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Permutation (Matrices) and Beyond

Matrices de Permutation et au-delà

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ABSTRACT. This is an exposition of my keynote talk given at The Third International Conference on Mathematics and Statistics: AUS-ICMS February 2020, Sharjah, UAE. It concerns permutations and permutation matrices, and some of their generalizations. It is not intended to be comprehensive in the topics discussed, but to highlight some aspects. The final section discusses a number of open problems and conjectures. A number of references are provided to get one started on a more comprehensive study.
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1. Introduction

The study of *permutations* is both ancient and modern. They can be viewed as the integers $1, 2, \ldots, n$ in some order or as $n \times n$ permutation matrices. Permutations can be regarded as data which is to be sorted. The explicit definition of the determinant uses permutations. An *inversion* of a permutation occurs when a larger integer precedes a smaller integer. Inversions can be used to define two *partial orders* on permutations, one weaker than the other. Partial orders have a unique minimal completion to a *lattice*, the *Dedekind-MacNeille completion*. Generalizations of permutation matrices determine related matrix classes, for instance, *alternating sign matrices* (*ASMs*) which arose independently in the mathematics and physics literature. Permutations may contain certain *patterns*, e.g. three integers in increasing order; avoiding such patterns determines certain permutation classes. Similar restrictions can be placed more generally on $(0, 1)$-matrices. The convex hull of $n \times n$ permutation matrices is the *polytope* of $n \times n$ doubly stochastic matrices. In a similar way we get *ASM polytopes*. We shall explore these and other ideas and their connections.

2. Permutations and Permutation Matrices

Permutations can be modeled in two basic ways:

- As a listing of a set of n elements, usually take to be the integers $\{1, 2, \ldots, n\}$, in some order, e.g. if $n = 6$, $(3, 6, 1, 5, 2, 4)$.
- As a permutation matrix, e.g. for the permutation $(3, 6, 1, 5, 2, 4)$ we have:

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \text{ often written as } \left[\begin{array}{c|c|c|c|c|c} & & 1 & & & \\ \hline & & & & & 1 \\ \hline 1 & & & & & \\ \hline & & & & 1 & \\ \hline & 1 & & & & \\ \hline & & & 1 & & \end{array}\right],$$

where empty positions are interpreted as 0's.

So a permutation consists of the integers $\{1, 2, \ldots, n\}$ in some order, and as a result some pairs of the integers may be out of order. How should this be measured? A natural way is by the number of such pairs.

Let $\sigma = (k_1, k_2, \ldots, k_n)$ be a permutation of $\{1, 2, \ldots, n\}$. Then (k_p, k_q) is an *inversion* of σ provided

$p < q$ and $k_p > k_q$ (a pair of integers out of their natural order).

The transformation $(k_p, k_q) \rightarrow (k_q, k_p)$ applied to σ is a *transposition*. Returning to our example, we have $(3, \mathbf{6}, 1, 5, \mathbf{2}, 4) \rightarrow (3, \mathbf{2}, 1, 5, \mathbf{6}, 4)$:

$$\begin{bmatrix} & & 1 & & & \\ & \mathbf{0} & & & & \mathbf{1} \\ 1 & & & & & \\ & & & & 1 & \\ & \mathbf{1} & & & & \mathbf{0} \\ & & & 1 & & \end{bmatrix} \rightarrow \begin{bmatrix} & & 1 & & & \\ & \mathbf{1} & & & & \mathbf{0} \\ 1 & & & & & \\ & & & & 1 & \\ & \mathbf{0} & & & & \mathbf{1} \\ & & & 1 & & \end{bmatrix}.$$

A transposition can always be chosen to reduce the *number* of inversions but not necessarily the *set of inversions* $\mathcal{I}(\sigma)$ by 1. The transposition

$$(\mathbf{3}, 4, 1, \mathbf{2}) \rightarrow (\mathbf{2}, 4, 1, \mathbf{3})$$

reduces the *number of inversions* from 4 to 3 but not the set of inversions by 1:

$$\{(3, 1), (3, 2), (4, 1), (4, 2)\} \rightarrow \{(2, 1), (4, 1), (4, 3)\}.$$

An *adjacent inversion* is of the form (k_p, k_{p+1}) with $k_p > k_{p+1}$. The total effect of the corresponding *adjacent transposition* is to remove one inversion from $\mathcal{I}(\sigma)$:

$$(3, \mathbf{4}, \mathbf{1}, 2) \rightarrow (3, \mathbf{1}, \mathbf{4}, 2), \quad \{(3, 1), (3, 2), (4, 1), (4, 2)\} \rightarrow \{(3, 1), (3, 2), (4, 2)\}.$$

If there is an inversion of σ (so σ is not the identity $(1, 2, \ldots, n)$), then there must be an adjacent inversion.

