k))}{\|\boldsymbol{e}_n\|_r} \\
&\leq \|\boldsymbol{x}\|_r \sup_{n\in\mathrm{Ran}(\varphi)} \frac{\|u\cdot \boldsymbol{e}_n\circ\varphi\|_s}{\|\boldsymbol{e}_n\|_r} = \|\boldsymbol{x}\|_r \sup_{n\in\mathrm{Ran}(\varphi)} \frac{\|W_{u,\varphi}(\boldsymbol{e}_n)\|_s}{\|\boldsymbol{e}_n\|_r}
\end{aligned}$$

and therefore

$$\begin{aligned}
\|W_{u,\varphi}(\boldsymbol{x})\|_s &= \sup_{k\in\mathbb{N}} |u(k)|\,|x(\varphi(k))|\,s(k) \\
&\leq \|\boldsymbol{x}\|_r \sup_{n\in\mathrm{Ran}(\varphi)} \frac{\|W_{u,\varphi}(\boldsymbol{e}_n)\|_s}{\|\boldsymbol{e}_n\|_r}.
\end{aligned}$$

The hypothesis and this last fact tell us that the operator $W_{u,\varphi}$ is continuous from $l^\infty(\boldsymbol{r})$ into $l^\infty(\boldsymbol{s})$. Furthermore, the above argument also proves that

$$\|W_{u,\varphi}\| = \sup_{n\in\mathrm{Ran}(\varphi)} \frac{\|W_{u,\varphi}(\boldsymbol{e}_n)\|_s}{\|\boldsymbol{e}_n\|_r}$$

since $\|W_{u,\varphi}(\boldsymbol{e}_n)\|_s = 0$ when $n \notin \mathrm{Ran}(\varphi)$. The proof of the theorem is now complete. $\square$

3. On the compactness

The aim of this section is to obtain a characterization for the compactness of the operator $W_{u,\varphi} : l^{\infty}(\boldsymbol{r}) \to l^{\infty}(\boldsymbol{s})$ in terms of the canonical basis. To this end, we first establish the following result which could have some interest by itself. A much more general result can be found in [1]. We include a proof for benefit of the reader.

Theorem 3.1. *Let $\boldsymbol{r}, \boldsymbol{s}$ be two weights. Let $\boldsymbol{u} = \{u(k)\}$ be a complex sequence, $\varphi : \mathbb{N} \to \mathbb{N}$ a function and suppose that the operator $W_{u,\varphi} : l^{\infty}(\boldsymbol{r}) \to l^{\infty}(\boldsymbol{s})$ is continuous. The weighted composition operator $W_{u,\varphi} : l^{\infty}(\boldsymbol{r}) \to l^{\infty}(\boldsymbol{s})$ is compact if and only if for all bounded-norm sequence $(\boldsymbol{x}_n) \subset l^{\infty}(\boldsymbol{r})$ such that $\boldsymbol{x}_n \to 0$ pointwise ($\lim_{n\to\infty} x_n(m) = 0$ for all $m \in \mathbb{N}$), we have*

$$\lim_{n\to\infty} \|W_{u,\varphi}(\boldsymbol{x}_n)\|_s = 0. \tag{3.1}$$

Proof. Let us suppose first that $W_{u,\varphi} : l^{\infty}(\boldsymbol{r}) \to l^{\infty}(\boldsymbol{s})$ is a compact operator. Let $(\boldsymbol{x}_n) \subset l^{\infty}(\boldsymbol{r})$ be a bounded-norm sequence such that $x_n \to 0$ pointwise and suppose that the condition (3.1) is false. Then there exists an $\varepsilon > 0$ and a subsequence $(\boldsymbol{x}_{n_k})$ of $(\boldsymbol{x}_n)$ such that

$$\|W_{u,\varphi}(\boldsymbol{x}_{n_k})\|_s \geq \varepsilon \tag{3.2}$$

for all $k \in \mathbb{N}$. Next, the compactness of $W_{u,\varphi}$ implies that, by passing to a subsequence, if necessary, we can suppose that $(W_{u,\varphi}(\boldsymbol{x}_{n_k}))$ converges in $l^{\infty}(\boldsymbol{s})$ and there exists $\boldsymbol{y} \in l^{\infty}(\boldsymbol{s})$ such that

$$\lim_{k\to\infty} \|W_{u,\varphi}(\boldsymbol{x}_{n_k}) - \boldsymbol{y}\|_s = 0.$$

We are going to prove that $\boldsymbol{y} = \boldsymbol{0}$ (the null sequence), that is, $y(m) = 0$ for all $m \in \mathbb{N}$. Indeed, by definition of $\|\cdot\|_s$, we can write

$$|u(m)\, x_{n_k}(\varphi(m)) - y(m)| \leq \frac{1}{s(m)} \|W_{u,\varphi}(\boldsymbol{x}_{n_k}) - \boldsymbol{y}\|_s$$

and hence

$$|y(m)| = \lim_{k\to\infty} |u(m)\, x_{n_k}(\varphi(m)) - y(m)| \leq \lim_{k\to\infty} \frac{1}{s(m)} \|W_{u,\varphi}(\boldsymbol{x}_{n_k}) - \boldsymbol{y}\|_s = 0,$$

where we have used that $(\boldsymbol{x}_n)$ converges pointwise to zero. Thus, we have obtained

$$\lim_{k\to\infty} \|W_{u,\varphi}(\boldsymbol{x}_{n_k})\|_s = 0,$$

which leads a contradiction to (3.2). This prove the implication.

