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Initial value problem for the nonconservative zero-pressure gas dynamics system

Problème de la valeur initiale pour le système dynamique des gaz à pression nulle non conservateur

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ABSTRACT. In this article, we study initial value problem for the zero-pressure gas dynamics system in non-conservative form and the associated adhesion approximation. We use adhesion approximation and modified adhesion approximation in the construction of weak asymptotic solution. First we prove a general existence result for the adhesion model for the initial velocity component in H^s for $s > \frac{n}{2} + 1$ and the initial data for the density component being a C^1 function. Using this, we construct weak asymptotic solution for the system with initial velocity in $L^2 \cap L^\infty$ and the initial density being a bounded Borel measure. Then we make a detailed analysis of the explicit formula for the weak asymptotic solution and generalized solution for the plane-wave type initial data.
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1. Introduction

The zero-pressure gas dynamics system is an important system of partial differential equations which comes in application in cosmology and is closely related to the Zeldovich approximation ([20]). This system describes the evolution of matter in the expansion of universe as cold dust moving under gravity alone. The laws are given by a system of partial differential equations

$$\begin{aligned} &u_t + (u \cdot \nabla)u = 0, \\ &\rho_t + \nabla \cdot (\rho u) = 0 \end{aligned} \tag{1.1}$$

where u denotes the velocity of the particles and ρ is the density, $x \in \mathbb{R}^n$ denotes the space variable, $t > 0$ denotes time and $\nabla = (\partial_{x_1}, \partial_{x_2}, ...\partial_{x_n})$. For physical case, the space dimensions $n = 1, 2, 3$ are important. The Cauchy problem is to find u and ρ in suitable function space that satisfy the equation in $x \in \mathbb{R}^n, t > 0$ and the initial condition

$$u(x, 0) = u_0(x)\ ,\ \rho(x, 0) = \rho_0(x)\ ,\ x \in \mathbb{R}^n. \tag{1.2}$$

The mathematical problem is to identify the space and to find the solution, prove uniqueness, continuous dependence on data and derive qualitative properties of the solution such as regularity, asymptotic behavior of solution. All these questions are open except for the one dimensional case $n = 1$.

It is well known that the problem does not have smooth solution even if the initial data is smooth. The fastest particles overrun the slowest ones and the density becomes infinite in finite time, thus as in the case of most nonlinear systems, the existence of global smooth solutions for (1.1) even with smooth initial data (1.2) is not possible in general. The mapping from Lagrangian space to the Eulerian space given by

$$x = y + tu_0(y)$$

is bijective only for short time and

$$u(x,t) = u_0(y)\ ,\ \rho(x,t) = \rho_0(y) \det\left(\frac{\partial y_i}{\partial x_j}\right)$$

gives only short time existence of solution. The solution has to be understood in a weak sense and weak solutions are not unique. The unique physical solution has to be selected by giving additional conditions on the solutions. One such selection criteria is that it is the limit of a given physical regularization, such as the adhesion approximation.

The adhesion model corresponding to (1.1) was introduced by Gurbatov ([7]). The adhesion approximation describes the motion of the particles by the motion of sticky particles, wherein the velocity obeys the Burgers equation and the density is governed by the continuity equation

$$\begin{aligned} &u_t + (u\cdot\nabla)u = \epsilon\Delta u, \\ &\rho_t + \nabla\cdot(\rho u) = 0. \end{aligned} \tag{1.3}$$

One of the most important physical cases is when the velocity u can be represented in terms of a velocity potential ϕ, which was analysed by many authors (see [19, 12, 11, 8, 2, 1] and the references there). When the initial data is of the form $u(x,0) = \nabla\phi_0(x)$, the exact formula for the velocity u^ϵ can be written down using the Hopf-Cole transformation ([12, 20]). The equation for ρ is then a linear equation with smooth coefficients and can be solved explicitly. As $\epsilon \to 0$, the limit u of u^ϵ is a locally bounded BV function while the limit ρ of ρ^ϵ is a Radon measure. An additional difficulty is the non-conservative nature of the first equation. The products $(u.\nabla)u$ and ρu can no more be described in the sense of distributions. The idea therefore is to use the microscopic behavior of the adhesion approximation to make sense of the products involved and to formulate an appropriate notion of solution.

In the modified adhesion approximation, viscosity term is added in both equations, namely

$$\begin{aligned} &u^\epsilon_t + (u^\epsilon\cdot\nabla)u^\epsilon = \epsilon\Delta u^\epsilon, \\ &\rho^\epsilon_t + \nabla\cdot(\rho^\epsilon u^\epsilon) = \epsilon\Delta\rho^\epsilon. \end{aligned} \tag{1.4}$$

For the one space dimension case, Hopf-Cole transformation was used in [11] to write explicit solution of (1.4) with any bounded measurable initial data and vanishing viscosity solution was analysed. However for $n > 1$, even in the case of gradient type initial velocity, Hopf-Cole transformation does not help to write the explicit solution.

An important class of solutions which is of gradient type is that of the radial ones which take the form

$$u(x,t) = \frac{x}{t}\cdot a(r,t)\ ,\ \rho(x,t) = b(r,t)$$

with $r = |x|$ and $a : [0,\infty) \to \mathbb{R}$, $b : [0,\infty) \to \mathbb{R}$. The equation for a and b becomes the 2×2 system with initial condition at $t = 0$ and and a condition at the origin or a condition on the mass conservation. In [2], a formula was constructed for the solution for this case using Lax's fomula. A solution for the initial boundary value problem was constructed for the Burgers equation in [10].

One basic question is the formulation of the solution for such problems. In this connection, the first such formulation was given in Lefloch ([15]) and and a different formulation was given in Joseph ([11]) for the one dimensional 2×2 system. The approach of Lefloch was for a slightly general case and uses Lax formula, whereas that of Joseph was for one dimensional case of (1.1) for the Riemann problem and uses modified adhesion model.

The works of Danilov and Shelkovich ([3, 4, 5]) gave a new direction by giving new definitions of weak formulation of the solution in the framework of weak asymptotic method. There are different notions of solutions introduced in [1, 3, 4, 5, 17] and they contain interconnections and applications to different systems. First we explain these different notions.

Definition 1.1: A family of smooth functions $(u^\epsilon, \rho^\epsilon)_{\epsilon>0}$ is called a *weak asymptotic solution* of the system (1.1) with initial conditions $u(\cdot,0) = u_0(\cdot)$, $\rho(\cdot,0) = \rho_0(\cdot)$ provided as $\epsilon \to 0$, we have

$$\begin{aligned} u_t^\epsilon + (u^\epsilon \cdot \nabla)u^\epsilon &= o_{\mathfrak{D}'}(\mathbb{R}^n)(1), \\ \rho_t^\epsilon + \nabla \cdot (\rho^\epsilon u^\epsilon) &= o_{\mathfrak{D}'}(\mathbb{R}^n)(1), \\ u^\epsilon(x,0) - u_0(x) &= o_{\mathfrak{D}'}(\mathbb{R}^n)(1), \\ \rho^\epsilon(x,0) - \rho_0(x) &= o_{\mathfrak{D}'}(\mathbb{R}^n)(1). \end{aligned} \tag{1.5}$$

Here $o_{\mathfrak{D}'}(\mathbb{R}^n)(1)$ means a quantity that converges to 0 in the distributional sense in $\mathbb{R}^n$. The first two relations are required to hold uniformly in $t > 0$.

Definition 1.2: A distribution (ρ, u) is called a *generalized solution* of the Cauchy problem in the domain $\Omega_T = \{(x,t) : x \in \mathbb{R}^n, T > t > 0\}$ if it is the limit in distribution of a weakly asymptotic solution $(\rho^\epsilon, u^\epsilon)_{\epsilon>0}$ of the Cauchy problem as ϵ goes to zero.

Our aim is to use adhesion and modified adhesion approximations to construct weak asymptotic solutions for general L^∞ initial data. For the case when initial data is of plane wave type, we construct explicit formula for the weak asymptotic solution and as a limit, a generalized solution to the Cauchy problem for (1.1).

2. General initial data existence for adhesion model

In this section we prove the existence of solution for the initial value problem for adhesion approximation. The important physical case is for space dimensions $n = 1, 2, 3$. The case $n = 1$ is well understood and even explicit solutions are available, see [13, 15] and the references mentioned there, so we focus on the case $n \geq 2$. Consider the system

$$u_t + (u \cdot \nabla)u = \epsilon \Delta u \ , \ (x,t) \in \mathbb{R}^n \times (0,\infty) \tag{2.1}$$

with initial condition

$$u(x,0) = u_0(x) \ , \ x \in \mathbb{R}^n \tag{2.2}$$

where $u = (u_1, u_2, \ldots, u_n)$.

Theorem 2.1. *Let $s > n/2 + 1$ and T be any positive real number. Then for any given $\epsilon > 0$ and $u_0 \in H^s(\mathbb{R}^n)$, there exists a weak solution*

$$u^\epsilon \in C([0,T]; L^2(\mathbb{R}^n)) \cap L^\infty([0,T]; H^s(\mathbb{R}^n))$$

for (2.1) *which is a continuous function in* (t, x) *and continuously differentiable function in the space variable. Further for* $s_0 + 2l \leq s$*, there exists a constant* $N(T, s, \epsilon, ||u_0||_{L^\infty(\mathbb{R}^n)}, ||u_0||_{H^s(\mathbb{R}^n)}) > 0$ *satisfying*

$$\int_0^T \left|\left|\partial_t^l u_j^\epsilon(\cdot, t)\right|\right|^2_{H^{s_0}(\mathbb{R}^n)} dt \leq N(T, s, \epsilon, ||u_0||_{L^\infty(\mathbb{R}^n)}, ||u_0||_{H^s(\mathbb{R}^n)}).$$

By a weak solution, we mean that

$$\int_0^T \int_{\mathbb{R}^n} \left[-u_j \phi_t(x,t) + \sum_{i=1}^n u_i \partial_{x_i} u_j \phi(x,t) - \epsilon \sum_{i=1}^n u_j {\partial_{x_i}}^2 \phi(x,t) \right] dx dt = 0 \tag{2.3}$$

for all $\phi \in C_0^\infty(\mathbb{R}^n \times (0, T))$ and the initial condition is satisfied in the sense that

$$\lim_{t \to 0} \int_{\mathbb{R}^n} |u_j(x,t) - u_{0j}(x)|^2 dx = 0 \tag{2.4}$$

for each j. The proof of the theorem is based on a fixed point argument. First we observe that the system (2.1) can be written componentwise with initial velocity $u_0 = (u_{01}, u_{02}, \ldots, u_{0n})$:

$$\begin{aligned} (u_j)_t + \sum_{i=1}^n u_i \partial_{x_i} u_j &= \epsilon \sum_{i=1}^n {\partial_{x_i}}^2 u_j \\ u_j(x,0) &= u_{0j}(x) \end{aligned} \tag{2.5}$$

for $j = 1, 2, \ldots, n$. Define the following iteration:

$$\begin{aligned} &u_j^0 = u_{0j}, \\ &\partial_t u_j^k + \sum_{i=1}^n u_i^{k-1} \partial_{x_i} u_j^k = \epsilon \sum_{i=1}^n \partial_{x_i}^2 u_j^k, \\ &u_j^k(x,0) = u_{0j}(x). \end{aligned} \tag{2.6}$$

We first prove the following lemma.

Lemma 2.2. *Given any* $T > 0$*, there exist weak solutions* u^k *for each* $k = 1, 2, 3, \ldots$ *satisfying the following estimates:*

$$||u^k(x,t)||_{L^\infty(\mathbb{R}^n \times [0,\infty))} = ||u_0||_{L^\infty(\mathbb{R}^n)}, \tag{2.7}$$

$$\sup_{0 \leq t \leq T} ||u^k(\cdot,t)||_{L^2(\mathbb{R}^n)} \leq \exp\left(\frac{n||u_0||^2_{L^\infty}}{2\epsilon} T \right) . ||u_0||_{L^2(\mathbb{R}^n)}, \tag{2.8}$$

$$\sup_{0 \leq t \leq T} ||u^k(\cdot,t)||_{H^1(\mathbb{R}^n)} \leq \exp\left(\frac{n^2||u_0||^2_{L^\infty}}{2\epsilon} T \right) . ||u_0||_{H^1(\mathbb{R}^n)}, \tag{2.9}$$

$$\sup_{0 \leq t \leq T} \left|\left|u^k(\cdot,t)\right|\right|_{H^s(\mathbb{R}^n)} \leq M(s, \epsilon, T, ||u_0||_{H^s(\mathbb{R}^n)}). \tag{2.10}$$

For $s_0 + 2l \leq s$,

$$\int_0^T \left|\left|\partial_t^l u_j^k(\cdot,t)\right|\right|^2_{H^{s_0}(\mathbb{R}^n)} dt \leq N(s_0, \epsilon, ||u_0||_{L^\infty}, ||u_0||_{H^{s_0+2l}}, T), \tag{2.11}$$

$$\left|\left|\partial_{x_i} u^{k+1}(\cdot,t)\right|\right|_{L^\infty(\mathbb{R}^n)} \leq C(t,\epsilon,u_0)\ ,\ i = 1,2,\ldots,n \tag{2.12}$$

where

$$\begin{aligned} C(t,\epsilon,u_0) := & \frac{1}{\sqrt{t\epsilon}}. \left|\left||y|.e^{-|y|^2}\right|\right|_{L^2}. ||u_0||_{L^2} \\ & + \frac{1}{\epsilon}. ||u_0||^2_{L^\infty}. \left|\left||y|.e^{-|y|^2}\right|\right|_{L^2}. ||u_0||_{H^1(\mathbb{R}^n)}. \exp\left(\frac{n^2. ||u_0||^2_{L^\infty}.T}{2\epsilon}\right)\sqrt{t}. \end{aligned} \tag{2.13}$$

Proof. For $j = 1,2,\ldots,n$ and $k = 1$, there exists weak solution

$$u_j^1 \in C^1[0,T; H^s(\mathbb{R}^n)] \cap L^\infty(0,T; H^{s-1}(\mathbb{R}^n))$$

to the equation (2.6) and by the maximum principle, we have

$$\sup_{x\in\mathbb{R}^n} |u_j^1(x,t)| = \sup_{x\in\mathbb{R}^n} |u_{0j}(x)|$$

for each $j = 1,2,\ldots,n$. By induction, there exist weak solution

$$u_j^k \in C^1[0,T; H^s(\mathbb{R}^n)] \cap L^\infty(0,T; H^{s-1}(\mathbb{R}^n))$$

for (2.6). Here we show that the estimates are independent of $k = 1,2,\ldots$. The L^∞ estimate for u^k follows from (2.6) by maximum principle.