A basic fact is: *A permutation σ of $\{1, 2, \ldots, n\}$ is uniquely determined by its set $\mathcal{I}(\sigma)$ of inversions* (*use induction on the location of* 1 *in the permutation*) *but not, in general, by its set of adjacent inversions.* For example, permutations $(4, 1, 2, 3)$ and $(2, 4, 1, 3)$ have exactly one adjacent inversion, namely $(4, 1)$ in both instances, but their sets of inversions are different: $\{(4, 1), (4, 2), (4, 3)\}$ and $\{(2, 1), (4, 1), (4, 3)\}$, respectively.

How might we better compare two permutations, other than by using the number of inversions? We can do so by using two fairly natural partial orders [4]:

- *Weak Bruhat Order* on the set $\mathcal{S}_n$ of permutations of $\{1, 2, \ldots, n\}$:

  $\pi_1 \preceq_b \pi_2$ provided that $\mathcal{I}(\pi_1) \subseteq \mathcal{I}(\pi_2)$.

  This is equivalent to: π_1 can be obtained from π_2 by a sequence of adjacent transpositions each thereby reducing the set of of inversions by exactly 1. This partially ordered set is denoted by $(\mathcal{S}_n, \preceq_b)$.

- *Bruhat Order* on permutations of $\{1, 2, \ldots, n\}$:

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  provided that π_1 can be obtained from π_2 by a sequence of transpositions each reducing the number of inversions by exactly 1 (but not necessarily reducing the set of inversions by 1). This partially ordered set is denoted by $(\mathcal{S}_n, \preceq_B)$.

So $\pi_1 \preceq_b \pi_2$ *implies* $\pi_1 \preceq_B \pi_2$*, but not conversely.* It is 'Set Containment' versus 'Number'.

For example,

- $(4, \mathbf{2}, 1, \mathbf{3}) \preceq_B (4, \mathbf{3}, 1, \mathbf{2})$ (one transposition) where $\mathcal{I}((4,3,1,2)) = \{(4,3), (4,1), (4,2), (3,1), (3,2)\}$ (5 inversions) while $\mathcal{I}((4,2,1,3)) = \{(4,2), (4,1), (4,3), (2,1)\}$ (4 inversions).
- $(4, \mathbf{2}, 1, \mathbf{3}) \not\preceq_b (4, \mathbf{3}, 1, \mathbf{2})$, since $\mathcal{I}((4,2,1,3)) = \{(4,2), (4,1), (4,3), (2,1)\} \not\subseteq \mathcal{I}((4,3,1,2)) = \{(4,3), (4,1), (4,2), (3,1), (3,2)\}$.

Diagrams of the Bruhat order and weak Bruhat order on the permutations of $\{1, 2, 3, 4\}$ can be found in [4].

A (finite) *lattice* is a (finite) partially ordered set in which any two elements have an LUB and a GLB (for finite partially ordered sets LUBs (resp. GLBs) guarantee GLBs (resp. LUBs). The Bruhat order is not a lattice, for instance, with $n = 4$, GLB(4312,4231) is not defined. The weak Bruhat order on $\mathcal{S}_n$ is a lattice since each permutation of $\mathcal{S}_n$ can be identified with its set of inversions with the partial order being set-containment.

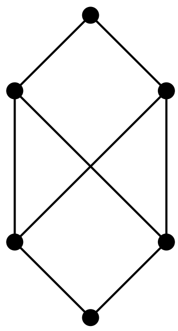
3. Bruhat orders

We now discuss the these two Bruhat orders in somewhat more detail.

The diagram of the Bruhat order on the permutations of order 3, $(\mathcal{S}_3, \preceq_B)$, is given below. This is not a lattice order since $(2,3,1)$ and $(3,1,2)$ do not have a meet, and $(2,1,3)$ and $(1,3,2)$ do not have a join.

$$L_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} : (3,2,1)$$

$$(2,3,1) : \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} : (3,1,2)$$



$$(2,1,3) : \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} : (1,3,2)$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : (1,2,3)$$

In both orders, the *identity permutation* $\iota_n = (1, 2, \ldots, n)$ is the unique minimal element ($\mathcal{I}(\iota_n) = \emptyset$), and the *anti-identity permutation* $\zeta_n = (n, n-1, \ldots, 2, 1)$ is the unique maximal element ($\mathcal{I}(\zeta_n) = \{(i,j) : i > j\}$). The *cover relation*, π_2 covers π_1 (or π_1 is covered by π_2), in these partially ordered sets means

- $(\mathcal{S}_n, \preceq_b)$: $\pi_1 \preceq_b \pi_2$ where $\mathcal{I}(\pi_1)$ is obtained from $\mathcal{I}(\pi_1)$ by removing one inversion.
- $(\mathcal{S}_n, \preceq_B)$: $\pi_1 \preceq_B \pi_2$ where π_1 has exactly 1 fewer inversion.