Conversely, suppose now that the relation (3.1) holds for all norm-bounded sequence $(\boldsymbol{x}_n) \subset l^{\infty}(\boldsymbol{r})$ such that $\boldsymbol{x}_n \to 0$ pointwise. We shall prove that $W_{u,\varphi} : l^{\infty}(\boldsymbol{r}) \to l^{\infty}(\boldsymbol{s})$ is a compact operator. To this end, let $(\boldsymbol{z}_n)$ any norm-bounded sequence in $l^{\infty}(\boldsymbol{r})$. Notice that for $m \in \mathbb{N}$ fixed, we have

$$|z_n(m)| \leq \frac{1}{r(m)} \|\boldsymbol{z}_n\|_r$$

and this last relation tell us that the numerical sequence of the m-th components, namely $\{z_n(m)\}_{n=1}^{\infty}$ is bounded in $\mathbb{C}$. Thus, Bolzano–Weierstrass theorem implies that there exists a subquence $\{z_{n_k}(m)\}_{k=1}^{\infty}$ of $\{z_n(m)\}_{n=1}^{\infty}$ and $y(m) \in \mathbb{C}$ such that

$$\lim_{k\to\infty} |z_{n_k}(m) - y(m)| = 0.$$

Next, using a diagonal argument we can built a subsequence $\{\boldsymbol{w}_{n_k}\}_{k=1}^{\infty}$ of $\{\boldsymbol{z}_n\}_{n=1}^{\infty}$ and a numerical sequence $\boldsymbol{y} = \{y(m)\}$ such that

$$\lim_{k\to\infty} |w_{n_k}(m) - y(m)| = 0$$

for all $m \in \mathbb{N}$. We can see that $\boldsymbol{y} \in l^{\infty}(\boldsymbol{r})$ since

$$\begin{aligned} |y(m)| r(m) &\leq |w_{n_k}(m) - y(m)| r(m) + |w_{n_k}(m)| r(m) \\ &\leq |w_{n_k}(m) - y(m)| r(m) + \|\boldsymbol{w}_{n_k}\|_r \end{aligned}$$

and $(\boldsymbol{w}_{n_k})$ is norm-bounded in $l^{\infty}(\boldsymbol{r})$. Thus, the sequence $(\boldsymbol{w}_{n_k} - \boldsymbol{y})$ is norm-bounded and converges to zero pointwise; so the hypothesis implies that

$$\lim_{k\to\infty} \|W_{u,\varphi}(\boldsymbol{w}_{n_k} - \boldsymbol{y})\|_s = 0$$

which proves that the operator $W_{u,\varphi} : l^{\infty}(\boldsymbol{r}) \to l^{\infty}(\boldsymbol{s})$ is compact.. $\square$

Now, we can enunciate and prove the main result of this section.

Theorem 3.2. *Let $\boldsymbol{r}, \boldsymbol{s}$ be two weights. Let $\boldsymbol{u} = \{u(k)\}$ be a complex sequence, $\varphi : \mathbb{N} \to \mathbb{N}$ a function and suppose that the operator $W_{u,\varphi} : l^{\infty}(\boldsymbol{r}) \to l^{\infty}(\boldsymbol{s})$ is continuous. The weighted composition operator $W_{u,\varphi} : l^{\infty}(\boldsymbol{r}) \to l^{\infty}(\boldsymbol{s})$ is compact if and only if*

$$\lim_{n\to\infty} \frac{\|W_{u,\varphi}(\boldsymbol{e}_n)\|_s}{\|\boldsymbol{e}_n\|_r} = 0. \tag{3.3}$$

Proof. Suppose first that $W_{u,\varphi} : l^{\infty}(\boldsymbol{r}) \to l^{\infty}(\boldsymbol{s})$ is a compact operator. By Theorem 3.1 we have

$$\lim_{n\to\infty} \|W_{u,\varphi}(\boldsymbol{x}_n)\|_s = 0$$

for all bounded-norm sequence $(\boldsymbol{x}_n) \subset l^{\infty}(\boldsymbol{r})$ such that $\boldsymbol{x}_n \to 0$ pointwise. In particular, the sequence $(\boldsymbol{x}_n)$ defined by

$$x_n = \frac{e_n}{\|\boldsymbol{e}_n\|_r}$$

is bounded and clearly for $m \in \mathbb{N}$ fixed we have

$$\lim_{n\to\infty} x_n(m) = \lim_{n\to\infty} \frac{e_n(m)}{\|\boldsymbol{e}_n\|_r} = 0$$

and therefore

$$0 = \lim_{n\to\infty} \|W_{u,\varphi}(\boldsymbol{x}_n)\|_s = \lim_{n\to\infty} \frac{\|W_{u,\varphi}(e_n)\|_s}{\|\boldsymbol{e}_n\|_r}.$$

This proves (3.3).

Conversely, suppose that the relation (3.3) holds, we are going to use Theorem 3.1 to prove that the operator $W_{u,\varphi} : l^{\infty}(\boldsymbol{r}) \to l^{\infty}(\boldsymbol{s})$ is compact. Thus, let $(\boldsymbol{x}_n)$ be any bounded sequence in $l^{\infty}(\boldsymbol{r})$ converging pointwise to zero, we shall prove that

$$\lim_{n\to\infty} \|W_{u,\varphi}(\boldsymbol{x}_n)\|_s = 0.$$

To this end, first we see that there exists an $M > 0$ such that $\|\boldsymbol{x}_n\|_r \leq M$ for all $n \in \mathbb{N}$. Next, we denote the sequence $\boldsymbol{x}_n \in l^\infty(\boldsymbol{r})$ by $\boldsymbol{x}_n = \{x_n(k)\}$ and we observe that by hypothesis, for $\varepsilon > 0$ given, there exists $H \in \mathbb{N}$ such that

$$\frac{\|W_{u,\varphi}(e_n)\|_s}{\|\boldsymbol{e}_n\|_r} < \varepsilon$$

for all $n \geq H$. Next, we consider the sets

$$\begin{aligned} A &= \{k \in \mathbb{N} : \varphi(k) \leq H\}, \\ B &= \{k \in \mathbb{N} : \varphi(k) > H\}. \end{aligned}$$