Next we get L^2- Sobolev estimates for u^k. For this, we multiply u_j^k on both sides in the second equation of (2.6) and integrate over $\mathbb{R}^n$, getting

$$\frac{1}{2}\partial_t\left[\int_{\mathbb{R}^n} (u_j^k)^2 dx\right] + \sum_{i=1}^n \int_{\mathbb{R}^n} u_i^{k-1} u_j^k \partial_{x_i} u_j^k dx = \epsilon \sum_{i=1}^n \int_{\mathbb{R}^n} \left(\partial^2_{x_i} u_j^k\right) u_j^k dx.$$

After using integration by parts, we get

$$\begin{aligned} \partial_t\left[\int_{\mathbb{R}^n} (u_j^k)^2 dx\right] + 2\epsilon \sum_{i=1}^n \int_{\mathbb{R}^n} (\partial_{x_i} u_j^k)^2 dx &= -2\sum_{i=1}^n \int_{\mathbb{R}^n} u_i^{k-1} u_j^k \partial_{x_i} u_j^k dx \\ &\leq 2||u_0||_{L^\infty} \sum_{i=1}^n \int_{\mathbb{R}^n} \left|u_j^k \partial_{x_i} u_j^k\right| dx \\ &\leq \sum_{i=1}^n \int_{\mathbb{R}^n} \left[||u_0||^2_{L^\infty} \frac{(u_j^k)^2}{\epsilon} + \epsilon(\partial_{x_i} u_j^k)^2\right] dx. \end{aligned}$$

Simplifying, we get

$$\partial_t\left[\int_{\mathbb{R}^n} (u_j^k)^2 dx\right] + \epsilon \sum_{i=1}^n \int_{\mathbb{R}^n} (\partial_{x_i} u_j^k)^2 dx \leq \frac{||u_0||^2_{L^\infty}}{\epsilon} \sum_{i=1}^n \int_{\mathbb{R}^n} (u_j^k)^2 dx.$$

Summation over the index j in the above equation gives

$$\partial_t \left[\sum_{j=1}^{n} \int_{\mathbb{R}^n} (u_j^k)^2 dx \right] + \epsilon \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} (\partial_{x_i} u_j^k)^2 dx \leq \frac{n||u_0||^2_{L^\infty}}{\epsilon} \sum_{j=1}^{n} \int_{\mathbb{R}^n} (u_j^k)^2 dx.$$

It follows that

$$\partial_t \left[\sum_{j=1}^{n} \int_{\mathbb{R}^n} (u_j^k)^2 dx \right] \leq \frac{n||u_0||^2_{L^\infty}}{\epsilon} \int_{\mathbb{R}^n} \sum_{j=1}^{n} (u_j^k)^2 dx.$$

Thus, we get

$$\sum_{j=1}^{n} \int_{\mathbb{R}^n} (u_j^k)^2 dx \leq \exp\left(\frac{n||u_0||^2_{L^\infty}}{\epsilon} t \right) \sum_{j=1}^{n} \int_{\mathbb{R}^n} (u_{0j})^2 dx, \tag{2.14}$$

which proves (2.8). Next, we differentiate the second equation in (2.6) w.r.t. x_l to get

$$\partial_t \left(\partial_{x_l} u_j^k \right) + \partial_{x_l} \left[\sum_{i=1}^{n} u_i^{k-1} \partial_{x_i} u_j^k \right] = \epsilon \sum_{i=1}^{n} \partial^2_{x_i} \partial_{x_l} (u_j^k).$$

Multiplying both sides of the above equation with $\partial_{x_l} u_j^k$ and integrating over $\mathbb{R}^n$, we get

$$\frac{1}{2} \partial_t \left[\int_{\mathbb{R}^n} (\partial_{x_l} u_j^k)^2 dx \right] + \sum_{i=1}^{n} \int_{\mathbb{R}^n} \partial_{x_l} \left[u_i^{k-1} \partial_{x_i} u_j^k \right] \partial_{x_l} u_j^k dx$$
$$= \epsilon \sum_{i=1}^{n} \int_{\mathbb{R}^n} \left(\partial_{x_l} u_j^k \right) \partial^2_{x_i} \left(\partial_{x_l} u_j^k \right) dx.$$

After using integration by parts, this equation becomes

$$\frac{1}{2} \partial_t \left[\int_{\mathbb{R}^n} (\partial_{x_l} u_j^k)^2 dx \right] - \sum_{i=1}^{n} \int_{\mathbb{R}^n} u_i^{k-1} \left(\partial_{x_i} u_j^k \right) \left(\partial^2_{x_l} u_j^k \right) dx$$
$$= -\epsilon \sum_{i=1}^{n} \int_{\mathbb{R}^n} \partial_{x_i} (\partial_{x_l} u_j^k) \partial_{x_i} (\partial_{x_l} u_j^k) dx.$$

This implies that

$$\partial_t \left[\int_{\mathbb{R}^n} (\partial_{x_l} u_j^k)^2 dx \right] + 2\epsilon \sum_{i=1}^{n} \int_{\mathbb{R}^n} \left(\partial_{x_i} \partial_{x_l} u_j^k \right)^2 dx$$
$$= 2 \sum_{i=1}^{n} \int_{\mathbb{R}^n} u_i^{k-1} \left(\partial_{x_i} u_j^k \right) \left(\partial^2_{x_l} u_j^k \right) dx$$
$$\leq \sum_{i=1}^{n} \int_{\mathbb{R}^n} ||u_0||^2_{L^\infty} \frac{\left(\partial_{x_i} u_j^k \right)^2}{\alpha} dx + n\alpha \int_{\mathbb{R}^n} \left(\partial^2_{x_l} u_j^k \right)^2 dx,$$

where $\alpha > 0$ will be suitably chosen below. Summing over the indices l and j, we get

$$\partial_t \left[\sum_{l,j} \int_{\mathbb{R}^n} \left(\partial_{x_l} u_j^k\right)^2 dx \right] + 2\epsilon \sum_{i,j,l} \int_{\mathbb{R}^n} \left(\partial_{x_i}\partial_{x_l} u_j^k\right)^2 dx$$

$$\leq \sum_{i,j} \int_{\mathbb{R}^n} n||u_0||_{L^\infty}^2 \frac{\left(\partial_{x_i} u_j^k\right)^2}{\alpha} dx + n\alpha \sum_{l,j} \int_{\mathbb{R}^n} \left(\partial_{x_l}^2 u_j^k\right)^2 dx.$$

For $\alpha = \frac{\epsilon}{n}$, we get

$$\partial_t \left[\sum_{l,j} \int_{\mathbb{R}^n} \left(\partial_{x_l} u_j^k\right)^2 dx \right] + 2\epsilon \sum_{i,j,l} \int_{\mathbb{R}^n} \left(\partial_{x_i}\partial_{x_l} u_j^k\right)^2 dx$$

$$\leq \sum_{i,j} \int_{\mathbb{R}^n} n^2||u_0||_{L^\infty}^2 \frac{(\partial_{x_i} u_j^k)^2}{\epsilon} dx + \epsilon \sum_{l,j} \int_{\mathbb{R}^n} \left(\partial_{x_l}^2 u_j^k\right)^2 dx$$

$$\leq \sum_{i,j} \int_{\mathbb{R}^n} n^2||u_0||_{L^\infty}^2 \frac{(\partial_{x_i} u_j^k)^2}{\epsilon} dx + \epsilon \sum_{i,j,l} \int_{\mathbb{R}^n} \left(\partial_{x_i}\partial_{x_l} u_j^k\right)^2 dx.$$

Employing the same analysis as before, we get the estimates

$$\sum_{i,j} \int_{\mathbb{R}^n} \left(\partial_{x_i} u_j^k\right)^2 dx \leq \exp\left(\frac{n^2||u_0||_{L^\infty}^2}{\epsilon} t\right) \sum_{i,j} \int_{\mathbb{R}^n} \left(\partial_{x_i} u_{0j}\right)^2 dx. \tag{2.15}$$

From (2.14) and (2.15), we have (2.9).

Now we get higher order estimates. We prove it by the method of induction on s. We have this estimate for $s = 0$ and $s = 1$. We show that for each $j, k = 1, 2, \ldots$

$$\sup_{0 \leq t \leq T} \left|\left|u_j^k(\cdot, t)\right|\right|_{H^s(\mathbb{R}^n)} \leq M(s, \epsilon, T, ||u_0||_{H^s(\mathbb{R}^n)}),$$

where $M(s, \epsilon, T, ||u_0||_{H^s(\mathbb{R}^n)})$ is a smooth function of $\epsilon > 0,\ T > 0,\ s > 0$ and $||u_0||_{H^s(\mathbb{R}^n)}$.

Now assume that such an estimate is true for $s - 1$. We show that u_j^k satisfies the estimate (2.10).

Let α be any multi-index in $\mathbb{R}^n$. Apply the operator ∂^α on both sides of the second equation in (2.6) to get

$$\partial_t \left(\partial^\alpha u_j^k\right) + \partial^\alpha \left[\sum_{i=1}^n u_i^{k-1} \partial_{x_i} u_j^k \right] = \epsilon \sum_{i=1}^n \partial_{x_i}^2 \partial^\alpha (u_j^k).$$

Multiplying $\partial^\alpha u_j^k$ to both sides of this equation and integrating over $\mathbb{R}^n$, we get

$$\frac{1}{2}\partial_t \left[\int_{\mathbb{R}^n} (\partial^\alpha u_j^k)^2 dx \right] + \sum_{i=1}^n \int_{\mathbb{R}^n} \partial^\alpha \left(u_i^{k-1} \partial_{x_i} u_j^k\right) \partial^\alpha u_j^k dx$$

$$= \epsilon \sum_{i=1}^n \int_{\mathbb{R}^n} (\partial^\alpha u_j^k) \partial_{x_i}^2 (\partial^\alpha u_j^k) dx.$$

Using integration by parts, we get, for any multi-index β in $\mathbb{R}^n$ with $|\beta| = 1$,

$$\begin{aligned}
&\partial_t \left[\int\limits_{\mathbb{R}^n} \left(\partial^\alpha u_j^k\right)^2 dx \right] + 2\epsilon \sum_{i=1}^n \int\limits_{\mathbb{R}^n} \left(\partial_{x_i}(\partial^\alpha u_j^k)\right)^2 dx \\
&= -2 \sum_{i=1}^n \int\limits_{\mathbb{R}^n} \partial^\alpha \left(u_i^{k-1} \partial_{x_i} u_j^k\right) \partial^\alpha u_j^k dx \\
&= (-1)^{|\beta|+1} 2 \sum_{i=1}^n \int\limits_{\mathbb{R}^n} \partial^{\alpha-\beta} \left(u_i^{k-1} \partial_{x_i} u_j^k\right) \partial^\alpha \left(\partial^\beta u_j^k\right) dx \\
&\leq \frac{n}{\epsilon} \sum_{i=1}^n \int\limits_{\mathbb{R}^n} \left[\partial^{\alpha-\beta} \left(u_i^{k-1} \partial_{x_i} u_j^k\right)\right]^2 dx + \frac{\epsilon}{n} \sum_{i=1}^n \int\limits_{\mathbb{R}^n} \left[\partial^\alpha \left(\partial^\beta u_j^k\right)\right]^2 dx \\
&\leq \frac{n}{\epsilon} \sum_{i=1}^n \int\limits_{\mathbb{R}^n} \left[\partial^{\alpha-\beta} \left(u_i^{k-1} \partial_{x_i} u_j^k\right)\right]^2 dx + 2\epsilon \int\limits_{\mathbb{R}^n} \left[\partial^\beta \left(\partial^\alpha u_j^k\right)\right]^2 dx.
\end{aligned}$$