Example: $(4, 5, 3, 2, 1) \preceq_B (4, 2, 3, 5, 1)$.

Both Bruhat orders on $\mathcal{S}_n$ are *graded* by the number of inversions. The grade is the **level** in the diagram of the partially ordered set, beginning with the identity permutation at level 0.

There is an equivalent, though not obvious, way to define the Bruhat order. For an $m \times n$ matrix $A = [a_{ij}]$, define its *sum-matrix* or Σ-*matrix* $\Sigma(A) = [\sigma_{ij}(A)]$ by

$$\sigma_{ij} = \sigma_{ij}(A) = \sum_{1 \le k \le i, 1 \le l \le j} a_{ij} \quad (1 \le i \le m, 1 \le j \le n),$$

the sum of the entries of the leading $i \times j$ submatrix of A. (If A is a permutation matrix, this is the same as the rank of the leading $i \times j$ submatrix of A.)

Example: $A = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & 3 & 1 & 2 \\ 3 & 5 & 1 & 2 \end{bmatrix} \to \Sigma(A) = \begin{bmatrix} 1 & 4 & 6 & 10 \\ 1 & 7 & 10 & 16 \\ 4 & 15 & 19 & 27 \end{bmatrix}$

We then have the following equivalent way to determine Bruhat oder.

Theorem: For $n \times n$ permutation matrices P and Q, we have

$P \preceq_B Q$ if and only if $\Sigma(P) \ge \Sigma(Q)$ (entrywise).

□

4. Dedekind-MacNeille completion of the Bruhat order

The following theorem is due to MacNeille [20].

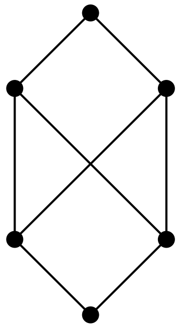
Theorem: (Dedekind-MacNeille Completion of a Partially Ordered Set.) *Let $(P, \le_P)$ be a finite partially ordered set. Then there exists a unique minimal lattice $(L, \le_L)$ such that $P \subseteq L$ and for $a, b \in P$, $a \le_P b$ if and only if $a \le_L b$.* □

$(L, \le_L)$ is the *Dedekind-MacNeille completion* of $(P, \le_P)$. The Dedekind-MacNeille completion of the rational numbers with the usual order gives the real numbers with $\pm\infty$. If one has a favorite partially ordered set which is not a lattice, one can try to find its Dedekind-MacNeille completion.

Recall the Bruhat order on the permutations of order 3, $(\mathcal{S}_3, \preceq_B)$, which, for convenience is repeated below. Recall also that $(\mathcal{S}_3, \preceq_B)$ is not a lattice.

$$L_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} : (3,2,1)$$

$$(2,3,1) : \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} : (3,1,2)$$



$$(2,1,3) : \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} : (1,3,2)$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : (1,2,3)$$

We determine the Dedekind-MacNeille Completion of $(\mathcal{S}_3 \preceq_B)$. As already pointed out the permutation matrices, given with their sum-matrices,

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \overset{\Sigma_1}{\to} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \overset{\Sigma_2}{\to} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix},$$

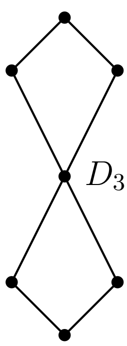
do not have a meet: With $\Sigma_3 = \min\{\Sigma_1, \Sigma_2\} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$, there does not exist a permutation matrix with this Σ_3. The problem is the 1 in the $(2,2)$-position of Σ_3. But

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -\mathbf{1} & 1 \\ 0 & 1 & 0 \end{bmatrix} \overset{\Sigma_3}{\to} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

The Dedekind-MacNeille Completion of $(\mathcal{S}_3, \preceq_B)$ is given below.

$$L_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$



$$D_3 \text{ where } D_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

There are two ways to describe the Dedekind-MacNeille completion of $(\mathcal{S}_n, \preceq_B)$, the first of which identifies permutation matrices with their Σ-matrices.

Theorem: (Lascoux & Schűtzenberger [19]): (Version# 1) The Dedekind-MacNeille completion of $(\mathcal{S}_n, \preceq_B)$ is $(\Sigma_n, \leq)$ where Σ_n is the set of $n \times n$ nonnegative integral matrices $X = [x_{ij}]$ satisfying

(i) For each i, the integers in row i and column i are taken from $\{1, 2, \ldots, i\}$ beginning with 0 or 1 and ending with i;

(ii) For each i, the integers in row i and column i are nondecreasing;

(iii) Two consecutive entries in a row or column are either equal or there is an increase of 1.