If $k \in A$, we can write

$$\begin{aligned} |u(k)|\,|x_n(\varphi(k))|\,s(k) &= |u(k)|\,|x_n(\varphi(k))|\,s(k)\,\frac{r(\varphi(k))}{r(\varphi(k))} \\ &\leq \left(\sup_{1\leq m\leq H} r(m)\,|x_n(m)|\right)\frac{|u(k)|\,s(k)}{r(\varphi(k))} \\ &= \left(\sup_{1\leq m\leq H} r(m)\,|x_n(m)|\right)\frac{|u(k)|\,s(k)}{r(\varphi(k))}e_{\varphi(k)}(\varphi(k)) \\ &\leq \left(\sup_{1\leq m\leq H} r(m)\,|x_n(m)|\right)\sup_{n\in\mathrm{Ran}(\varphi)}\frac{|u(k)|\,s(k)}{r(n)}e_n(\varphi(k)) \\ &\leq \left(\sup_{1\leq m\leq H} r(m)\,|x_n(m)|\right)\sup_{n\in\mathrm{Ran}(\varphi)}\frac{\|W_{u,\varphi}(e_n)\|_s}{\|\boldsymbol{e}_n\|_r}. \end{aligned}$$

Also, since all convergent sequence is bounded, there exists $L_{u,\varphi} > 0$ such that

$$L_{u,\varphi} = \sup_{n\in\mathrm{Ran}(\varphi)}\frac{\|W_{u,\varphi}(\boldsymbol{e}_n)\|_s}{\|\boldsymbol{e}_n\|_r} < \infty.$$

Thus, since $x_n \to 0$ pointwise, we can find $N_H \in \mathbb{N}$ such that

$$\sup_{1\leq m\leq H} r(m)\,|x_n(m)| < \varepsilon$$

whenever $n \geq N_H$. We conclude that

$$\sup_{k\in A}|u(k)|\,|x_n(\varphi(k))|\,s(k) < L_{u,\varphi}\varepsilon. \tag{3.4}$$

for all $n \geq N_H$.

Next, we consider the case when $k \in B$. Here we have $\varphi(k) \geq H+1$ and we can write

$$\begin{aligned} |u(k)|\,|x_n(\varphi(k))|\,s(k) &= |u(k)|\,|x_n(\varphi(k))|\,s(k)\,\frac{r(\varphi(k))}{r(\varphi(k))} \\ &\leq \|\boldsymbol{x}_n\|_r\,|u(k)|\,s(k)\,\frac{e_{\varphi(k)}(\varphi(k))}{r(\varphi(k))} \\ &\leq \|\boldsymbol{x}_n\|_r \sup_{n\geq H+1}|u(k)|\,s(k)\,\frac{e_n(\varphi(k))}{r(n)} \\ &\leq \|\boldsymbol{x}_n\|_r \sup_{n\geq H+1}\frac{\|W_{u,\varphi}(\boldsymbol{e}_n)\|_s}{\|\boldsymbol{e}_n\|_r} \leq M\varepsilon. \end{aligned}$$

Therefore, we obtain

$$\sup_{k\in B} |u(k)| |x_n(\varphi(k))| s(k) < M\varepsilon. \tag{3.5}$$

From (3.4) and (3.5) we conclude

$$\|W_{u,\varphi}(\boldsymbol{x}_n)\|_s = \sup_{k\in\mathbb{N}} |u(k)| |x_n(\varphi(k))| s(k) < (L_{u,\varphi} + M)\varepsilon$$

whenever $n \geq N_H$. This proves that

$$\lim_{n\to\infty} \|W_{u,\varphi}(\boldsymbol{x}_n)\|_s = 0$$

and the operator $W_{u,\varphi} : l^\infty(\boldsymbol{r}) \to l^\infty(\boldsymbol{s})$ is compact. □

4. Bounded below weighted composition operators on $l^\infty(\boldsymbol{r})$

In this section we give a characterization for bounded below weighted composition operators between weighted sequence spaces in terms of the canonical basis. We recall that a continuous linear operator $T : X \to Y$ with X, Y Banach spaces is called *bounded below* if there exists a constant $L > 0$ such that

$$\|Tx\|_Y \geq L \|x\|_X$$

for all $x \in X$. In the context of analytic functions, the characterization is given in terms of the so called sampling sets (see [8, 4, 10] and the reference therein). In the context of weighted sequence spaces, this does not make any sense because if a subset $H \subset \mathbb{N}$ satisfies

$$\sup_{k\in H} |x(k)| r(k) \geq L \|\boldsymbol{x}\|_r$$

for all $\boldsymbol{x} \in l^\infty(\boldsymbol{r})$ and for some $L > 0$, then we can conclude that $H = \mathbb{N}$. However, for each $\varepsilon > 0$ we set

$$S_\varepsilon = \left\{ k \in \mathbb{N} : \frac{|u(k)| s(k)}{r(\varphi(k))} \geq \varepsilon \right\}$$

and we have the following result.