Since $|\beta| = 1$, we get

$$\partial_t \left[\int\limits_{\mathbb{R}^n} \left(\partial^\alpha u_j^k\right)^2 dx \right] \leq \frac{n}{\epsilon} \sum_{i=1}^n \int\limits_{\mathbb{R}^n} \left[\partial^{\alpha-\beta} \left(u_i^{k-1} \partial_{x_i} u_j^k\right)\right]^2 dx.$$

Summing over the indices α for $|\alpha| \leq s$, we obtain

$$\begin{aligned}
\partial_t \left[\sum_{|\alpha| \leq s} \int\limits_{\mathbb{R}^n} \left(\partial^\alpha u_j^k\right)^2 dx \right] &\leq \frac{n}{\epsilon} \sum_{i=1}^n \left|\left|u_i^{k-1}(\partial_{x_i} u_j^k)\right|\right|_{s-1}^2 \\
&\leq \frac{n}{\epsilon} \sum_{i=1}^n \left|\left|u_i^{k-1}\right|\right|_{s-1}^2 \left|\left|\partial_{x_i} u_j^k\right|\right|_{s-1}^2 \\
&\leq \frac{n}{\epsilon} \sum_{i=1}^n \left|\left|u_i^{k-1}\right|\right|_{s-1}^2 \left|\left|u_j^k\right|\right|_s^2,
\end{aligned}$$

from which we get

$$\partial_t \left(\left|\left|u^k\right|\right|_s^2 \right) \leq \frac{n}{\epsilon} \left|\left|u^{k-1}\right|\right|_{s-1}^2 \cdot \left|\left|u^k\right|\right|_s^2.$$

This implies that

$$\left|\left|u^k(\cdot, t)\right|\right|_s^2 \leq \exp\left(\int\limits_0^t \frac{n}{\epsilon} \left|\left|u^{k-1}(\cdot, \tau)\right|\right|_{s-1}^2 d\tau \right) ||u_0||_s^2. \tag{2.16}$$

Using the induction hypothesis, we have

$$\sup_{0 \leq t \leq T} \left|\left|u^k(\cdot, t)\right|\right|_s \leq \exp\left(\frac{n}{\epsilon} T \cdot M(s-1, \epsilon, T, ||u_0||_{H^{s-1}(\mathbb{R}^n)}) \right) ||u_0||_s. \tag{2.17}$$

Now higher order estimates follow from (2.16) and (2.17). Mixed derivatives with respect to x and t can be calculated using the second equation in (2.6). For example, we use (2.6) to get

$$||\partial_t u_j^k||_{L^2(\mathbb{R}^n\times[0,T])} \leq \sum_{i=1}^n ||u_i^{k-1}\partial_{x_i} u_j^k||_{L^2(\mathbb{R}^n\times[0,T])} + \epsilon \sum_{i=1}^n ||\partial_{x_i}^2 u_j^k||_{L^2(\mathbb{R}^n\times[0,T])}.$$

Using (2.10) and (2.17), we get an estimate of the form

$$||\partial_t u_j^k(\cdot,t)||_{L^2(\mathbb{R}^n\times[0,T])} \leq N(\epsilon, ||u_0||_{L^\infty}, ||u_0||_{H^2}, T),$$

which is bounded independent of k. In general, for $s_0 + 2l \leq s$, we will get

$$\int_0^T ||\partial_t^l u_j^k(\cdot,t)||^2_{H^{s_0}(\mathbb{R}^n)} dt \leq N(s_0, \epsilon, ||u_0||_{L^\infty}, ||u_0||_{H^{s_0+2l}}, T).$$

Estimation of $||\partial_{x_i} u^{k+1}(\cdot,t)||_{L^\infty(\mathbb{R}^n)}$.

Next, we derive the following estimate for fixed $T > 0$ and any $t \in (0,T]$:

$$||\partial_{x_i} u^{k+1}(\cdot,t)||_{L^\infty(\mathbb{R}^n)} \leq C(t,\epsilon,u_0)\ ,\ i = 1,2,\ldots,n \tag{2.18}$$

where

$$\begin{aligned} C(t,\epsilon,u_0) &:= \frac{1}{\sqrt{t\epsilon}}.\left|\left||y|.e^{-|y|^2}\right|\right|_{L^2}.\,||u_0||_{L^2} \\ &\quad + \frac{1}{\sqrt{\epsilon}}\,||u_0||_{L^\infty}.\left|\left||y|.e^{-|y|^2}\right|\right|_{L^2}.\,||u_0||_{H^1}.\exp\left(\frac{n^2.\,||u_0||^2_{L^\infty}.T}{2\epsilon}\right)\sqrt{t}. \end{aligned} \tag{2.19}$$

We have

$$\begin{aligned} u_t^{k+1} &= \epsilon.\Delta u^{k+1} - (u^k\cdot\nabla)u^{k+1} \\ u(x,0) &= u_0(x) \end{aligned} \tag{2.20}$$

To derive the above estimate, we use the following representation formula:

$$\begin{aligned} u^{k+1}(x,t) &= \frac{1}{(4\pi t\epsilon)^{\frac{n}{2}}}\int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{4t\epsilon}\right) u_0(y)dy \\ &\quad + \int_0^t\int_{\mathbb{R}^n} \frac{\exp\left(-\frac{|x-y|^2}{4(t-s)\epsilon}\right)}{[4\pi(t-s)\epsilon]^{\frac{n}{2}}}(u^k\cdot\nabla)u^{k+1}(y,s)dyds. \end{aligned} \tag{2.21}$$

Differentiating the above formula w.r.t. x_i for each i, we get

$$\begin{aligned} &\partial_{x_i} u^{k+1}(x,t) \\ &= \frac{1}{(4\pi t\epsilon)^{\frac{n}{2}}}\int_{\mathbb{R}^n} -\left(\frac{x_i-y_i}{\sqrt{2t\epsilon}}\right).\frac{1}{\sqrt{2t\epsilon}}\exp\left(-\frac{|x-y|^2}{4t\epsilon}\right)u_0(y)dy \\ &+ \int_0^t\int_{\mathbb{R}^n} -\left(\frac{x_i-y_i}{\sqrt{2(t-s)\epsilon}}\right).\frac{1}{\sqrt{2(t-s)\epsilon}}.\frac{\exp\left(-\frac{|x-y|^2}{4(t-s)\epsilon}\right)}{[4\pi(t-s)\epsilon]^{\frac{n}{2}}}(u^k\cdot\nabla)u^{k+1}(y,s)dyds. \end{aligned} \tag{2.22}$$

Thus for fixed $T > 0$, we have the following estimate for each $t \in (0, T]$:

$$
\begin{aligned}
&\left|\left|\partial_{x_i} u^{k+1}(\cdot, t)\right|\right|_{L^\infty} \\
&\leq \frac{1}{(2\pi)^{\frac{n}{2}}} \cdot \left[\frac{1}{\sqrt{t\epsilon}} \cdot \left|\left| y_i . e^{-|y|^2} \right|\right|_{L^2} \cdot ||u_0||_{L^2} \right. \\
&\left. + \int_0^t \frac{1}{\sqrt{(t-s)\epsilon}} \cdot ||u_0||_{L^\infty} \cdot \left|\left| y_i . e^{-|y|^2} \right|\right|_{L^2} \cdot \left|\left| \nabla u^{k+1}(\cdot, s) \right|\right|_{L^2} \right] \\
&\leq \frac{1}{(2\pi)^{\frac{n}{2}}} \left[\frac{1}{\sqrt{t\epsilon}} \cdot \left|\left| y_i . e^{-|y|^2} \right|\right|_{L^2} . ||u_0||_{L^2} \right. \\
&\left. + \left(\exp\left(\frac{n^2 \, ||u_0||^2_{L^\infty} . T}{2\epsilon} \right) ||u_0||_{H^1(\mathbb{R}^n)} \right) ||u_0||_{L^\infty} \left|\left| y_i . e^{-|y|^2} \right|\right|_{L^2} \int_0^t \frac{1}{\sqrt{(t-s)\epsilon}} ds \right] \\
&\leq \frac{1}{\sqrt{t\epsilon}} \cdot \left|\left| |y| . e^{-|y|^2} \right|\right|_{L^2} \cdot ||u_0||_{L^2} \\
&+ \frac{1}{\sqrt{\epsilon}} \cdot ||u_0||_{L^\infty} \cdot \left|\left| |y| . e^{-|y|^2} \right|\right|_{L^2} \cdot ||u_0||_{H^1(\mathbb{R}^n)} \cdot \exp\left(\frac{n^2 . \, ||u_0||^2_{L^\infty} . T}{2\epsilon} \right) \sqrt{t}.
\end{aligned}
\tag{2.23}
$$

□

Proof of Theorem 2.1. We are going to choose a suitable $T_0 \in (0, 1)$ so that we get a contraction in $L^2([0, T_0] \times \mathbb{R}^n)$. For each $k \in \{2, 3, 4, \dots\}$, we have

$$
\begin{aligned}
u_t^{k+1} &= \epsilon . \Delta u^{k+1} - (u^k \cdot \nabla) u^{k+1} \\
u_t^k &= \epsilon . \Delta u^k - (u^{k-1} \cdot \nabla) u^k
\end{aligned}
\tag{2.24}
$$

Subtracting the second equation from the first one and multiplying the resulting equation with $(u^{k+1} - u^k)$ followed with integration by parts, we get

$$
\begin{aligned}
&\frac{1}{2} \partial_t \left[\int_{\mathbb{R}^n} \left| u^{k+1} - u^k \right|^2 dx \right] + \epsilon . \int_{\mathbb{R}^n} \left| \nabla (u^{k+1} - u^k) \right|^2 dx \\
&= \int_{\mathbb{R}^n} \left[u^k \cdot \nabla (u^{k+1} - u^k) \right] \cdot (u^{k+1} - u^k) dx + \int_{\mathbb{R}^n} (u^k - u^{k-1}) \cdot \nabla u^k \cdot (u^{k+1} - u^k) dx \\
&\leq ||u_0||_\infty \left|\left| \nabla (u^{k+1} - u^k) \right|\right|_2 \left|\left| u^{k+1} - u^k \right|\right|_2 + \left|\left| u^k - u^{k-1} \right|\right|_2 \left|\left| \nabla u^k \right|\right|_\infty \left|\left| u^{k+1} - u^k \right|\right|_2 \\
&\leq ||u_0||_\infty \left|\left| \nabla (u^{k+1} - u^k) \right|\right|_2 \left|\left| u^{k+1} - u^k \right|\right|_2 + C(t, \epsilon, u_0) \left|\left| u^k - u^{k-1} \right|\right|_2 \left|\left| u^{k+1} - u^k \right|\right|_2 \\
&\leq \frac{\epsilon}{2} \cdot \left(\left|\left| \nabla (u^{k+1} - u^k) \right|\right|_{L^2} \right)^2 + \frac{||u_0||^2_{L^\infty}}{2\epsilon} \cdot \left(\left|\left| u^{k+1} - u^k \right|\right|_{L^2} \right)^2 \\
&\quad + C(t, \epsilon, u_0) . \left|\left| u^k - u^{k-1} \right|\right|_{L^2} \cdot \left|\left| u^{k+1} - u^k \right|\right|_{L^2}
\end{aligned}
\tag{2.25}
$$

for any $t \in (0, T_0]$. This gives

$$\begin{aligned}
&\partial_t \left[\left|\left|u^{k+1} - u^k\right|\right|_{L^2}^2 \right] + \epsilon . \int_{\mathbb{R}^n} |\nabla(u^{k+1} - u^k)|^2 dx \\
&= \partial_t \left[\int_{\mathbb{R}^n} |u^{k+1} - u^k|^2 dx \right] + \epsilon . \int_{\mathbb{R}^n} |\nabla(u^{k+1} - u^k)|^2 dx \\
&\leq \frac{||u_0||_{L^\infty}^2}{\epsilon} . \left|\left|u^{k+1} - u^k\right|\right|_{L^2}^2 + 2C(t, \epsilon, u_0) . \left|\left|u^k - u^{k-1}\right|\right|_{L^2} \left|\left|u^{k+1} - u^k\right|\right|_{L^2} .
\end{aligned} \tag{2.26}$$

In particular, we have

$$\begin{aligned}
&\partial_t \left[\left|\left|u^{k+1} - u^k\right|\right|_{L^2}^2 \right] \\
&\leq \frac{||u_0||_{L^\infty}^2}{\epsilon} . \left|\left|u^{k+1} - u^k\right|\right|_{L^2}^2 + 2C(t, \epsilon, u_0) . \left|\left|u^k - u^{k-1}\right|\right|_{L^2} \left|\left|u^{k+1} - u^k\right|\right|_{L^2} .
\end{aligned} \tag{2.27}$$