The (lattice) partial order is the entrywise order. $\square$

Theorem: (Lascoux & Schűtzenberger [19]): (Version 2) The MacNeille completion of $(\mathcal{S}_n, \preceq_B)$ is $(\mathcal{A}_n, \preceq_B)$ where $\mathcal{A}_n$ is the set of $n \times n$ *alternating sign matrices*, that is, $(0, 1, -1)$-matrices where the ± 1's in each row and column alternate, ignoring 0's, and start and end with a 1, and the partial order $\preceq_B$ in $(\mathcal{A}_n, \preceq_B)$ is: $A_1 \preceq_B A_2$ provided A_1 can be gotten from A_2 by transformations obtained by adding 2×2 submatrices of the form $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ where all intermediate matrices are ASMs. $\square$

Note that $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, and thus these transformations are *transpositions* where -1s are now allowed in the result. All $n \times n$ ASMs can be obtained from I_n by a sequence of transpositions with all intermediary matrices ASMs (see e.g. [11]).

5. Alternating Sign Matrices: ASMs

As a side remark we point out the matrices of 0's, 1's, and -1's occur frequently in combinatorial matrix settings. We mention the well-known adjacency matrices of signed graphs and Hadamard matrices, and matrices that arise in the notion of sign-solvability of linear systems concerning to what extent the signs of the solution of a linear system can be determined by the signs of the coefficients [16].

Two examples of ASMs that are not permutation matrices are:

$$\left[\begin{array}{c|c|c|c|c|c} & & 1 & & & \\ \hline & 1 & -1 & & 1 & \\ \hline 1 & -1 & & 1 & -1 & 1 \\ \hline & & 1 & -1 & 1 & \\ \hline & 1 & -1 & 1 & & \\ \hline & & 1 & & & \end{array}\right], \quad \left[\begin{array}{c|c|c|c|c|c} & 1 & & & & \\ \hline 1 & -1 & 1 & & & \\ \hline & 1 & -1 & 1 & & \\ \hline & & 1 & -1 & 1 & \\ \hline & & & 1 & -1 & 1 \\ \hline & & & & 1 & \end{array}\right].$$

Some Basic Properties of ASMs are the following:

(i) The partial row and column sums starting from the first or last entry equal 0 or 1, with the full row and column sums equal to 1.

(ii) The ASM property is preserved under the dihedral group of order 8 (symmetries of a square), but not under arbitrary (simultaneous) row and columns permutations.

(iii) The largest number of nonzeros (so ±1's) in an $n \times n$ ASM occurs for the so-called *diamond ASMs*. If n is even, there are two diamond matrices; if n is odd, there is only one [15]. The 5×5 diamond ASM and the two 6×6 diamond ASMs are:

$$D_5 = \left[\begin{array}{c|c|c|c|c} & & 1 & & \\ \hline & 1 & -1 & 1 & \\ \hline 1 & -1 & 1 & -1 & 1 \\ \hline & 1 & -1 & 1 & \\ \hline & & 1 & & \end{array}\right],$$

$$D_6 = \left[\begin{array}{c|c|c|c|c|c} & & 1 & & & \\ \hline & 1 & -1 & 1 & & \\ \hline 1 & -1 & 1 & -1 & 1 & \\ \hline & 11 & -1 & 1 & -1 & 1 \\ \hline & & 1 & -1 & 1 & \\ \hline & & & 1 & & \end{array}\right], D_6' = \left[\begin{array}{c|c|c|c|c|c} & & & 1 & & \\ \hline & & 1 & -1 & 1 & \\ \hline & 1 & -1 & 1 & -1 & 1 \\ \hline 1 & -1 & 1 & -1 & 1 & \\ \hline & 1 & -1 & 1 & & \\ \hline & & 1 & & & \end{array}\right]$$

A bijection between the two versions of the MacNeille Completion is given by:

(i) If A is an $n \times n$ ASM, Then $\Sigma(A)$ satisfies the defining conditions of Σ_n, that is,

- For each i, the integers in row i and column i are taken from $\{1, 2, \ldots, i\}$ beginning with 0 or 1 and ending with i,
- For each i, the integers in row i and column i are nondecreasing.
- Two consecutive entries in a row or in a column are either equal or increase by 1.

(ii) Given a matrix $X = [x_{ij}] \in \Sigma_n$, then $A = [a_{ij}]$ is an $n \times n$ ASM where

$$a_{ij} = x_{ij} + x_{i-1,j-1} - x_{i-1,j} - x_{i,j-1},$$

where $x_{i0} = x_{0j}$ are defined to be 0.

The number of $n \times n$ permutation matrices is $n!$. It is natural to ask how many $n \times n$ ASMs are there, an obviously more difficult question.

For small n, the number of $n \times n$ ASMs is: $1, 2, 7, 42, 429, 7436, \ldots$.