Theorem 4.1. *Let $\boldsymbol{r}, \boldsymbol{s}$ be two weights. Let $\boldsymbol{u} = \{u(k)\}$ be a complex sequence, $\varphi : \mathbb{N} \to \mathbb{N}$ a function and suppose that the operator $W_{u,\varphi} : l^\infty(\boldsymbol{r}) \to l^\infty(\boldsymbol{s})$ is continuous. The following statements are equivalent:*

1. *The weighted composition operator $W_{u,\varphi} : l^\infty(\boldsymbol{r}) \to l^\infty(\boldsymbol{s})$ is bounded below,*
2. *There exists $\varepsilon > 0$ such that the set $\varphi(S_\varepsilon) = \mathbb{N}$,*
3. $$\inf_{n\in\mathbb{N}} \frac{\|W_{u,\varphi}(\boldsymbol{e}_n)\|_s}{\|\boldsymbol{e}_n\|_r} > 0.$$

Proof. **[(2)⇒(1)]** Let us suppose first that there exists $\varepsilon > 0$ such that $\varphi(S_\varepsilon) = \mathbb{N}$. Then for all $\boldsymbol{x} \in l^\infty(\boldsymbol{r})$ we have

$$\|\boldsymbol{x}\|_r = \sup_{k\in\mathbb{N}} |x(k)| r(k) = \sup_{m\in S_\varepsilon} |x(\varphi(m))| r(\varphi(m))$$

$$
\begin{aligned}
&= \sup_{m \in S_\varepsilon} \frac{r(\varphi(m))}{|u(m)|\, s(m)} |u(m)|\, |x(\varphi(m))|\, s(m) \\
&\leq \frac{1}{\varepsilon} \sup_{m \in S_\varepsilon} |u(m)|\, |x(\varphi(m))|\, s(m) \leq \frac{1}{\varepsilon} \|W_{u,\varphi}(\boldsymbol{x})\|_s
\end{aligned}
$$

and this proves that the operator $W_{u,\varphi} : l^\infty(\boldsymbol{r}) \to l^\infty(\boldsymbol{s})$ is bounded below.

[(1)⇒(2)] Suppose now that $W_{u,\varphi} : l^\infty(\boldsymbol{r}) \to l^\infty(\boldsymbol{s})$ is bounded below. There exists $L > 0$ such that

$$
\|W_{u,\varphi}(\boldsymbol{x})\|_s \geq L \|\boldsymbol{x}\|_r
$$

for all $\boldsymbol{x} \in l^\infty(\boldsymbol{r})$. In particular, if $\|\boldsymbol{x}\|_r = 1$, then

$$
\|W_{u,\varphi}(\boldsymbol{x})\|_s = \sup_{k \in \mathbb{N}} |u(k)|\, |x(\varphi(k))|\, s(k) \geq L
$$

and hence, by definition of supremum, there exists $k_x \in \mathbb{N}$ such that

$$
|u(k_x)|\, |x(\varphi(k_x))|\, s(k_x) \geq \frac{L}{2}
$$

which in turn implies that

$$
\frac{|u(k_x)|\, s(k_x)}{r(\varphi(k_x))} |x(\varphi(k_x))|\, r(\varphi(k_x)) \geq \frac{L}{2} \tag{4.1}
$$

and since $|x(\varphi(k_x))|\, r(\varphi(k_x)) \leq \|\boldsymbol{x}\|_r = 1$, we have

$$
\frac{|u(k_x)|\, s(k_x)}{r(\varphi(k_x))} \geq \frac{L}{2}.
$$

Thus, putting $\varepsilon = \frac{1}{2}L$ we have $k_x \in S_\varepsilon$. Next, we use the fact that $W_{u,\varphi} : l^\infty(\boldsymbol{r}) \to l^\infty(\boldsymbol{s})$ is a continuous operator and also we can write

$$
\begin{aligned}
\frac{|u(k_x)|\, s(k_x)}{r(\varphi(k_x))} &\leq \frac{\sup_{k:\varphi(k)=\varphi(k_x)} |u(k)|\, s(k)}{r(\varphi(k_x))} = \frac{\left\|W_{u,\varphi}\left(\boldsymbol{e}_{\varphi(k_x)}\right)\right\|_s}{\left\|\boldsymbol{e}_{\varphi(k_x)}\right\|_r} \\
&\leq \sup_{n \in \mathrm{Ran}(\varphi)} \frac{\|W_{u,\varphi}(\boldsymbol{e}_n)\|_s}{\|\boldsymbol{e}_n\|_r} \leq \|W_{u,\varphi}\| = L_{u,\varphi}.
\end{aligned}
$$

This last fact and the relation (4.1) allow us to write

$$
\frac{L}{2} \leq \frac{|u(k_x)|\, s(k_x)}{r(\varphi(k_x))} |x(\varphi(k_x))|\, r(\varphi(k_x)) \leq L_{u,\varphi} |x(\varphi(k_x))|\, r(\varphi(k_x))
$$

and therefore

$$
\sup_{n \in \varphi(S_\varepsilon)} |x(n)|\, r(n) \geq |x(\varphi(k_x))|\, r(\varphi(k_x)) \geq \frac{L}{2L_{u,\varphi}} > 0 \tag{4.2}
$$

for all $\boldsymbol{x} \in l^\infty(\boldsymbol{r})$ such that $\|\boldsymbol{x}\|_r = 1$. Finally, if there exists a $k_0 \in \mathbb{N} - \varphi(S_\varepsilon)$, then the sequence

$$
\boldsymbol{x}_0 = \frac{\boldsymbol{e}_{k_0}}{r(k_0)}
$$

has norm equal to 1 and also it satisfies

$$
\sup_{k \in \varphi(S_\varepsilon)} |x_0(k)|\, r(k) = 0
$$

which is a contradiction to (4.2). We conclude that $\varphi(S_\varepsilon) = \mathbb{N}$ and the proof of the implication is now complete.