Integrating over [0,t] for each $t \in (0, T_0]$, we get

$$\begin{aligned}
&\left|\left|(u^{k+1} - u^k)(\cdot, t)\right|\right|_{L^2}^2 \\
&\leq \frac{||u_0||_{L^\infty}^2}{\epsilon} . \left(\sup_{[0,T_0]} \left|\left|(u^{k+1} - u^k)(\cdot, s)\right|\right|_{L^2} \right)^2 .T_0 \\
&+ \int_0^{T_0} 2C(s, \epsilon, u_0) . \sup_{[0,T_0]} \left|\left|u^k - u^{k-1}\right|\right|_{L^2} . \sup_{[0,T_0]} \left|\left|u^{k+1} - u^k\right|\right|_{L^2} ds \\
&\leq \frac{||u_0||_{L^\infty}^2}{\epsilon} . \left(\sup_{[0,T_0]} \left|\left|(u^{k+1} - u^k)(\cdot, s)\right|\right|_{L^2} \right)^2 .T_0 \\
&+ \alpha(\epsilon, u_0, T_0) . \sup_{[0,T_0]} \left|\left|u^k - u^{k-1}\right|\right|_{L^2} . \sup_{[0,T_0]} \left|\left|u^{k+1} - u^k\right|\right|_{L^2} ,
\end{aligned} \tag{2.28}$$

where

$$\begin{aligned}
&\alpha(\epsilon, u_0, T_0) \\
&:= \frac{4}{\sqrt{\epsilon}} \left|\left| |y| e^{-|y|^2} \right|\right|_{L^2(\mathbb{R}^n)} ||u_0||_{H^1(\mathbb{R}^n)} T_0^{\frac{1}{2}} \left[1 + ||u_0||_{L^\infty} .T_0 \exp\left(\frac{n^2 ||u_0||_{L^\infty}^2 T_0}{2\epsilon} \right) \right] .
\end{aligned} \tag{2.29}$$

Hence, we get

$$\begin{aligned}
&\left(1 - \frac{||u_0||_{L^\infty}^2}{\epsilon} T_0 \right) . \sup_{[0,T_0]} \left|\left|(u^{k+1} - u^k)(\cdot, s)\right|\right|_{L^2} \\
&\leq \alpha(\epsilon, u_0, T_0) . \sup_{[0,T_0]} \left|\left|(u^k - u^{k-1})(\cdot, s)\right|\right|_{L^2} .
\end{aligned} \tag{2.30}$$

We now take $T = T_0 \in (0, 1)$ small enough so that

$$0 \leq \frac{\alpha(\epsilon, u_0, T_0)}{1 - \frac{||u_0||_{L^\infty}^2}{\epsilon} .T_0} < \frac{1}{2} \tag{2.31}$$

Then we will get a contraction in $C([0,T_0]\times L^2(\mathbb{R}^n))$. Also, we have the following estimates for $t\geq 0$ uniformly in k:

$$\begin{aligned}&\left|\left|u^k(\cdot,t)\right|\right|_{L^\infty(\mathbb{R}^n)}\leq ||u_0||_{L^\infty(\mathbb{R}^n)}\,,\\&\left|\left|u^k(\cdot,t)\right|\right|_{H^1(\mathbb{R}^n)}\leq \exp\left(\frac{n^2||u_0||^2_{L^\infty}.t}{2\epsilon}\right).\,||u_0||_{H^1(\mathbb{R}^n)}\,.\end{aligned}\tag{2.32}$$

The estimates for the solution follows from the corresponding estimates for the approximations u^k (2.7)-(2.12) as these are uniform in k.

Existence upto T. Once we have local existence of the solution, its global existence follows by a standard argument since the limit function u satisfies all the estimates mentioned in the above lemma. Indeed, we can continue the solution with the initial data starting from T_0. This process can be continued as long as the estimates in the lemma hold for the limit function. Thus the maximal time of existence of solution cannot be less than T. □

Remark : If $u_0\in H^1(R^n)$, from the proof of theorem, we have contraction in $C([0,T_0],L^2(R^n))$. This together with estimate (2.7)-(2.9), give a weak solution because we can pass to the limit in $(u^{k-1}\cdot\nabla)u^k$ and the solution is in $C([0,T],L^2(R^n))\cap L^\infty([0,T],H^1(R^n))\cap L^\infty([0,T]\times R^n)$. Our interest is in smooth solutions since we are interested in weak asymptotic solution of (1.1), this is obtained from the estimates (2.10)-(2.12) and to get this estimates we need to add the condition $u_0\in H^s(R^n), s>n/2+1$. Indeed we use this result in the proof of next theorem with the assumption that initial data is in $H^s(R^n)$, for all s.

More precisely, we consider the equation (1.1) with initial condition (1.2). We prove the following result.

Theorem 2.3. *The family of solutions $(u^\epsilon,\rho^\epsilon)_{\epsilon>0}$ of the equation* (1.3) *with initial conditions*

$$u^\epsilon(x,0)=u_0^\epsilon(x)=(u_0*\eta_\epsilon)(x)\ ,\ \rho^\epsilon(x,0)=\rho_0^\epsilon(x)=(\rho_0*\eta_\epsilon)(x)$$

where $u_0\in L^\infty\cap L^2$, ρ_0 is a bounded Borel measure and η is the Friedrichs mollifier, is a weak asymptotic solution satisfying the inviscid equation

$$u_t+(u\cdot\nabla)u=0\ ,\ \rho_t+\nabla\cdot(\rho u)=0$$

with initial conditions

$$u(x,0)=u_0(x)\ ,\ \rho(x,0)=\rho_0(x).$$

Proof. Note that $u_0^\epsilon\in H^s(R^n)$ for all $s>0$ and by the previous theorem, $u^\epsilon\in H^s(R^n\times[0,T])$ for all $s>0$ and hence u^ϵ is a C^∞ function and the continuity equation for ρ^ϵ is a linear equation whose coefficients are infinitely differentiable functions with infinitely differentiable initial data. Hence its solution ρ^ϵ is again a C^∞ function. Indeed using the method of characteristics, the explicit solution of ρ^ϵ can be written as

$$\rho^\epsilon(x,t)=\rho_0^\epsilon(X^\epsilon(x,t,0))J^\epsilon(x,t,0),$$

where $X^\epsilon(x,t,\cdot)$ is the solution to

$$\begin{aligned}&\frac{dX^\epsilon(x,t,s)}{ds}=u^\epsilon(X^\epsilon(x,t,s),s)\ ,\ 0\leq s\leq t\\&X^\epsilon(x,t,t)=x\end{aligned}\tag{2.33}$$

and $J^\epsilon(x,t,0)$ is the Jacobian of $X^\epsilon(x,t,0)$ w.r.t. x. Since u^ϵ is a weak solution and ρ^ϵ is a classical solution, we have

$$\int_0^T \int_{\mathbb{R}^n} \left[u_t^\epsilon + (u^\epsilon \cdot \nabla)u^\epsilon\right] \phi(x,t)dxdt = \epsilon \int_0^T \int_{\mathbb{R}^n} u^\epsilon \Delta\phi dxdt$$

$$\int_0^T \int_{\mathbb{R}^n} \left[\rho_t^\epsilon + \nabla \cdot (\rho^\epsilon u^\epsilon)\right] \phi(x,t)dxdt = 0$$

for any $\phi \in C_c^\infty(\mathbb{R}^n \times (0,T))$. By maximum principle,

$$|u^\epsilon(x,t)| \le \max_{y\in\mathbb{R}^n} |u_0(y)|.$$

Therefore, (1.5) follows easily as ϵ tends to zero. □

3. The adhesion approximation - weak asymptotic solution

In this section, we construct weak asymptotic solution of the system (1.1) with the initial conditions being of plane-wave type. Let $\xi \in S^{n-1}$ (i.e., $\xi \in \mathbb{R}^n$ with $|\xi| = 1$). Consider (1.1) with the initial data for u and ρ being functions of $y = \xi \cdot x$, namely

$$u(x,0) = u_0(x) = \bar{u}_0(\xi \cdot x)\ ,\ \rho(x,0) = \rho_0(x) = \bar{\rho}_0(\xi \cdot x) \tag{3.1}$$

where $\bar{u}_0 : \mathbb{R} \to \mathbb{R}^n$ and $\bar{\rho}_0 : \mathbb{R} \to \mathbb{R}$ are given functions of a real variable. We prove the following result.

Theorem 3.1. *Assume $\bar{u}_0 \in L^\infty(\mathbb{R};\mathbb{R}^n)$ and $\bar{\rho}_0 \in L^\infty(\mathbb{R})$. Let $\bar{u}_0^\epsilon = \bar{u}_0 * \eta^\epsilon$ and $\bar{\rho}_0^\epsilon = \bar{\rho}_0 * \eta^\epsilon$, where η^ϵ is the usual Friedrichs mollifier in the space variable $y \in \mathbb{R}$. For $y \in \mathbb{R}$, $z \in \mathbb{R}$ and $t > 0$, let us define*

$$\theta(y,t,z) := \frac{(y-z)^2}{2t} + \int_0^z \xi \cdot \bar{u}_0^\epsilon(s)ds. \tag{3.2}$$

Then $(u^\epsilon, \rho^\epsilon)$ is given by

$$u^\epsilon(x,t) = \frac{\int_{-\infty}^{\infty} \bar{u}_0^\epsilon(z).\exp\left(-\frac{1}{2\epsilon}\theta(\xi\cdot x,t,z)\right)dz}{\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\epsilon}\theta(\xi\cdot x,t,z)\right)dz} \tag{3.3}$$

$$\rho^\epsilon(x,t) = \partial_y \left(\int_0^{z^\epsilon(y,t)} \bar{\rho}_0^\epsilon(s)ds \right)\Bigg|_{y=\xi\cdot x} \tag{3.4}$$

with $z^\epsilon(y,t) := X^\epsilon(y,t,0)$, where $X^\epsilon(y,t,\cdot)$ is the solution to the ODE

$$\begin{aligned} \frac{dX^\epsilon}{ds}(y,t,s) &= \xi \cdot u^\epsilon(X^\epsilon(y,t,s),s)\ ,\ s\in[0,t] \\ X^\epsilon(y,t,t) &= y \end{aligned} \tag{3.5}$$

Furthermore, $(u^\epsilon,\rho^\epsilon)_{\epsilon>0}$ given by (3.3) and (3.4) is a weak asymptotic solution to (1.1) with the prescribed initial conditions (3.1).

Proof. We look for plane wave-type solutions

$$\begin{aligned} u(x,t) &= \bar{u}(y,t) \\ \rho(x,t) &= \bar{\rho}(y,t) \end{aligned} \tag{3.6}$$

where $y = \xi \cdot x$ and we have suppressed the dependence of u^ϵ and ρ^ϵ on ϵ. Then the equation (1.1) reduces to

$$\begin{aligned} &\bar{u}_t + (\xi \cdot \bar{u})\, \bar{u}_y = \epsilon \bar{u}_{yy} \\ &\bar{\rho}_t + ((\xi \cdot \bar{u})\bar{\rho})_y = 0 \end{aligned} \tag{3.7}$$

The initial data (3.1) for $\bar{u}, \bar{\rho}$ become

$$\bar{u}(y,0) = \bar{u}_0^\epsilon(y)\ ,\ \bar{\rho}(y,0) = \bar{\rho}_0^\epsilon(y). \tag{3.8}$$

Multiply the first equation in (3.7) with ξ and denote $\sigma = \xi \cdot \bar{u}$. Then the above system is reduced to

$$\begin{aligned} &\sigma_t + \sigma\sigma_y = \epsilon\sigma_{yy}, \\ &\bar{u}_t + \sigma\bar{u}_y = \epsilon\bar{u}_{yy}, \\ &\bar{\rho}_t + (\sigma\bar{\rho})_y = 0. \end{aligned} \tag{3.9}$$

The initial conditions become

$$\sigma(y,0) = \xi \cdot \bar{u}_0^\epsilon(y)\ ,\ \bar{u}(y,0) = \bar{u}_0^\epsilon(y)\ ,\ \bar{\rho}(y,0) = \bar{\rho}_0^\epsilon(y). \tag{3.10}$$

This can be reduced to a linear equation by a generalized Hopf-Cole transformation in the following way:

If q is a solution to

$$\begin{aligned} &q_t + \frac{{q_y}^2}{2} = \epsilon q_{yy} \\ &q(y,0) = \int_0^y \xi \cdot \bar{u}_0^\epsilon(z)dz \end{aligned} \tag{3.11}$$

then the function σ defined by $\sigma := q_y$ is a solution to the first equation in (3.9) with initial condition

$$\sigma(y,0) = \xi \cdot \bar{u}_0^\epsilon(y). \tag{3.12}$$

Now set

$$r = e^{-\frac{q}{2\epsilon}}\ ,\ v = \bar{u}e^{-\frac{q}{2\epsilon}}. \tag{3.13}$$

This leads to

$$\begin{aligned} &r_t = \epsilon r_{yy}\ ,\ r(y,0) = e^{-\frac{1}{2\epsilon}\int_0^y \xi\cdot\bar{u}_0^\epsilon(z)dz}; \\ &v_t = \epsilon v_{yy}\ ,\ v(y,0) = \bar{u}_0^\epsilon(y)e^{-\frac{1}{2\epsilon}\int_0^y \xi\cdot\bar{u}_0^\epsilon(z)dz}. \end{aligned} \tag{3.14}$$