Mills, Robbins, and Rumsey [21] made a conjecture in 1983 which became a celebrated 1996 theorem of Zeilberger [23] (later and independently also proved by Kuperberg [18]): The number of $n \times n$ ASMs is

$$\frac{1!4!7!\cdots(3n-2)!}{n!(n+1)!(n+2)!\cdots(2n-1)!} = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} \sim \left(\frac{3\sqrt{3}}{4}\right)^{n^2}.$$

This sequence arose earlier in another context: Totally Symmetric Self-Complementary Plane Partitions (TSSCPPs) which we now very briefly describe. As an example of a TSSCPP is obtained by stacking the following 4×4 matrices to form a $4 \times 4 \times 4$ configuration:

$$\begin{array}{|c|c|c|c|}\hline x & x & x & x \\ \hline x & x & x & y \\ \hline x & x & x & y \\ \hline x & y & y & y \\ \hline\end{array}, \begin{array}{|c|c|c|c|}\hline x & x & x & x \\ \hline x & x & x & y \\ \hline x & x & y & y \\ \hline x & y & y & y \\ \hline\end{array}, \begin{array}{|c|c|c|c|}\hline x & x & x & y \\ \hline x & x & y & y \\ \hline x & y & y & y \\ \hline y & y & y & y \\ \hline\end{array}, \begin{array}{|c|c|c|c|}\hline x & x & x & y \\ \hline x & y & y & y \\ \hline x & y & y & y \\ \hline y & y & y & y \\ \hline\end{array}$$

$11, 5 \qquad 10, 6 \qquad 6, 10 \qquad 5, 11$

Andrews (1994) (see e.g. [5]) showed that the number of TSSCPPs in a $2n \times 2n \times 2n$ configuration equals

$$\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

the same number that was conjectured for the number of $n \times n$ ASMS. Zeilberger proved that the sequence counting ASMs is the same as the sequence counting TSSCPPs and then used Andrews formula. Note that no bijection between ASMs and TSSCPPs is known. Other combinatorial objects are know to be equinumerous with ASMs. Just recently a bijective proof has been announced by Ilse Fischer and Matjaz Konvalinka for the enumeration of another combinatorial object related to ASMs, and this may lead to other bijective proofs.

There is a bijection between ASMs and certain objects called *square ice configurations*, a system of water (H_2O) molecules frozen in a square lattice (see [5]). There are oxygen atoms at each vertex of an $n \times n$ lattice, with hydrogen atoms between successive oxygen atoms in a row or column, and on either vertical side of the lattice, but not on the two horizontal sides. For instance, with $n = 4$,

$$\begin{array}{ccccccccc} H & O & H & O & H & O & H & O & H \\ & H & & H & & H & & H & \\ H & & H & O & H & O & H & O & H \\ & H & & H & & H & & H & \\ H & O & H & O & H & O & H & O & H \\ & H & & H & & H & & H & \\ H & O & H & O & H & O & H & O & H \end{array}.$$

Each O is to be attached to two Hs (giving a water molecule H_2O) in a one to two bijection. There are six possible configurations in which an oxygen atom can be attached to two hydrogen atoms:

$$H \leftarrow O \rightarrow H \qquad \begin{array}{c} H \\ \uparrow \\ O \\ \downarrow \end{array}$$

$$\begin{array}{ccc} & & H \\ & & \uparrow \\ H & \leftarrow & O \end{array} \qquad \begin{array}{ccc} H & & \\ \uparrow & & \\ O & \rightarrow & H \end{array} \qquad \begin{array}{ccc} O & \rightarrow & H \\ \downarrow & & \\ H & & \end{array} \qquad \begin{array}{ccc} H & \leftarrow & O \\ & & \downarrow \\ & & H \end{array}.$$

Let the top left (horizontal) configuration correspond to $\mathbf{1}$ and the top right (vertical) configuration correspond to $-\mathbf{1}$. Let the other four (skew) configurations correspond to $\mathbf{0}$. Then we have

$$
\begin{array}{ccccccccccccccccc}
H & \leftarrow & O & & H & \leftarrow & O & & H & \leftarrow & O & \rightarrow & H & & O & \rightarrow & H \\
 & & \downarrow & & & & \downarrow & & & & & & & & \downarrow & & \\
 & & H & & & & H & & & & H & & & & H & & \\
 & & & & & & & & & & \uparrow & & & & & & \\
H & \leftarrow & O & \rightarrow & H & & O & \rightarrow & H & & O & & H & \leftarrow & O & \rightarrow & H \\
 & & & & & & \downarrow & & & & \downarrow & & & & & & \\
 & & H & & & & H & & & & H & & & & H & & \\
 & & \uparrow & & & & & & & & & & & & \uparrow & & \\
H & \leftarrow & O & & H & \leftarrow & O & \rightarrow & H & & O & \rightarrow & H & & O & \rightarrow & H \\
 & & & & & & & & & & \downarrow & & & & & & \\
 & & H & & & & H & & & & H & & & & H & & \\
 & & \uparrow & & & & \uparrow & & & & & & & & \uparrow & & \\
H & \leftarrow & O & & H & \leftarrow & O & & H & \leftarrow & O & \rightarrow & H & & O & \rightarrow & H
\end{array}
$$