[(2)⇒(3)] Let us suppose that there exists an $\varepsilon > 0$ such that the set $\varphi(S_\varepsilon) = \mathbb{N}$. Then for each $n \in \mathbb{N}$, we can find a $k_0 \in S_\varepsilon$ such that $n = \varphi(k_0)$. In particular, we have

$$\frac{|u(k_0)|\, s(k_0)}{r(\varphi(k_0))} > \varepsilon$$

and hence

$$\begin{aligned} \frac{\|W_{u,\varphi}(\boldsymbol{e}_n)\|_s}{\|\boldsymbol{e}_n\|_r} &= \frac{\sup_{k:\varphi(k)=n} |u(k)|\, s(k)}{r(n)} \\ &\geq \frac{|u(k_0)|\, s(k_0)}{r(\varphi(k_0))} > \varepsilon. \end{aligned}$$

This proves the implication.

[(3)⇒(2)] Conversely, if

$$\inf_{n\in\mathbb{N}} \frac{\|W_{u,\varphi}(\boldsymbol{e}_n)\|_s}{\|\boldsymbol{e}_n\|_r} > 0,$$

then there exists an $\varepsilon > 0$ such that

$$\frac{\sup_{k:\varphi(k)=n} |u(k)|\, s(k)}{r(n)} \geq \varepsilon$$

for all $n \in \mathbb{N}$. Thus, for each $n \in \mathbb{N}$, the definition of supremum, implies that there exists a $k_n \in \mathbb{N}$ such that $\varphi(k_n) = n$

$$\frac{|u(k_n)|\, s(k_n)}{r(\varphi(k_n))} > \frac{\varepsilon}{2}$$

which proves that $n \in \varphi\left(S_{\varepsilon/2}\right)$ and $\mathbb{N} = \varphi\left(S_{\varepsilon/2}\right)$. This concludes the proof of the theorem. □

Since all bounded below operator is also injective, the above result give us a light about the condition which characterize injective weighted composition operators between weighted sequence spaces. We recall that if $T : X \to Y$ is a linear operator, the kernel of T, denoted by $\mathrm{Ker}(T)$, is the set of all $x \in X$ such that $T(x) = 0$; and it is known that T is injective or 1-1 if and only if $\mathrm{Ker}(T) = \{0\}$. We have the following result.

Theorem 4.2. *Let $\boldsymbol{r}, \boldsymbol{s}$ be two weights. Let $\boldsymbol{u} = \{u(k)\}$ be a complex sequence, $\varphi : \mathbb{N} \to \mathbb{N}$ a function. The operator $W_{u,\varphi} : l^\infty(\boldsymbol{r}) \to l^\infty(\boldsymbol{s})$ is injective if and only if*

$$\frac{\|W_{u,\varphi}(\boldsymbol{e}_n)\|_s}{\|\boldsymbol{e}_n\|_r} > 0 \tag{4.3}$$

for all $n \in \mathbb{N}$.

Proof. Let us suppose first that the condition (4.3) holds for all $n \in \mathbb{N}$. In particular, we see that $\mathrm{Ran}(\varphi) = \mathbb{N}$ because if $n \notin \mathrm{Ran}(\varphi)$, then $\|W_{u,\varphi}(\boldsymbol{e}_n)\|_s = 0$. Consider now any $\boldsymbol{x} \in l^\infty(\boldsymbol{r})$ such that

$W_{u,\varphi}(\boldsymbol{x}) = \mathbf{0}$, we are going to prove that $\boldsymbol{x}$ is the null sequence, that is, $x(n) = 0$ for all $n \in \mathbb{N}$. Indeed, for each $n \in \mathbb{N}$, the hypothesis and the definition of the s-norm imply

$$\frac{\sup_{k:\varphi(k)=n} |u(k)|\, s(k)}{r(n)} > 0.$$

Thus, the definition of supremum guarantee that there exists $k_0 \in \mathbb{N}$ such that $\varphi(k_0) = n$ and

$$\frac{|u(k_0)|\, s(k_0)}{r(n)} > 0.$$

In particular, $|u(k_0)| > 0$; but we also have

$$|u(k)|\, |x(\varphi(k))| = 0 \tag{4.4}$$

for all $k \in \mathbb{N}$ since $W_{u,\varphi}(\boldsymbol{x}) = \mathbf{0}$. Therefore, for this k_0 we obtain

$$|u(k_0)|\, |x(n)| = 0$$

and $x(n) = 0$. This proves that $W_{u,\varphi} : l^\infty(\boldsymbol{r}) \to l^\infty(\boldsymbol{s})$ is an injective operator. The converse is evident. Indeed, if there exists $n \in \mathbb{N}$ such that

$$\frac{\|W_{u,\varphi}(\boldsymbol{e}_n)\|_s}{\|\boldsymbol{e}_n\|_r} = 0.$$

Then $\|W_{u,\varphi}(\boldsymbol{e}_n)\|_s = 0$ and $\mathrm{Ker}(W_{u,\varphi}) \neq \{0\}$; that is, $W_{u,\varphi} : l^\infty(\boldsymbol{r}) \to l^\infty(\boldsymbol{s})$ is not injective. $\square$

5. On the Closedness of the range

In this section we analyze when the weighted composition operator $W_{u,\varphi} : l^\infty(\boldsymbol{r}) \to l^\infty(\boldsymbol{s})$ has closed range. We recall that if $T : X \to Y$ is a linear operator with X, Y Banach spaces, the range of T is $T(X) = \{y \in Y : y = T(x) \text{ for some } x \in X\}$. Just like the other results of previous section, the criteria we have obtained is given in terms of the canonical basis. Furthermore, we know that if $n \notin \mathrm{Ran}(\varphi)$, then $\|W_{u,\varphi}(\boldsymbol{e}_n)\|_s = 0$, however, it could happen that there exists $n \in \mathrm{Ran}(\varphi)$ such that $\|W_{u,\varphi}(\boldsymbol{e}_n)\|_s = 0$. Hence, it is necessary to consider the set

$$S = \left\{n \in \mathbb{N} : \|W_{u,\varphi}(\boldsymbol{e}_n)\|_s \neq 0\right\} = \left\{n \in \mathrm{Ran}(\varphi) : \|W_{u,\varphi}(\boldsymbol{e}_n)\|_s \neq 0\right\}$$

and we have the following result.