To write down the solution, we introduce the function

$$\theta(y,t,z) := \frac{(y-z)^2}{2t} + \int_0^z \xi \cdot \bar{u}_0^\epsilon(s)ds.$$

Solving the heat equation, we get

$$r(y,t) = \frac{1}{(4\pi t\epsilon)^{1/2}} \int_{-\infty}^{\infty} e^{\frac{\theta(y,t,z)}{2\epsilon}} dz,$$

$$v(y,t) = \frac{1}{(4\pi t\epsilon)^{1/2}} \int_{-\infty}^{\infty} \bar{u}_0^\epsilon(z) e^{\frac{\theta(y,t,z)}{2\epsilon}} dz.$$

Using the above transformation, we have $\bar{u} = \frac{v}{r}$. On substituting the formula for v and r, we get

$$\bar{u}(y,t) = \frac{\int_{-\infty}^{\infty} \bar{u}_0^\epsilon(z). \exp\left(-\frac{1}{2\epsilon} \cdot \theta(y,t,z)\right) dz}{\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\epsilon} \cdot \theta(y,t,z)\right) dz}. \tag{3.15}$$

Using Leibnitz formula, it easily follows that $\bar{u}$ is a C^∞ solution. Now the equation for $\bar{\rho}$ is linear with smooth coefficients. It follows that if R solves

$$\begin{aligned} & R_t + \sigma R_y = 0 \\ & R(y,0) = \int_0^y \bar{\rho}_0^\epsilon(s) ds \end{aligned} \tag{3.16}$$

then

$$\bar{\rho} = R_y \tag{3.17}$$

is a solution to the last equation in (3.9), with the initial condition for $\bar{\rho}$ as in (3.10). By the method of characteristics, $R(y,t)$ is constant along the characteristics. So

$$R(y,t) = \int_0^{z^\epsilon(y,t)} \bar{\rho}_0^\epsilon(s) ds$$

with $z^\epsilon(y,t) = X^\epsilon(y,t,0)$, where $X^\epsilon(y,t,\cdot)$ is the solution to the ODE

$$\begin{aligned} & \frac{dX^\epsilon}{ds}(y,t,s) = \sigma(X^\epsilon(y,t,s),s) \ , \ s \in [0,t] \\ & X^\epsilon(y,t,s=t) = y. \end{aligned}$$

Note that by the formula, u^ϵ is a C^∞ function and it is bounded independent of ϵ and (x,t). Therefore we have

$$\epsilon \int_{\mathbb{R}^n} \Delta u^\epsilon(x,t)\eta(x)\, dx = \epsilon \int_{\mathbb{R}^n} u^\epsilon(x,t)\Delta\eta(x)\, dx = O(1)\epsilon$$

uniformly in t for every $\eta \in C_0^\infty(\mathbb{R}^n)$, whence it follows that u^ϵ satisfies the estimate for the weak asymptotic solution.

That $(u^\epsilon, \rho^\epsilon)_{\epsilon>0}$ satisfies the required estimates on the initial conditions can be seen easily from the identities

$$\begin{aligned} & u^\epsilon(x,0) - u_0(x) = \bar{u}_0^\epsilon(\xi \cdot x) - \bar{u}_0(\xi \cdot x) \\ & \rho^\epsilon(x,0) - \rho_0(x) = \bar{\rho}_0^\epsilon(\xi \cdot x) - \bar{\rho}_0(\xi \cdot x), \end{aligned}$$

since as $\epsilon \to 0$, $\bar{u}_0^\epsilon(\xi \cdot x) \to \bar{u}_0(\xi \cdot x)$ and $\bar{\rho}_0^\epsilon(\xi \cdot x) \to \bar{\rho}_0(\xi \cdot x)$ in the sense of distributions. □

4. Vanishing viscosity limit of adhesion approximation : generalized solution

In this section, we consider (1.1) with initial data of plane wave type (3.1). We assume $\bar{u}_0 \in W^{1,\infty}(\mathbb{R})$ and $\bar{\rho}_0 \in BV(\mathbb{R})$. We prove the following result.

Theorem 4.1. *For each $t > 0$ fixed, let $z(y,t)$ be a minimizer of*

$$\theta(y,t,z) = \frac{(y-z)^2}{2t} + \int_0^z \xi \cdot \bar{u}_0(s)ds \tag{4.1}$$

which is monotone increasing in y and is unique except for a countable number of points y. Then at these points, the limit $\lim_{\epsilon\to 0}(u^\epsilon(y,t), \rho^\epsilon(y,t))$ exists and is given by

$$(u(y,t), \rho(y,t)) = \left(\bar{u}_0(z(y,t)) \ , \ \partial_y \left(\int_0^{z(y,t)} \bar{\rho}_0(z)dz \right) \right). \tag{4.2}$$

Furthermore,

$$u(x,t) = \bar{u}_0(z(y,t))|_{y=\xi\cdot x} \ , \ \rho(x,t) = \partial_y \left(\int_0^{z(y,t)} \bar{\rho}_0(z)dz \right)\Bigg|_{y=\xi\cdot x} \tag{4.3}$$

is a generalized solution in the sense of Definition 1.2.

Proof. It is standard from the from the works of Hopf [9] and Lax [14] and can be found in Evans [6] that for each $t > 0$, the minimizer $z(y,t)$ is unique except for a countable number of points y. At these points, where we have a unique minimizer, we can apply Laplace asymptotic formula as given on page 205 in Evans [6] to get the formula for the limit $\lim_{\epsilon\to 0} \bar{u}^\epsilon(y,t) = \bar{u}_0(z(y,t))$.

To prove the limit for the density ρ, observe that the function $X^\epsilon(y,t,s)$ considered in the previous theorem is a Lipschitz continuous function of s and bounded uniformly in $\epsilon > 0$ on the interval $s \in [0,T]$. So by the Arzela-Ascoli theorem, this family of curves is compact in uniform norm on $[0,T]$ and the limit point set need not be singleton. In particular, we have a subsequence such that $z^\epsilon(y,t)$ converges to $z(y,t)$ as ϵ goes to zero and hence $R^\epsilon(y,t)$ converges to

$$R(y,t) = \int_0^{z(y,t)} \bar{\rho}_0(s)ds$$

as $\epsilon \to 0$. Since $X^\epsilon(y,t,0) = z^\epsilon(y,t)$ is a monotone function in each of the variables y and t, the same is true for $z(y,t)$. So $R(y,t)$ is a BV function of y. □

5. Formula for Riemann type initial data

In this section we consider the adhesion model

$$u_t^\epsilon + (u^\epsilon \cdot \nabla)u^\epsilon = \frac{\epsilon}{2}\Delta u^\epsilon, \ \ \rho_t^\epsilon + \nabla \cdot (\rho^\epsilon u^\epsilon) = 0 \tag{5.1}$$

with Riemann type initial data, namely

$$u_0(x) = u_L \cdot \chi_{\{\xi\cdot x<0\}} + u_R \cdot \chi_{\{\xi\cdot x>0\}} \ , \ \rho_0(x) = \rho_L \cdot \chi_{\{\xi\cdot x<0\}} + \rho_R \cdot \chi_{\{\xi\cdot x>0\}}. \tag{5.2}$$

We have the following formula for the vanishing viscosity limit.

Theorem 5.1. *Let u^ϵ and ρ^ϵ be the solutions given by* (5.1) *with initial data of Riemann type* (5.2) *and*

$$u(x,t) = \lim_{\epsilon\to 0} u^\epsilon(x,t) \ , \ \rho(x,t) = \lim_{\epsilon\to 0} \rho^\epsilon(x,t).$$

Then (u,ρ) has the following form:

Case 1. $u_L = u_R = u_0$ **:**

$$(u(x,t),\rho(x,t)) = \begin{cases} (u_0,\rho_L), & \xi\cdot(x-u_0t)<0 \\ \\ (u_0,\rho_R), & \xi\cdot(x-u_0t)>0 \end{cases} \tag{5.3}$$

Case 2. $\xi\cdot u_L < \xi\cdot u_R$ **:**

$$(u(x,t),\rho(x,t)) = \begin{cases} (u_L,\rho_L) \ , \ \xi\cdot(x-u_Lt)<0 \\ \\ (x/t,0) \ , \ \xi\cdot u_Lt < \xi\cdot x < \xi\cdot u_Rt \\ \\ (u_R,\rho_R) \ , \ \xi\cdot(x-u_Rt)>0 \end{cases} \tag{5.4}$$

Case 3. $\xi\cdot u_R < \xi\cdot u_L$ **:**

$$(u(x,t),\rho(x,t)) = \begin{cases} (u_L,\rho_L dx) \ , \ \xi\cdot x < st \\ \\ \left(\frac{1}{2}(u_L+u_R), \frac{1}{2}(u_L-u_R)(\rho_L+\rho_R)t\delta_{\xi\cdot x=st}\right) \ , \ \xi\cdot x = st \\ \\ (u_R,\rho_R dx) \ , \ \xi\cdot x > st \end{cases} \tag{5.5}$$

where $s = \xi\cdot\left(\frac{u_L+u_R}{2}\right)$.

Proof. This theorem does not directly follow from the previous theorem, because of the possible discontinuity in the initial data. We first introduce the following notations:

$$y = \xi\cdot x \ , \ \sigma_L = \xi\cdot u_L \ , \ \sigma_R = \xi\cdot u_R.$$

Also, we write u_j^ϵ and ρ^ϵ for $\epsilon>0$ in terms of the standard $erfc$ given by

$$erfc(y) = \int_y^\infty e^{-z^2}dz$$

and use its asymptotic expansion as $y \to \pm\infty$. To do this, first note that from the formula (3.15) (with ϵ replaced by $\frac{\epsilon}{2}$), we have

$$u^\epsilon(y,t) = \frac{\sigma_L \cdot A_L^\epsilon(y,t) + \sigma_R \cdot A_R^\epsilon(y,t)}{A_L^\epsilon(y,t) + A_R^\epsilon(y,t)}, \tag{5.6}$$

where

$$A_L^\epsilon(y,t) = \int_0^\infty e^{-\frac{1}{\epsilon}\left[\frac{(y+z)^2}{2t} - \sigma_L z\right]} dz\ ,\ A_R^\epsilon(y,t) = \int_0^\infty e^{-\frac{1}{\epsilon}\left[\frac{(y-z)^2}{2t} + \sigma_R z\right]} dz. \tag{5.7}$$

Since

$$A_R^\epsilon(y,t) = e^{\frac{\sigma_R^2 t}{2\epsilon} - \frac{\sigma_R y}{\epsilon}} \int_0^\infty e^{-\frac{(z-y+\sigma_R t)^2}{2t\epsilon}} dz = \sqrt{2t\epsilon}\, e^{\frac{(\sigma_R)^2 t}{2\epsilon} - \frac{\sigma_R y}{\epsilon}} \int_{\frac{-y+\sigma_R t}{\sqrt{2t\epsilon}}}^\infty e^{-z^2} dz, \tag{5.8}$$

writing in terms of *erfc* gives us

$$A_R^\epsilon(y,t) = \sqrt{2t\epsilon}\, e^{\frac{\sigma_R^2 t}{2\epsilon} - \frac{\sigma_R y}{\epsilon}} erfc\left(\frac{-y + \sigma_R t}{\sqrt{2t\epsilon}}\right). \tag{5.9}$$

Similarly,

$$A_L^\epsilon(y,t) = \sqrt{2t\epsilon}\, e^{\frac{\sigma_L^2 t}{2\epsilon} - \frac{\sigma_L y}{\epsilon}} erfc\left(\frac{y - \sigma_L t}{\sqrt{2t\epsilon}}\right). \tag{5.10}$$

Using the asymptotic expansions of the *erfc*, namely

$$\begin{aligned} erfc(y) &= \left(\frac{1}{2y} - \frac{1}{4y^3} + o\left(\frac{1}{y^3}\right)\right) e^{-y^2}\ ,\ y \to \infty \\ erfc(-y) &= \sqrt{\pi} - \left(\frac{1}{2y} - \frac{1}{4y^3} + o\left(\frac{1}{y^3}\right)\right) e^{-y^2}\ ,\ y \to \infty. \end{aligned} \tag{5.11}$$

in (5.9) and (5.10), we have the following as $\nu \to 0$:

$$A_R^\epsilon(y,t) \approx \begin{cases} \dfrac{t\epsilon}{-y + \sigma_R t} e^{-\frac{y^2}{2t\epsilon}}\ ,\ -y + \sigma_R t > 0 \\[2ex] \sqrt{\dfrac{\pi t\epsilon}{2}} e^{\frac{\sigma_R^2 t}{2\epsilon} - \frac{\sigma_R y}{\epsilon}}\ ,\ -y + \sigma_R t = 0 \\[2ex] \sqrt{2\pi t\epsilon}\, e^{\frac{(\sigma_R)^2 t}{2\epsilon} - \frac{\sigma_R y}{\epsilon}} + \left(\dfrac{t\epsilon}{-y + \sigma_R t}\right) e^{-\frac{y^2}{2t\epsilon}}\ ,\ -y + \sigma_R t < 0 \end{cases} \tag{5.12}$$

$$A_L^\epsilon(y,t) \approx \begin{cases} \dfrac{t\epsilon}{y - \sigma_L t} e^{-\frac{y^2}{2t\epsilon}}\ ,\ y - \sigma_L t > 0 \\[2ex] \sqrt{\dfrac{\pi t\epsilon}{2}} e^{\frac{\sigma_L^2 t}{2\epsilon} - \frac{\sigma_L y}{\epsilon}}\ ,\ y - \sigma_L t = 0 \\[2ex] \sqrt{2\pi t\epsilon}\, e^{\frac{(\sigma_L)^2 t}{2\epsilon} - \frac{\sigma_L y}{\epsilon}} + \dfrac{t\epsilon}{y - \sigma_L t} e^{-\frac{y^2}{2t\epsilon}}\ ,\ y - \sigma_L t < 0 \end{cases} \tag{5.13}$$

First we consider the case $\sigma_L < \sigma_R$ and prove (5.4). We have to treat different regions.