and this corresponds to the ASM:

$$
\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.
$$

6. Origin of ASMs

We now describe how ASM came about [21]. The λ-determinant arises by starting with

$$\det_\lambda \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} + \lambda a_{12}a_{21} \text{ (or we can start with } \det_\lambda[a_{11}] = a_{11})$$

and adapting the well-known Dodgson's condensation formula for determinants (which iteratively expresses a determinant in terms of 2×2 determinants) to the λ-determinant using the rule

$$\det_\lambda A = \frac{\det_\lambda A_{UL} \det_\lambda A_{LR} + \lambda \det_\lambda A_{UR} \det_\lambda A_{LL}}{\det_\lambda A_C}.$$

(A_{UL} is the $(n-1)\times(n-1)$ submatrix in upper left, A_{LR} in lower right, etc. and A_C is the $(n-2)\times(n-2)$ submatrix in the center.)

If $\lambda = -1$, we get Dodgson's formula for the ordinary determinant. If $n = 2$ (so C is empty), we get

$$\det_\lambda \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} + \lambda a_{12}a_{21},$$

(if $\lambda = -1$, we get the ordinary determinant). If $n = 3$ (so $C = [a_{22}]$) we get

$$\det_\lambda(A) = a_{11}a_{22}a_{33} + \lambda a_{12}a_{21}a_{33} + \lambda a_{11}a_{23}a_{32} + (\lambda^2 + \lambda)a_{12}a_{21}a_{22}^{-1}a_{23}a_{32}$$

$$+\lambda^2 a_{13}a_{21}a_{32} + \lambda^2 a_{12}a_{23}a_{31} + \lambda^3 a_{13}a_{22}a_{31},$$

(if $\lambda = -1$, we get the ordinary determinant since $\lambda^2 + \lambda = (-1)^2 + (-1) = 0$). One can calculate that for $n = 3$, we have

$$\det_\lambda(A) = a_{11}a_{22}a_{33} + \lambda a_{12}a_{21}a_{33} + \lambda a_{11}a_{23}a_{32} + (\lambda^2 + \lambda)a_{12}a_{21}a_{22}^{-1}a_{23}a_{32}$$
$$+\lambda^2 a_{13}a_{21}a_{32} + \lambda^2 a_{12}a_{23}a_{31} + \lambda^3 a_{13}a_{22}a_{31}.$$

If for each of the seven terms we replace entries in A by the corresponding exponent, we get the seven 3×3 ASMs. For instance,

$$(\lambda^2 + \lambda)a_{12}a_{21}a_{22}^{-1}a_{23}a_{32} \to \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

and the other terms give the six 3×3 permutation matrices. If $A = [a_{ij}]$ is an $n \times n$ matrix, then $\det_\lambda A$ is of the form

$$\sum_{B=[b_{ij}] \in \mathrm{ASM}_{n\times n}} p_B(\lambda) \prod_{i.j=1}^{n} a_{ij}^{b_{ij}}$$

where $p_B(\lambda)$ is a polynomial in λ. The number of terms is $|\mathrm{ASM}_{n\times n}|$ whose formula was given earlier.

7. Other Topics on Permutations

Permutation Patterns form a huge topic [5]. One particular theme is permutations *avoiding* certain patterns. We touch briefly on this topic.

Let $\sigma = (p_1, p_2, \ldots, p_k)$ be a permutation of $\{1, 2, \ldots, k\}$. Then a permutation $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ of $\{1, 2, \ldots, n\}$ *contains* σ provided there exists $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $\pi_{i_r} < \pi_{i_s}$ if and only if $p_r < p_s$. Otherwise, π *avoids* σ. If $k = 2$ and $\sigma = (2, 1)$, then the only permutation π that avoids σ is the identity permutation $(1, 2, \ldots, n)$. If $k = 2$ and $\sigma = (1, 2)$, then the only permutation π that avoids σ is the anti-identity permutation $(n, n-1, \ldots, 2, 1)$.

Patterns of length $k = 3$ are already of some interest. There are 3 possibilities: $\sigma = (1, 2, 3), (1, 3, 2), (2, 1, 3)$ but under reversal and complementation, there are only two non-equivalent patterns, namely, $(1, 2, 3)$ and $(3, 1, 2)$. The permutation $(3, 4, 5, 1, 2, 6, 7)$ is 321-avoiding in that there does not exist a decreasing subsequence of length 3. The permutation $(2, 1, 3, 5, 4, 6)$ is 312-avoiding; there is no subsequence of the form L(arge), S(mall), M(edium).