Theorem 5.1. *Let $\boldsymbol{r}, \boldsymbol{s}$ be two weights. Let $\boldsymbol{u} = \{u(k)\}$ be a complex sequence, $\varphi : \mathbb{N} \to \mathbb{N}$ a function and suppose that the operator $W_{u,\varphi} : l^\infty(\boldsymbol{r}) \to l^\infty(\boldsymbol{s})$ is continuous. The weighted composition operator $W_{u,\varphi} : l^\infty(\boldsymbol{r}) \to l^\infty(\boldsymbol{s})$ has closed range if and only if*

$$\inf_{n \in S} \frac{\|W_{u,\varphi}(\boldsymbol{e}_n)\|_s}{\|\boldsymbol{e}_n\|_r} > 0. \tag{5.1}$$

Proof. Let us suppose that the condition (5.1) holds. Then there exists $\delta > 0$ such that

$$\frac{\|W_{u,\varphi}(\boldsymbol{e}_n)\|_s}{\|\boldsymbol{e}_n\|_r} > \delta$$

for all $n \in S = \left\{n \in \mathbb{N} : \|W_{u,\varphi}(\boldsymbol{e}_n)\|_s \neq 0\right\}$. If $\boldsymbol{y} \in \overline{W_{u,\varphi}(l^\infty(\boldsymbol{r}))}$, then there exists a sequence $(\boldsymbol{y}_m) \subset W_{u,\varphi}(l^\infty(\boldsymbol{r}))$ such that

$$\lim_{m\to\infty} \|\boldsymbol{y}_m - \boldsymbol{y}\|_s = 0.$$

Also, there exists a sequence $(\boldsymbol{x}_m) \subset l^\infty(\boldsymbol{r})$ such that $\boldsymbol{y}_m = W_{u,\varphi}(\boldsymbol{x}_m)$ for all $m \in \mathbb{N}$. Thus, for any $k \in \mathbb{N}$, we have

$$y_m(k) = u(k)\, x_m(\varphi(k))$$

hence, we can suppose that $x_m(n) = 0$ for all $n \notin \operatorname{Ran}(\varphi)$. Furthermore, if $n \in \operatorname{Ran}(\varphi)$ and $n \notin S$, then

$$0 = \|W_{u,\varphi}(\boldsymbol{e}_n)\|_s = \sup_{k\in\mathbb{N}} |u(k)|\, |e_n(\varphi(k))|$$

and we obtain

$$|u(k)| = 0$$

for all $k \in \mathbb{N}$ such that $\varphi(k) = n$. Thus, we also can suppose that $x_m(n) = 0$ for all $n \notin S$. With these conditions, now we shall prove that $(\boldsymbol{x}_m)$ is a Cauchy sequence in $l^\infty(\boldsymbol{r})$. Indeed,

$$\|\boldsymbol{x}_H - \boldsymbol{x}_m\|_r = \sup_{n\in S} |x_H(n) - x_m(n)|\, r(n);$$

but if $n \in S$, then $n \in \operatorname{Ran}(\varphi)$ and there exists $k \in \mathbb{N}$ such that $\varphi(k) = n$; in fact, by hypothesis we have $\frac{\|W_{u,\varphi}(\boldsymbol{e}_n)\|_s}{\|\boldsymbol{e}_n\|_r} > \delta$, and this means that we can find $k \in \mathbb{N}$ such that $\varphi(k) = n$ and

$$\frac{|u(k)|\, s(k)}{r(n)} > \delta$$

and we can write

$$\begin{aligned} |x_H(n) - x_m(n)|\, r(n) &= |u(k)\, x_H(\varphi(k))\, s(k) - u(k)\, x_m(\varphi(k))\, s(k)|\, \frac{r(n)}{|u(k)|\, s(k)} \\ &\leq \frac{1}{\delta} \|W_{u,\varphi}(\boldsymbol{x}_H) - W_{u,\varphi}(\boldsymbol{x}_m)\|_s = \frac{1}{\delta} \|\boldsymbol{y}_H - \boldsymbol{y}_m\|_s . \end{aligned}$$

This proves that

$$\|\boldsymbol{x}_H - \boldsymbol{x}_m\|_r \leq \frac{1}{\delta} \|\boldsymbol{y}_H - \boldsymbol{y}_m\|_s$$

and $(\boldsymbol{x}_m)$ is a Cauchy sequence in $l^\infty(\boldsymbol{r})$ which is a Banach space. Thus, there exists $\boldsymbol{x} \in l^\infty(\boldsymbol{r})$ such that

$$\lim_{m\to\infty} \|\boldsymbol{x}_m - \boldsymbol{x}\|_r = 0.$$

Clearly, $x(n) = 0$ for all $n \notin S$ and since

$$\begin{aligned} \|\boldsymbol{y} - W_{u,\varphi}(\boldsymbol{x})\|_s &\leq \|\boldsymbol{y}_m - \boldsymbol{y}\|_s + \|\boldsymbol{y}_m - W_{u,\varphi}(\boldsymbol{x})\|_s \\ &\leq \|\boldsymbol{y}_m - \boldsymbol{y}\|_s + \|W_{u,\varphi}\|\, \|\boldsymbol{x}_m - \boldsymbol{x}\|_r , \end{aligned}$$

we conclude that $\|\boldsymbol{y} - W_{u,\varphi}(\boldsymbol{x})\|_s = 0$ and $\boldsymbol{y} \in W_{u,\varphi}(l^\infty(\boldsymbol{r}))$. This proves the implication.