Region 1 : $y \leq \sigma_L t$

Suppose $y < \sigma_L t$. Since $\sigma_L < \sigma_R$, we will have $y < \sigma_R t$ and so $-y + \sigma_R t > 0$ in this region. Using (5.12) and (5.13) in (5.6), we have

$$u^\epsilon(y,t) \approx \frac{\sigma_L . \left[\sqrt{2\pi t\epsilon}\, e^{\frac{\sigma_L^2 t}{2\epsilon} - \frac{\sigma_L y}{\epsilon}} + \frac{t\epsilon}{y-\sigma_L t} e^{-\frac{y^2}{2t\epsilon}}\right] + \sigma_R . \left[\frac{t\epsilon}{-y+\sigma_R t} e^{-\frac{y^2}{2t\epsilon}}\right]}{\sqrt{2\pi t\epsilon}\, e^{\frac{\sigma_L^2 t}{2\epsilon} - \frac{\sigma_L y}{\epsilon}} + \frac{t\epsilon}{y-\sigma_L t} e^{-\frac{y^2}{2t\epsilon}} - \frac{t\epsilon}{y-\sigma_R t} e^{-\frac{y^2}{2t\epsilon}}}, \tag{5.14}$$

so that

$$u^\epsilon(y,t) \approx \frac{\sigma_L . \sqrt{2\pi} + \left[\sigma_L . \frac{\sqrt{t\epsilon}}{y-\sigma_L t} - \sigma_R . \frac{\sqrt{t\epsilon}}{y-\sigma_R t}\right] e^{-\frac{(y-\sigma_L t)^2}{2t\epsilon}}}{\sqrt{2\pi} + \left[\frac{\sqrt{t\epsilon}}{y-\sigma_L t} - \frac{\sqrt{t\epsilon}}{y-\sigma_R t}\right] e^{-\frac{(y-\sigma_L t)^2}{2t\epsilon}}}. \tag{5.15}$$

When $y = \sigma_L t$, we have $-y + \sigma_R t > 0$. Using (5.10), (5.12) and rearranging the terms, we get

$$u^\epsilon(y,t) \approx \frac{\sigma_L . \sqrt{2\pi} - \sigma_R . \frac{\sqrt{t\epsilon}}{y-\sigma_R t}}{\sqrt{2\pi} - \frac{\sqrt{t\epsilon}}{y-\sigma_R t}}. \tag{5.16}$$

Combining all these, we have

$$\lim_{\epsilon \to 0} u^\epsilon(y,t) = \sigma_L \ , \ y \leq \sigma_L t. \tag{5.17}$$

Region 2 : $\sigma_L t < y < \sigma_R t$

In this case, we have $-y + \sigma_R t > 0$ and $y - \sigma_L t > 0$. Using (5.12) and (5.13) in (5.6), we get

$$u^\epsilon(y,t) \approx \frac{\sigma_L \cdot \left[\frac{t\epsilon}{y-\sigma_L t} e^{-\frac{y^2}{2t\epsilon}}\right] + \sigma_R \cdot \left[\frac{t\epsilon}{-y+\sigma_R t} e^{-\frac{y^2}{2t\epsilon}}\right]}{\frac{t\epsilon}{y-\sigma_L t} e^{-\frac{y^2}{2t\epsilon}} - \frac{t\epsilon}{y-\sigma_R t} e^{-\frac{y^2}{2t\epsilon}}} = \frac{\frac{\sigma_L}{y-\sigma_L t} - \frac{\sigma_R}{y-\sigma_R t}}{\frac{1}{y-\sigma_L t} - \frac{1}{y-\sigma_R t}}. \tag{5.18}$$

Simplifying this, we get

$$\lim_{\epsilon \to 0} u^\epsilon(y,t) = y/t, \ \ \sigma_L t < y < \sigma_R t. \tag{5.19}$$

Region 3 : $y \geq \sigma_R t$

First we suppose $y > \sigma_R t$. In this case, we have $-y + \sigma_R t < 0$ and $y - \sigma_L t > 0$. Using (5.12) and (5.13) in (5.6), we get

$$u^\epsilon(y,t) \approx \frac{\sigma_L . \left[\frac{\sqrt{t\epsilon}}{y-\sigma_L t} e^{-\frac{(y-\sigma_R t)^2}{2t\epsilon}}\right] + \sigma_R . \left[\sqrt{2\pi} - \frac{\sqrt{t\epsilon}}{y-\sigma_R t} e^{-\frac{(y-\sigma_R t)^2}{2t\epsilon}}\right]}{\frac{\sqrt{t\epsilon}}{y-\sigma_L t} e^{-\frac{(y-\sigma_R t)^2}{2t\epsilon}} + \sqrt{2\pi} - \frac{\sqrt{t\epsilon}}{y-\sigma_R t} e^{-\frac{(y-\sigma_R t)^2}{2t\epsilon}}}. \tag{5.20}$$

If we have $y = \sigma_R t$, then $y - \sigma_L t > 0$. Using (5.9) and (5.13) in (5.6), we get

$$u^\epsilon(y,t) \approx \frac{\sigma_L . \frac{\sqrt{t\epsilon}}{y-\sigma_L t} + \sigma_R . \sqrt{2\pi}}{\frac{\sqrt{t\epsilon}}{y-\sigma_L t} + \sqrt{2\pi}}. \tag{5.21}$$

From (5.20) and (5.21), we get

$$\lim_{\epsilon\to 0} u^\epsilon(y,t) = \sigma_R\ ,\ y \geq \sigma_R t \tag{5.22}$$

Combining (5.17), (5.19) and (5.22), we get (5.4).

Now we shall take the case $\sigma_L > \sigma_R$ and prove (5.5). Based on the way we use (5.9) and (5.13), there are three regions to consider here, namely Region 1 ($\{Y \leq \sigma_R T\}$), Region 2 ($\{Y \geq \sigma_L T\}$) and Region 3 ($\{\sigma_R T < Y < \sigma_L T\}$). The first two regions are treated exactly as Regions 1 and 3 of the rarefaction case ($\sigma_L < \sigma_R$) . We will have

$$\lim_{\epsilon\to 0} u^\epsilon(y,t) = \begin{cases} \sigma_L,\ y \leq \sigma_R t \\ \\ \sigma_R,\ y \geq \sigma_L t \end{cases} \tag{5.23}$$

So we consider the remaining region $\{\sigma_R T < Y < \sigma_L T\}$. Using (5.12) and (5.13) in the formula (5.6) and rearranging the terms, we get

$$\begin{aligned} &u^\epsilon(y,t) \\ &\approx \frac{\sigma_L\left(1+\left[\frac{\sqrt{t\epsilon}}{y-\sigma_L t}+\frac{\sqrt{t\epsilon}}{-y+\sigma_R t}\right]e^{-\frac{(y-\sigma_L t)^2}{2\epsilon t}}\right)+\sigma_R\left(e^{\frac{(\sigma_L-\sigma_R)}{\epsilon}\left(y-\frac{\sigma_L+\sigma_R}{2}t\right)}\right)}{1+\left[\frac{\sqrt{t\epsilon}}{y-\sigma_L t}+\frac{\sqrt{t\epsilon}}{-y+\sigma_R t}\right]e^{-\frac{(y-\sigma_L t)^2}{2\epsilon t}}+e^{\frac{(\sigma_L-\sigma_R)}{\epsilon}\left(y-\frac{\sigma_L+\sigma_R}{2}t\right)}}. \end{aligned} \tag{5.24}$$

Since at the present case $\sigma_L - \sigma_R > 0$, it follows that

$$\lim_{\epsilon\to 0} u^\epsilon(y,t) = \begin{cases} \sigma_L, & \sigma_R t < y < \left(\frac{\sigma_L+\sigma_R}{2}\right).t \\ \frac{\sigma_L+\sigma_R}{2}, & y = \frac{\sigma_L+\sigma_R}{2}.t \\ \sigma_R, & \left(\frac{\sigma_L+\sigma_R}{2}\right).t < y < \sigma_L t \end{cases} \tag{5.25}$$

Combining (5.23) and (5.25), we get (5.5).

Now we find the limit $\lim_{\epsilon\to 0} R^\epsilon(y,t)$. For this, we first show that u^ϵ converges uniformly on compact sets away from $y = st$ for the case $\sigma_L > \sigma_R$ and away from $y = \sigma_L t$ and $y = \sigma_R t$ for the case $\sigma_L < \sigma_R$.

Suppose $K \subseteq \mathbb{R}^2$ is a compact subset of $\{(Y,T) : Y - \sigma_L.T < 0\}$. Then we can find $A > 0$ satisfying

$$|Y| + |T| \leq A < \infty\ ,\ (Y,T) \in K$$

Fix $\eta \in \left(0, \frac{1}{4}\right)$. We have that

$$erfc(y) = \left(\frac{1}{2y} - \frac{1}{4y^3} + o\left(\frac{1}{y^3}\right)\right).\exp(-y^2)\ as\ y \to \infty$$

Therefore, we can find $M > 0$ such that

$$\left|\frac{erfc(y).\exp(y^2) - \left(\frac{1}{2y} - \frac{1}{4y^3}\right)}{\frac{1}{y^3}}\right| < \eta\ ,\ y > M$$

Set

$$w_\epsilon(y,t) = -\frac{y - \sigma_L.t}{\sqrt{2t\epsilon}}\ ,\ (y,t) \in K\ ,\ \epsilon > 0$$

Fix $(y,t) \in K$ and write $w_\epsilon = w_\epsilon(y,t)$. If

$$C := \inf\{-(Y - \sigma_L.T) : (Y,T) \in K\},$$

then

$$|w_\epsilon| \geq \frac{C}{\sqrt{2A\epsilon}} > M$$

provided $\epsilon < \epsilon_1 := \frac{1}{2A}.\left(\frac{C}{M}\right)^2$. Therefore

$$|w_\epsilon|^2.\left|w_\epsilon.\left(\sqrt{\pi}.\exp(w_\epsilon^2) - \frac{A_L^\epsilon(y,t).\exp\left(\frac{y^2}{2t\epsilon}\right)}{\sqrt{2t\epsilon}}\right) - \frac{1}{2}\right| < \eta + \frac{1}{4} < \frac{1}{2}$$

whenever $\epsilon < \epsilon_1$, and hence,

$$\left|w_\epsilon.\left(\sqrt{\pi}.\exp(w_\epsilon^2) - \frac{A_L^\epsilon(y,t).\exp\left(\frac{y^2}{2t\epsilon}\right)}{\sqrt{2t\epsilon}}\right) - \frac{1}{2}\right| < \frac{1}{2|w_\epsilon|^2}$$

$$\leq \frac{1}{2}\left(\frac{\sqrt{2A\epsilon}}{C}\right)^2 < \eta$$

provided $\epsilon < \epsilon_2 := \min\left\{\epsilon_1, \frac{C^2\eta}{A}\right\}$. Proceeding similarly, we will get

$$\left|1 - \frac{A_L^\epsilon(y,t)}{\sqrt{2\pi t\epsilon}.\exp\left(-\frac{1}{2t\epsilon}\left(2t.\sigma_L.y - \sigma_L^2.t^2\right)\right)}\right| < \frac{1}{|w_\epsilon|}.\frac{1}{\sqrt{\pi}}.\exp\left(-w_\epsilon^2\right)$$

$$\leq \frac{1}{|w_\epsilon|} \leq \frac{\sqrt{2A\epsilon}}{C} < \eta$$

provided $\epsilon < \epsilon_3 := \min\left\{\epsilon_2, \frac{C^2\eta^2}{2A}\right\}$. Since ϵ_3 is independent of our choice of (y,t), it follows that

$$\lim_{\epsilon\to 0+}\left(\frac{A_L^\epsilon(y,t)}{\sqrt{2\pi t\epsilon}.\exp\left(-\frac{1}{2t\epsilon}\left(2t.\sigma_L.y - \sigma_L^2.t^2\right)\right)}\right) = 1$$

uniformly in K. We can similarly justify uniform convergence for the remaining cases.

Next, we shall find the limit of R^ϵ as $\epsilon \to 0$. First we consider the shock case.