The number of σ-avoiding permutations is the same in all cases of $k = 3$, namely,

$$C_n := \frac{\binom{2n}{n}}{n+1}, \qquad \text{the ubiquitous } n\text{th Catalan number.}$$

Let π be a permutation of $\{1, 2, \ldots, n\}$. Then π is a 312-*avoiding permutation* provided π has no subsequence a, b, c with $a > b, a > c, b < c$. As an $n \times n$ permutation matrix, a 312-avoiding permutation is one having no 3×3 submatrix of the form

$$\left[\begin{array}{c|c|c} & & 1 \\ \hline 1 & & \\ \hline & 1 & \end{array}\right] = \left[\begin{array}{c|c|c} & & \mathrm{L} \\ \hline \mathrm{S} & & \\ \hline & \mathrm{M} & \end{array}\right].$$

Similar statements can be made for the other patterns of length 3. This suggests generalizations to arbitrary $m \times n$ $(0,1)$-matrices. For instance, the matrices below are 312-avoiding.

$$\left[\begin{array}{c|c|c|c|c} 1 & 1 & 1 & 1 & \\ \hline & 1 & 1 & 1 & \\ \hline 1 & 1 & & 1 & \\ \hline 1 & & & 1 & 1 \\ \hline 1 & & & 1 & 1 \end{array}\right], \quad \left[\begin{array}{c|c|c|c|c|c} 1 & 1 & 1 & 1 & & \\ \hline & 1 & 1 & 1 & & \\ \hline 1 & 1 & & 1 & & \\ \hline 1 & & & 1 & 1 & 1 \\ \hline 1 & & & 1 & 1 & 1 \\ \hline 1 & & & & & 1 \end{array}\right].$$

In general, an $m \times n$ 312-avoiding (0,1)-matrix A contains at most $2(m+n-2)$ 1's; if A contains fewer than $2(m+n-2)$ 1's, then it is always possible to change a 0 to a 1 resulting also in a 312-avoiding matrix [7].

Continuous analogue of $n \times n$ permutation matrices are the $n \times n$ *doubly stochastic matrices*: nonnegative entries with all row and column sums equal to 1. For example,

$$\begin{bmatrix} .5 & .2 & .3 \\ .3 & .4 & .3 \\ .2 & .4 & .4 \end{bmatrix}.$$

By Birkhoff's theorem, the set Ω_n of $n \times n$ doubly stochastic matrices is the convex hull of the set $\mathcal{P}_n$ of $n \times n$ permutation matriuces and these are its extreme points. The dimension of Ω_n is given $n \times n$ permutation matrices $\mathcal{P}_n$ and these are its extreme points. The dimension of Ω_n is given by $\dim \Omega_n = (n-1)^2$, that is, $\mathcal{P}_n$ has a linear span of dimension $(n-1)^2+1$. Edges and faces, in general, of Ω_n have been characterized and correspond to certain $n \times n$ (0,1)-matrices.

There are several known bases of the linear span $\langle \mathcal{P}_n \rangle$ of the $n \times n$ permutation matrices.

- (Farahat and Mirsky [2]): the identity permutation ι_n, all 2-cycles, all 3-cycles of the form $1 \to i \to j \to 1$ where $1 < i < j \leq n$ (the $n \times n$ permutation matrices C_{1ij} with $1 < i < j \leq n$). See also [14] for other bases.
- There is a basis of 123-avoiding permutations and also a basis of 312-avoiding permutations [8].

Every ASM is a ± 1 linear combination of permutation matrices and so the dimension of the linear span of the $n \times n$ ASMs $\mathcal{A}_n$ is also $(n-1)^2+1$. The convex hull Λ_n of $\mathcal{A}_n$ has the following linear characterization: $\mathcal{A}_n$ consists of all $n \times n$ matrices $A = [a_{ij}]$ with row and column sums equal to 1, and satisfying

$$\sum_{j=1}^{q} a_{ij}, \sum_{j=q+1}^{n} a_{ij} \geq 0 \text{ (all } q \text{ and } i\text{) with similar inequalities for columns.}$$

We have that $\dim(\Lambda_n) = (n-1)^2$ and the set of extreme points of Λ_n is $\mathcal{A}_n$. Edges of Λ_n have been characterized. See [3, 9, 22].

There are higher dimensional analogues of permutation matrices and alternating sign matrices [10]. These $n \times n \times n$ arrays with all line sums (three directions) equal to 1. For example,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \nearrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \nearrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

which is really the Latin Square

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix},$$

where the integers in the 3×3 array represent the height of a corresponding 1 in the $3 \times 3 \times 3$ array. An example of a 3-dimensional ASM is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \nearrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \nearrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

We have avoided some technicalities in hope that we might have enticed some readers to explore these ideas in more detail; the selected references below can be used to get one started. So that concludes our brief and selective story. In the final section we discuss some open problems.