Conversely, suppose now that the operator $W_{u,\varphi} : l^{\infty}(\boldsymbol{r}) \to l^{\infty}(\boldsymbol{s})$ has closed range. We consider the set

$$X = \{\boldsymbol{x} \in l^{\infty}(\boldsymbol{r}) : x(k) = 0 \text{ for all } k \notin S\}.$$

Clearly, X is a closed subspace of $l^{\infty}(\boldsymbol{r})$. Indeed, if $\boldsymbol{x} \in \overline{X}$, then there exists a sequence $(\boldsymbol{x}_n) \subset X$ such that

$$\lim_{n\to\infty} \|\boldsymbol{x}_n - \boldsymbol{x}\|_r = 0.$$

Thus, by definition of the $\boldsymbol{r}$-norm, for each $k \notin S$ we have

$$|x_n(k) - x(k)| \leq \frac{1}{r(k)} \|\boldsymbol{x}_n - \boldsymbol{x}\|_r$$

and we arrive to the conclusion $\boldsymbol{x} \in X$. In particular, $(X, \|\cdot\|_r)$ is a Banach space. Furthermore, if $\boldsymbol{y} \in W_{u,\varphi}(l^{\infty}(\boldsymbol{r}))$, then there exists $\boldsymbol{x} \in l^{\infty}(\boldsymbol{r})$ such that

$$y(k) = u(k)\, x(\varphi(k))$$

for all $k \in \mathbb{N}$. Thus, if we consider the sequence

$$\widehat{\boldsymbol{x}} = \boldsymbol{x} \cdot \mathbf{1}_S,$$

where $\mathbf{1}_S$ is the sequence defined by

$$1_S(k) = \begin{cases} 1, & k \in S, \\ 0, & \text{otherwise}, \end{cases}$$

then we see that $\widehat{\boldsymbol{x}} \in X$ and for any $k \in \mathbb{N}$ such that $\varphi(k) \in S$ we have

$$u(k)\, \widehat{x}(\varphi(k)) = u(k)\, x(\varphi(k)) \cdot 1_S(\varphi(k)) = u(k)\, x(\varphi(k));$$

while for $k \in \mathbb{N}$ such that $n = \varphi(k) \notin S$ we can see

$$u(k)\, \widehat{x}(\varphi(k)) = u(k)\, x(\varphi(k)) \cdot 1_S(\varphi(k)) = 0;$$

and also

$$u(k)\, x(\varphi(k)) = u(k)\, x(n) = 0$$

since $\boldsymbol{x} \in X$. Summarizing,

$$y(k) = u(k)\, \widehat{x}(\varphi(k))$$

for all $k \in \mathbb{N}$. This means that $\boldsymbol{y} = W_{u,\varphi}(\widehat{\boldsymbol{x}})$ and therefore $W_{u,\varphi}(l^{\infty}(\boldsymbol{r})) = W_{u,\varphi}(X)$. In particular, the hypothesis tell us that the operator $W_{u,\varphi} : X \to l^{\infty}(\boldsymbol{s})$ has closed range.

Next, we shall prove that $W_{u,\varphi} : X \to l^{\infty}(\boldsymbol{s})$ is injective. Indeed, if $\boldsymbol{x} \in X$ is such that $W_{u,\varphi}(\boldsymbol{x}) = \mathbf{0}$, then

$$u(k)\, x(\varphi(k)) = 0 \tag{5.2}$$

for all $k \in \mathbb{N}$. We want to see that $x(n) = 0$ for all $n \in \mathbb{N}$. We know that $x(n) = 0$ for all $n \notin S$. Now, if $n \in S$, then $n \in \mathrm{Ran}(\varphi)$ and also

$$\sup_{k:\varphi(k)=n} |u(k)|\, s(k) > 0.$$

In particular, by definition of supremum, there exists $k \in \mathbb{N}$ such that $\varphi(k) = n$ and

$$|u(k)|\, s(k) > 0$$

hence $|u(k)| > 0$ and for this k, substituting in (5.2), we obtain $x(n) = 0$. Hence $\boldsymbol{x} = \mathbf{0}$ and the operator $W_{u,\varphi} : X \to l^{\infty}(\boldsymbol{s})$ is $1-1$ and it has closed range. Therefore, this operator is bounded below and there exists a $\delta > 0$ such that

$$\|W_{u,\varphi}(\boldsymbol{x})\|_s \geq \delta \|\boldsymbol{x}\|_r$$

for all $\boldsymbol{x} \in X$. In particular, for any $n \in S$ it is obvious that $e_n(k) = 0$ for all $k \notin S$. Hence $\boldsymbol{e}_n \in X$ for all $n \in S$ and then

$$\|W_{u,\varphi}(\boldsymbol{e}_n)\|_s \geq \delta \|\boldsymbol{e}_n\|_r$$

for all $n \in S$ and the proof is now complete. □

Comment. From the proof of the second part of the above theorem, we could establish that

$$W_{u,\varphi}(l^{\infty}(\boldsymbol{r})) = W_{u,\varphi}(X) = W_{u,\varphi}(\{\boldsymbol{x} \in l^{\infty}(\boldsymbol{r}) : x(k) = 0 \text{ for all } k \notin S\}).$$

In fact, every $y \in W_{u,\varphi}(l^{\infty}(\boldsymbol{r}))$ can be written as

$$y = W_{u,\varphi}(x \cdot 1_S)$$

for some $\boldsymbol{x} \in l^{\infty}(\boldsymbol{r})$. For this reason, we also have the following consequence which characterize weighted composition operators acting between weighted sequence spaces with finite dimensional range.