Case $\sigma_L > \sigma_R$:

Consider any point $(y,t) \in \mathbb{R} \times (0,\infty)$ and let us assume that $y - s.t < 0$. Our aim is to show that there exists $\epsilon_1 > 0$ satisfying

$$(X^\epsilon(y,t,\tau),\tau) \in \{(Y,T) : Y - s.T < 0\}\ ,\ \epsilon \in (0,\epsilon_1)\ ,\ 0 \leq \tau \leq t.$$

Choose $r > 0$ with $y - s.t < -r$. We choose $\delta \in (0, r/t)$ so that

$$y - \sigma_L.t < y - (s+\delta).t < y - (s-\delta).t < -r$$

(Equivalently, we choose $\delta \in (0, r/t)$ so that $\delta < \sigma_L - s$ and $\delta < -\frac{(y-s.t)+r}{t}$) .

Consider the closed trapezium Λ formed by the four lines $T = 0, Y = s.T - r, Y - \sigma_L.T = (y - \sigma_L.t) - r$ and $T = t + r'$ (where $r' > 0$ is so chosen that a trapezium can be formed with these four lines). Then u^ϵ uniformly converges to σ_L in Λ as $\epsilon \longrightarrow 0+$ and hence, we can find $\epsilon_1 > 0$ such that

$$(s <)\ \sigma_L - \delta < u^\epsilon(Y, T) < \sigma_L + \delta\ ,\ (Y, T) \in \Lambda\ ,\ 0 < \epsilon < \epsilon_1 \tag{5.26}$$

Fix $\epsilon \in (0, \epsilon_1)$. Let us suppose that for some point $t' \in [0, t)$, we have

$$X^\epsilon(y, t, t') = s.t' - r$$

Since $X^\epsilon(y, t, t) = y$, we can use the mean-value theorem on the function $X^\epsilon(y, t, \cdot)$ and get a point $p \in (t', t)$ satisfying

$$\frac{dX^\epsilon(y, t, \cdot)}{d\tau}\Big|_{\tau=p} = \frac{y - (s.t' - r)}{t - t'}.$$

This implies that

$$\begin{aligned} u^\epsilon\left(X^\epsilon(y, t, p), p\right) = \frac{y - (s.t' - r)}{t - t'} &= \frac{(y - (s - \delta).t + r) + (s - \delta).(t - t') - \delta.t'}{t - t'} \\ &< \frac{0 + (s - \delta).(t - t') + 0}{t - t'} = s - \delta < s, \end{aligned}$$

which is a contradiction to (5.26).

Next, let us suppose that

$$X^\epsilon(y, t, t'') = \sigma_L.t'' + (y - \sigma_L.t) - r$$

for some $t'' \in [0, t)$. As above, we can use the mean-value theorem to get

$$\frac{dX^\epsilon(y, t, \cdot)}{d\tau}\Big|_{\tau=q} = \frac{y - [\sigma_L.t'' + (y - \sigma_L.t) - r]}{t - t''}$$

for some point $q \in (t'', t)$. This implies that

$$u^\epsilon\left(X^\epsilon(y, t, q), q\right) = \frac{\sigma_L.(t - t') + r}{t - t''} > \sigma_L + \frac{r}{t - t''} > \sigma_L + \frac{r}{t} > \sigma_L + \delta,$$

which contradicts (5.26) again.

Let us now consider

$$t^* := \inf\{t' \in [0, t] : (X^\epsilon(y, t, \tau), \tau) \in \Lambda,\ \ t' \leq \tau \leq t\}$$

We have that $X^\epsilon(y, t, \cdot)$ is a continuous function on $[0, t]$ with $X^\epsilon(y, t, t) = y$. Therefore, the above infimum exists in $\mathbb{R}$ and $0 \leq t^* < t$. By definition, there exists a sequence (t_n) in the set $\{t' \in [0, t] : (X^\epsilon(y, t, \tau), \tau$ such that $t_n \to t^*$. Thus, each t_n satisfies

$$X^\epsilon(y, t, \tau), \tau) \in \Lambda,\ \ t_n \leq \tau \leq t$$

Consider any point $\tau \in (t^*, t]$. Then there exists $N \in \mathbb{N}$ such that

$$t^* \leq t_N < \tau \leq t$$

and hence the point $(X^\epsilon(y, t, \tau), \tau)$ belongs to Λ. Since each $X^\epsilon(y, t, t_n), t_n)$ belongs to the set Λ, we can let $n \to \infty$ and get $(X^\epsilon(y, t, t^*), t^*) \in \Lambda$. Therefore, we have

$$X^\epsilon(y, t, \tau), \tau) \in \Lambda,\ \ t^* \leq \tau \leq t$$

Now the point $(X^\epsilon(y,t,t^*),t^*)$ is either an interior point or a boundary point of Λ. It cannot be an interior point because then we could use its definition together with the continuity of $X^\epsilon(y,t,\cdot)$ on $[0,t]$ to get a contradiction. Thus $(X^\epsilon(y,t,t^*),t^*)$ has to lie on one of the four lines $T=0$, $Y=\sigma_L.T-r$, $Y-\sigma_R.T=(y-\sigma_R.t)-r$ and $T=t+r'$. Now the point cannot lie on the last line since $t^*<t$ and neither on the second and the third lines by the arguments provided above. Therefore, we must have $t^*=0$ and hence, it follows that

$$(X^\epsilon(y,t,\tau),\tau)\in\Lambda,\ \ 0\le\tau\le t$$

Next, we consider the case $y-s.t>0$. Here, our aim is to show that there exists $\epsilon_2>0$ satisfying

$$(X^\epsilon(y,t,\tau),\tau)\in\{(Y,T):Y-s.T>0\}\ ,\ \epsilon\in(0,\epsilon_2)\ ,\ 0\le\tau\le t$$

Choose $r>0$ with $r<y-s.t$. We choose $\delta\in(0,r/t)$ such that

$$r<y-(s+\delta).t<y-(s-\delta).t<y-\sigma_R.t$$

(Equivalently, we want $\delta\in(0,r/t)$ to satisfy $\delta<s-\sigma_R$ and $\delta<\dfrac{(y-s.t)-r}{t}$).

Consider the closed trapezium Λ formed by the four lines $T=0$, $Y=\sigma_R.T+r$, $Y-\sigma_L.T=(y-\sigma_L.t)+r$ and $T=t+r''$ (where $r''>0$ is so chosen that a trapezium can be formed with the above four lines). Then u^ϵ uniformly converges to σ_R in Λ as $\epsilon\longrightarrow 0+$ and therefore, we can find $\epsilon_2>0$ such that

$$\sigma_R-\delta<u^\epsilon(Y,T)<\sigma_R+\delta\ ,\ (Y,T)\in\Lambda\ ,\ 0<\epsilon<\epsilon_2 \tag{5.27}$$

Fix $\epsilon\in(0,\epsilon_2)$ and let us suppose that

$$X^\epsilon(y,t,t')=s.t'+r$$

for some $t'\in[0,t)$. Since $X^\epsilon(y,t,t)=y$, we can use the mean-value theorem on $X^\epsilon(y,t,\cdot)$ and get

$$\frac{dX^\epsilon(y,t,\cdot)}{d\tau}\Big|_{\tau=p}=\frac{y-(s.t'+r)}{t-t'}$$

for some point $p\in(t',t)$. This implies that

$$\begin{aligned}u^\epsilon\left(X^\epsilon(y,t,p),p\right)=\frac{y-(s.t'+r)}{t-t'}&=\frac{(y-(s+\delta).t-r)+(s+\delta).(t-t')+\delta.t'}{t-t'}\\&>\frac{0+(s+\delta).(t-t')+0}{t-t'}=s+\delta>\sigma_R+\delta,\end{aligned}$$

which is a contradiction to (5.27). Next, let us suppose that

$$X^\epsilon(y,t,t'')=\sigma_R.t''+(y-\sigma_R.t)+r$$

for some $t''\in[0,t)$. As above, we can use the mean-value theorem to get

$$\frac{dX^\epsilon(y,t,\cdot)}{d\tau}\Big|_{\tau=q}=\frac{y-[\sigma_R.t''+(y-\sigma_R.t)+r]}{t-t''}$$

for some point $q\in(t'',t)$. This implies that

$$u^\epsilon\left(X^\epsilon(y,t,q),q\right)=\frac{\sigma_R.(t-t'')-r}{t-t''}=\sigma_R-\frac{r}{t-t''}<\sigma_R-\frac{r}{t}<\sigma_R-\delta,$$

which again contradicts (5.27). Now we can use similar arguments as in the preceding case and conclude that

$$(X^\epsilon(y,t,\tau),\tau) \in \Lambda\ ,\ 0 \leq \tau \leq t$$

Consider any point (y_0,t_0) in the region $\{(Y,T) : Y < sT\}$. Then the characteristic curves $\{(X^\epsilon(y_0,t_0,\tau),\tau)$ $0 \leq \tau \leq t_0\}$ passing through (y_0,t_0) lie in a compact subset of the same region for sufficiently small $\epsilon > 0$ and we can pass to the limit in the equation for characteristics

$$X^\epsilon(y_0,t_0,p) = y_0 - \int_p^{t_0} u^\epsilon(X^\epsilon(y_0,t_0,\tau),\tau)d\tau$$

for any $p \in [0,t_0]$ as $\epsilon \to 0+$, getting

$$X(y_0,t_0,p) = y_0 - \int_p^{t_0} \sigma d\tau = y_0 - \sigma_L(t_0 - p),$$

so that $X(y_0,t_0,0) = y_0 - \sigma_L t_0$. Thus we have

$$\lim_{\epsilon\to 0} R^\epsilon(y_0,t_0) = \rho_L.(y_0 - \sigma_L t_0).$$

Similarly, if $y_0 > st_0$, then

$$\lim_{\epsilon\to 0} R^\epsilon(y_0,t_0) = \rho_R.(y_0 - \sigma_R t_0).$$

Now we consider the rarefaction case.

Case $\sigma_L < \sigma_R$:

Suppose $(y,t) \in \mathbb{R} \times (0,\infty)$ and let us assume that $y - \sigma_L.t < 0$. Our aim is to show that there exists $\epsilon_1 > 0$ such that

$$(X^\epsilon(y,t,\tau),\tau) \in \{(Y,T) : Y - \sigma_L.T < 0\}\ ,\ \epsilon \in (0,\epsilon_1)\ ,\ 0 \leq \tau \leq t$$

Choose $r > 0$ with $y - \sigma_L.t < -r$. We choose $\delta > 0$ such that

$$y - \sigma_R.t < y - (\sigma_L + \delta).t < y - (\sigma_L - \delta).t < -r$$

(Equivalently, $\delta \in (0, r/t)$ should satisfy $\delta < \sigma_R - \sigma_L$ and $\delta < -\dfrac{(y - \sigma_L.t) + r}{t}$).

Consider the closed trapezium Λ formed by the four lines $T = 0$, $Y = \sigma_L.T - r$, $Y - \sigma_R.T = (y - \sigma_R.t) - r$ and $T = t + r'$ (where $r' > 0$ is so chosen that a trapezium can be formed with the four lines). Then u^ϵ uniformly converges to σ_L in Λ as $\epsilon \longrightarrow 0+$ and therefore, we can find $\epsilon_1 > 0$ such that

$$\sigma_L - \delta < u^\epsilon(Y,T) < \sigma_L + \delta\ (< \sigma_R)\ ,\ (Y,T) \in \Lambda\ ,\ 0 < \epsilon < \epsilon_1 \tag{5.28}$$

Fix $\epsilon \in (0,\epsilon_1)$. Let us suppose that

$$X^\epsilon(y,t,t') = \sigma_L.t' - r$$

for some $t' \in [0,t)$. Since $X^\epsilon(y,t,t) = y$, we can use the mean-value theorem on the function $X^\epsilon(y,t,\cdot)$ and get

$$\frac{dX^\epsilon(y,t,\cdot)}{d\tau}\Big|_{\tau=p} = \frac{y-(\sigma_L.t'-r)}{t-t'}$$

for some point $p \in (t',t)$, which implies that

$$u^\epsilon\left(X^\epsilon(y,t,p),p\right) = \frac{y-(\sigma_L.t'-r)}{t-t'} = \frac{(y-(\sigma_L-\delta).t+r)+(\sigma_L-\delta).(t-t')-\delta.t'}{t-t'}$$
$$< \frac{0+(\sigma_L-\delta).(t-t')+0}{t-t'} = \sigma_L-\delta,$$

which is a contradiction to (5.28).