8. Coda

As discussed earlier the Bruhat orders on the set $\mathcal{P}_n$ of permutations of order n and the set $\mathcal{A}_n$ of $n \times n$ ASMs are graded by rank (number of inversions) where the minimal rank is 0 in both and the maximum rank is $\binom{n}{2}$ for $\mathcal{P}_n$ and is $\binom{n+1}{3}$ for $\mathcal{A}_n$.

It is well known that the maximum number of elements of a given rank in $\mathcal{P}_n$ is the largest of the binomial coefficients $\binom{n}{k}$ and this occurs for $k = \lfloor \frac{n}{2} \rfloor$ and $k = \lceil \frac{n}{2} \rceil$. For $\mathcal{P}_n$ these rank numbers $0, 1, 2, \ldots, \binom{n}{2}$ are *unimodal*, that is, they increase to the maximum and then decrease. For $\mathcal{A}_n$ it has been *conjectured* in [15] that the rank numbers are also unimodal for $n \geq 6$ with the maximum equal to

$$\frac{\binom{n+1}{3}}{2}, \qquad \text{if } n \not\equiv 2 \bmod 4,$$

$$\frac{\binom{n+1}{3} \pm 1}{2}, \qquad \text{otherwise.}$$

The conjecture is false for $n = 5$ [15].

An *antichain* in a partially ordered set is a set of elements no two of which are comparable in the partial order. In particular, the elements of a fixed rank form an antichain. Hence the maximum cardinality of an antichain is at least as large as the maximum cardinality of the set of elements of a fixed rank. In the case of $\mathcal{P}_n$, the maximum cardinality of an antichain equals the largest of the rank numbers, that is, $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. One way to prove this is by constructing a *symmetric chain decomposition* of $\mathcal{P}_n$, that is, a partition of the elements of $\mathcal{P}_n$ into chains of successive ranks which are symmetric with respect to rank; see e.g. Section 5.6 of [6]. In [15] it has been *conjectured* that $\mathcal{A}_n$ has a symmetric chain decomposition for $n \geq 6$. If this is true, then the maximum cardinality of an antichain of $\mathcal{A}_n$ equals the maximum of its rank numbers.

In an ASM the first and last nonzero in each row and column is a $+1$, and the first and last rows and columns contain only one $+1$. In [12], ASMs were generalized to allow the first and last elements in each row and column to be arbitrarily described. Let $u = (u_1, u_2, \ldots, u_m), u' = (u'_1, u'_2, \ldots, u'_m), v =$

$(v_1, v_2, \ldots, v_n)$, and $v' = (v_1, v_2, \ldots, v'_n)$ be vectors of ± 1's. Then a $(u, u'; v, v')$-ASM is an $m \times n$ matrix $A = [a_{ij}]$ of 0's, 1's, and -1's such that the 1's and -1's in each of

$$(v_i, a_{i1}, a_{i2}, \ldots, a_{in}, v'_i) \text{ and } (u_j, a_{1j}, a_{2j}, \ldots, a_{mj}, u'_j), \quad (1 \leq i \leq m, 1 \leq j \leq n)$$

alternate. If u, u', v, v' are vectors of all -1's, then a $(u, u'; v, v')$-ASM is an ordinary ASM. Necessary and sufficient conditions for the existence of a $(u, u'; v, v')$-ASM are obtained in [3]. A special case is obtained by taking $m = n$, and $u' = u$ and $v' = v$; then nonemptiness is equivalent to u and v containing the same number of $+1$'s. In [13] the Bruhat order on the set of $n \times n$ ASMs is extended to $(u, u; v, v)$-ASMs with the result also being a graded distributive lattice. A *problem* mentioned is that of determining the *join-irreducible elements* of this lattice, that is, the elements that cannot be expressed as the join of two elements strictly below it. The reason for the interest in join-irreducible elements is that a distributive lattice is isomorphic to the lattice of subsets of its join-irreducible elements partially ordered by set-inclusion. Another interesting question is to investigate when the lattice of $(u, u; v, v)$-ASMs is self-dual.

Two $n \times n$ ASMs $A = [a_{ij}]$ and $B = [b_{ij}]$ are called *orthogonal ASM-mates* provided their scalar product $A \cdot B = \sum_{i,j} a_{ij} b_{ij}$ equals 1. If A and B are permutation matrices, then orthogonality means that A and B have exactly one common position with a 1. As shown in [10] the ASM

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

does not have an orthogonal ASM-mate. The ASMs

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

are pairwise orthogonal ASM-mates. It is an interesting *problem* to determine:

(i) Which $n \times n$ ASMs have an orthogonal ASM-mate?

(ii) Which $n \times n$ ASMs have a permutation matrix as an orthogonal ASM-mate.

Other questions concerning 3-dimensional ASMs and the notion of 3-dimensional latin squares are explored in [10].

Finally we conclude with the following suggestion: Consider your favorite partially ordered set and try to determine its Dedekind-MacNeille Completion.

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