Corollary 5.2. *Let $\boldsymbol{r}, \boldsymbol{s}$ be two weights. Let $\boldsymbol{u} = \{u(k)\}$ be a complex sequence, $\varphi : \mathbb{N} \to \mathbb{N}$ a function and suppose that the operator $W_{u,\varphi} : l^{\infty}(\boldsymbol{r}) \to l^{\infty}(\boldsymbol{s})$ is continuous. The weighted composition operator $W_{u,\varphi} : l^{\infty}(\boldsymbol{r}) \to l^{\infty}(\boldsymbol{s})$ has finite dimensional range if and only if $S = \{n \in \mathbb{N} : \|W_{u,\varphi}(\boldsymbol{e}_n)\|_s \neq 0\} = \{n \in \mathrm{Ran}(\varphi) : \|W_{u,\varphi}(\boldsymbol{e}_n)\|_s \neq 0\}$ is a finite set.*

Finally, our results also allow us to characterize when a weighted composition operator acting between weighted sequence spaces is upper semi-Fredholm. We recall that an operator $T : X \to Y$, with X and Y Banach spaces, is said to be *upper semi-Fredholm* if it has finite dimensional kernel and $T(X)$ is a closed subspace of Y.

Corollary 5.3. *Let $\boldsymbol{r}, \boldsymbol{s}$ be two weights. Let $\boldsymbol{u} = \{u(k)\}$ be a complex sequence, $\varphi : \mathbb{N} \to \mathbb{N}$ a function and suppose that the operator $W_{u,\varphi} : l^{\infty}(\boldsymbol{r}) \to l^{\infty}(\boldsymbol{s})$ is continuous. The weighted composition operator $W_{u,\varphi} : l^{\infty}(\boldsymbol{r}) \to l^{\infty}(\boldsymbol{s})$ is upper semi-Fredholm if and only if $Z = \mathbb{N} - S$ is a finite set and there exists $\delta > 0$ such that*

$$\frac{\|W_{u,\varphi}(\boldsymbol{e}_n)\|_s}{\|\boldsymbol{e}_n\|_r} > \delta$$

for all $n \in S$.

Proof. The result follows from the fact that $\mathrm{Ker}(W_{u,\varphi})$ is finite dimensional if and only if $Z = \{n \in \mathbb{N} : \|W_{u,\varphi}(\boldsymbol{e}_n)\|_s = 0\}$ is a finite set. □

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References

[1] A. A. Albanese, J. Bonet and W. J. Ricker. Multiplier and averaging operators in the Banach spaces $ces(p)$, $1 < p < \infty$. *Positivity* **23** (2019), no. 1, 177-–193.

[2] G. M. Antón Marval, R. E. Castillo and J. C. Ramos-Fernández. Maximal functions and properties of the weighted composition operators acting on the Korenblum, α-Bloch and α-Zygmund spaces. *Cubo* **19** (2017), no. 1, 39—51.

[3] J. W. Carlson. Weighted composition operators on l^2. Thesis (Ph.D.)-Purdue University. *ProQuest LLC, Ann Arbor, MI,* 1985. 90 pp.

[4] M. Contreras and A. Hernández-Díaz. Weighted composition operators in weighted Banach spaces of analytic functions. *J. Austral. Math. Soc. Ser. A* **69** (2000), 41–60.

[5] H. Hudzik, R. Kumar and R. Kumar. Matrix multiplication operators on Banach function spaces. *Proc. Indian Acad. Sci. Math Sci.* **116** (2006), no. 1, 71–81.

[6] D. M. Luan and L. H. Khoi. Weighted composition operators on weighted sequence spaces. *Contemporary Math.* **645** (2015), 199–215.

[7] D. M. Luan and L. H. Khoi. Hilbert spaces of entire functions and composition operators. *Complex Anal. Oper. Theory*, **10**, (2016), no. 1, 213–230.

[8] A. Montes-Rodríguez. Weighted composition operators on weighted Banach spaces of analytic functions. *J. London Math. Soc.* (2) **61** (2000), 872–884.

[9] K. Raj, B. S. Komal and V. Khosla. On operators of weighted composition on Orlicz sequence spaces. *Int. J. Contemp. Math. Sci.*, **5** (2010), no. 37-40, 1961–1968.

[10] J. C. Ramos-Fernández. Composition operators between μ-Bloch spaces. *Extracta Math.* **26** (2011), no. 1, 75—88.

[11] J. C. Ramos Fernández. On the norm and the essential norm of weighted composition operators acting on the weighted Banach space of analytic functions. *Quaest. Math.* **39** (2016), no. 4, 497–509.

[12] J. C. Ramos-Fernández and M. Salas-Brown. On multiplication operators acting on Köthe sequence spaces. *Afr. Mat.* **28** (2017), no. 3-4, 661–667.

[13] A. L. Shields. Weighted shift operators and analytic function theory. *Topics in operator theory*, pp. 49–128. Math. Surveys, No. 13, *Amer. Math. Soc., Providence, R.I.*, 1974.

[14] R. K. Singh and J. S. Manhas. Composition operators on function spaces. North-Holland Mathematics Studies, 179, *North-Holland Publishing Co., Amsterdam*, 1993.

[15] J. J. Williams and Q. Ye. Infinite matrices bounded on weighted l^1 spaces. *Linear Algebra Appl.* **438** (2013), no. 12, 4689-—4700.