Next, let us assume that

$$X^\epsilon(y,t,t'') = \sigma_R.t'' + (y-\sigma_R.t) - r$$

for some $t'' \in [0,t)$. As above, we can use the mean-value theorem to get

$$\frac{dX^\epsilon(y,t,\cdot)}{d\tau}\Big|_{\tau=q} = \frac{y-[\sigma_R.t''+(y-\sigma_R.t)-r]}{t-t''}$$

for some point $q \in (t'',t)$. This implies that

$$u^\epsilon\left(X^\epsilon(y,t,q),q\right) = \frac{\sigma_R.(t-t'')+r}{t-t''} > \frac{\sigma_R.(t-t'')+0}{t-t''} = \sigma_R,$$

which is a contradiction to (5.28) again. Proceeding as before, we will again be able to conclude that

$$(X^\epsilon(y,t,\tau),\tau) \in \Lambda\ ,\ 0 \le \tau \le t$$

Next, we consider the case $y-\sigma_R.t > 0$. Here, our aim is to show that there exists $\epsilon_2 > 0$ such that

$$(X^\epsilon(y,t,\tau),\tau) \in \{(Y,T) : Y-\sigma_R.T > 0\}\ ,\ \epsilon \in (0,\epsilon_2)\ ,\ 0 \le \tau \le t$$

Choose $r > 0$ with $r < y-\sigma_R.t$. We choose $\delta > 0$ satisfying

$$r < y-(\sigma_R+\delta).t < y-(\sigma_R-\delta).t < y-\sigma_L.t$$

(Equivalently, $\delta > 0$ should satisfy $\delta < \sigma_R-\sigma_L$ and $\delta < \dfrac{(y-\sigma_R.t)-r}{t}$).

Consider the trapezium Λ formed by the four lines $T=0$, $Y=\sigma_R.T+r$, $Y-\sigma_L.T=(y-\sigma_L.t)+r$ and $T=t+r''$ (where $r''>0$ is so chosen that a trapezium can be formed with the above four lines). Then u^ϵ uniformly converges to σ_R in Λ as $\epsilon \to 0+$ and hence, we can find $\epsilon_2 > 0$ such that

$$\sigma_R-\delta < u^\epsilon(Y,T) < \sigma_R+\delta\ ,\ (Y,T)\in\Lambda,\ 0<\epsilon<\epsilon_2 \tag{5.29}$$

Fix $\epsilon \in (0,\epsilon_2)$ and let us suppose that

$$X^\epsilon(y,t,t') = \sigma_R.t' + r$$

for some $t' \in [0,t)$. Since $X^\epsilon(y,t,t)=y$, we can use the mean-value theorem on $X^\epsilon(y,t,\cdot)$ and get a point $p \in (t',t)$ satisfying

$$\frac{dX^\epsilon(y,t,\cdot)}{d\tau}\Big|_{\tau=p} = \frac{y-(\sigma_R.t'+r)}{t-t'},$$

which implies that

$$u^{\epsilon}\left(X^{\epsilon}(y,t,p),p\right) = \frac{y-(\sigma_R.t'+r)}{t-t'} = \frac{(y-(\sigma_R+\delta).t-r)+(\sigma_R+\delta).(t-t'')+\delta.t''}{t-t'}$$
$$> \frac{0+(\sigma_R+\delta).(t-t')+0}{t-t'} = \sigma_R+\delta,$$

which is a contradiction to (5.29).

Next, suppose that

$$X^{\epsilon}(y,t,t'') = \sigma_L.t'' + (y-\sigma_L.t) + r$$

for some $t'' \in [0,t)$. As above, we can use the mean-value theorem to get a point $q \in (t',t)$ satisfying

$$\left.\frac{dX^{\epsilon}(y,t,\cdot)}{d\tau}\right|_{\tau=q} = \frac{y-[\sigma_L.t''+(y-\sigma_L.t)+r]}{t-t''}$$

This implies that

$$u^{\epsilon}\left(X^{\epsilon}(y,t,q),q\right) = \frac{\sigma_L.(t-t'')-r}{t-t''} < \frac{\sigma_L.(t-t'')-0}{t-t''} = \sigma_L < \sigma_R - \delta,$$

giving us a contradiction to (5.29) again. As before, we can now show that

$$(X^{\epsilon}(y,t,\tau),\tau) \in \Lambda\ ,\ 0 \le \tau \le t$$

Now the argument for passing to the limit for R^{ϵ} is the same as in the previous case in the regions $\{(Y,T) : Y < \sigma_L T\}$ and $\{(Y,T) : Y > \sigma_R T\}$. We will get

$$\lim_{\epsilon\to 0} R^{\epsilon}(y,t) = \rho_L(y-\sigma_L t)\ ,\ y < \sigma_L t$$

and

$$\lim_{\epsilon\to 0} R^{\epsilon}(y,t) = \rho_R(y-\sigma_R t)\ ,\ y > \sigma_R t.$$

For any point (y,t) in the region $\{(Y,T) : \sigma_L T < Y < \sigma_R T\}$, we will have

$$\lim_{\epsilon\to 0} X^{\epsilon}(y,t,0) = 0,$$

because the characteristic curves do not cross. Thus

$$\lim_{\epsilon\to 0} R^{\epsilon}(y,t) = 0.$$

□

6. Modified adhesion model

In the modified adhesion model, both the equations for velocity and density have a viscosity term, namely

$$\begin{aligned} u_t + (u\cdot\nabla)u &= \frac{\epsilon}{2}\Delta u,\\ \rho_t + \nabla\cdot(\rho u) &= \frac{\epsilon}{2}\Delta\rho. \end{aligned} \tag{6.1}$$

Here we again take the initial data to be of plane wave type

$$u(x,0) = \bar{u}_0(y)\ ,\ \rho(x,0) = \bar{\rho}_0(y) \tag{6.2}$$

where $y = \xi\cdot x$. We look for solutions as functions of $y = \xi\cdot x$, i.e.,

$$u(x,t) = \bar{u}(y,t)\ ,\ \rho(x,t) = \bar{\rho}(y,t).$$

Theorem 6.1. *Assume $\bar{u}_0 \in L^\infty(\mathbb{R})$ and $\bar{\rho}_0 \in L^\infty(\mathbb{R})$. Let $\bar{u}_0^\epsilon = \bar{u}_0 * \eta^\epsilon$ and $\bar{\rho}_0^\epsilon = \bar{\rho}_0 * \eta^\epsilon$, where η^ϵ is the usual Friedrichs mollifier in the space variable $y \in \mathbb{R}$. For $y \in \mathbb{R}, z \in \mathbb{R}$ and $t > 0$, let*

$$\theta(y,t,z) := \frac{(y-z)^2}{2t} + \int_0^z \xi \cdot \bar{u}_0^\epsilon(s)ds. \tag{6.3}$$

Then $(u^\epsilon, \rho^\epsilon)$ is given by

$$u^\epsilon(x,t) = \bar{u}^\epsilon(y,t)|_{y=\xi\cdot x} = \frac{\int_{-\infty}^{\infty} \bar{u}_0^\epsilon(z).\exp\left(-\frac{\theta(\xi\cdot x,t,z)}{\epsilon}\right)dz}{\int_{-\infty}^{\infty} \exp\left(-\frac{\theta(\xi\cdot x,t,z)}{\epsilon}\right)dz}, \tag{6.4}$$

$$\rho^\epsilon(x,t) = \bar{\rho}^\epsilon(y,t)|_{y=\xi\cdot x} = \partial_y \left(\frac{\int_{-\infty}^{\infty} \left(\int_0^z \bar{\rho}_0^\epsilon(\tau)d\tau\right) \exp\left(-\frac{\theta(y,t,z)}{\epsilon}\right)dz}{\int_{-\infty}^{\infty} \exp\left(-\frac{\theta(y,t,z)}{\epsilon}\right)dz} \right)\Bigg|_{y=\xi.x}. \tag{6.5}$$

Furthermore, $(u^\epsilon, \rho^\epsilon)_{\epsilon>0}$ is a weak asymptotic solution to (1.1) *with the prescribed initial conditions* (6.2).

Vanishing viscosity limit : If we assume $\bar{u}_0 \in W^{1,\infty}(\mathbb{R})$ and $\bar{\rho}_0 \in BV(\mathbb{R})$, we have the following formula for the vanishing viscosity limit. For each fixed $y \in \mathbb{R}$ and $t > 0$, let $z(y,t)$ be a minimizer of $\min_{z\in\mathbb{R}} \theta(y,t,z)$, which is monotone increasing in y and is unique except for a countable number of points y. Then at these points, the limit $\lim_{\epsilon\to 0}(\bar{u}^\epsilon(y,t), \bar{\rho}^\epsilon(y,t))$ exists and is given by

$$(\bar{u}(y,t), \bar{\rho}(y,t)) = \left(\bar{u}_0(z(y,t)) \,,\; \partial_y \left(\int_0^{z(y,t)} \bar{\rho}_0(z)dz \right) \right). \tag{6.6}$$

Furthermore,

$$u(x,t) = \bar{u}_0(z(y,t))|_{y=\xi\cdot x} \,,\; \rho(x,t) = \partial_y \left(\int_0^{z(y,t)} \bar{\rho}_0(z)dz \right)\Bigg|_{y=\xi\cdot x} \tag{6.7}$$

is a generalized solution in the sense of Definition 1.2.

Proof. We look for solutions of the form

$$u(x,t) = \bar{u}(y,t) \,,\; \rho(x,t) = \bar{\rho}(y,t)$$

where $y = \xi \cdot x$. The equations become

$$\bar{u}_t + (\bar{u}^2/2)_y = \frac{\epsilon}{2}\bar{u}_{yy} \,,\; \bar{\rho}_t + (\bar{\rho}\bar{u})_y = \frac{\epsilon}{2}\bar{\rho}_{yy}.$$

Setting $\sigma = \xi \cdot \bar{u}$ and $R(y,t) = \int_0^y \bar{\rho}(z,t)dz$, the equations become

$$\begin{aligned} \sigma_t + \sigma\sigma_y &= \frac{\epsilon}{2}\sigma_{yy}, \\ \bar{u}_t + \sigma\bar{u}_y &= \frac{\epsilon}{2}\bar{u}_{yy}, \\ R_t + \sigma R_y &= \frac{\epsilon}{2}\sigma_{yy} \end{aligned}$$

and the initial conditions become

$$\sigma(y,0)=\sigma_0(y)=\xi\cdot\bar{u}_0^\epsilon(y)\ ,\ \bar{u}(y,0)=\bar{u}_0^\epsilon(y)\ ,\ R(y,0)=\int\limits_0^y\bar{\rho}_0^\epsilon(z)dz.$$

Now we make the generalized Hopf-Cole transformation

$$\sigma=q_y\ ,\ r=e^{-\frac{q}{\epsilon}}\ ,\ v=\bar{u}e^{-\frac{q}{\epsilon}}\ ,\ s=Re^{-\frac{q}{\epsilon}}.$$

Then r,v,s satisfy the heat equation

$$h_t=\frac{\epsilon}{2}h_{yy}$$

with initial conditions

$$r(y,0)=\exp\left(-\frac{1}{\epsilon}\int\limits_0^y\xi\cdot\bar{u}_0^\epsilon(z)dz\right),$$

$$v(y,0)=\bar{u}_0^\epsilon(y)\exp\left(-\frac{1}{\epsilon}\int\limits_0^y\xi\cdot\bar{u}_0^\epsilon(z)dz\right),$$

$$s(y,0)=\int\limits_0^y\bar{\rho}_0^\epsilon(z)\exp\left(-\frac{1}{\epsilon}\int\limits_0^z\xi\cdot\bar{u}_0^\epsilon(\tau)d\tau\right)dz.$$

Solving for r,v,s, we have

$$r(y,t)=\frac{1}{(2\pi t\epsilon)^{1/2}}\int\limits_{-\infty}^{\infty}e^{\frac{\theta(y,t,z)}{\epsilon}}dz,$$

$$v(y,t)=\frac{1}{(2\pi t\epsilon)^{1/2}}\int\limits_{-\infty}^{\infty}\bar{u}_0^\epsilon(z)e^{\frac{\theta(y,t,z)}{\epsilon}}dz,$$

$$s(y,t)=\frac{1}{(2\pi t\epsilon)^{1/2}}\int\limits_{-\infty}^{\infty}\left(\int\limits_0^z\bar{\rho}_0^\epsilon(\tau)d\tau\right)e^{\frac{\theta(y,t,z)}{\epsilon}}dz.$$

Now from the transformations, we have $\bar{u}(y,t)=\frac{v(y,t)}{r(y,t)}$ and $R(y,t)=\frac{s(y,t)}{r(y,t)}$. Substituting the formulas for v,r,s, we get the explicit formulas (6.4)-(6.5) for u^ϵ and ρ^ϵ as stated in the theorem.

To show that $(u^\epsilon,\rho^\epsilon)_{\epsilon>0}$ satisfies the equation in the weak asymptotic sense, we note that by Leibnitz rule, these functions are C^∞. Also u^ϵ is bounded and ρ^ϵ is the derivative of R^ϵ which is bounded uniformly for any $\epsilon>0$. So both $\epsilon\Delta u^\epsilon$ and $\epsilon\Delta\rho^\epsilon$ go to zero in the distributional sense in $\mathbb{R}^n$ uniformly in $t>0$.

Now we compute the vanishing viscosity limit for special initial data $\bar{u}_0\in W^{1,\infty}(\mathbb{R})$ and $\bar{\rho}_0\in BV(\mathbb{R})$. Let $(u^\epsilon,\rho^\epsilon)$ be the formula given by (6.3), (6.4) and (6.5) with $(\bar{u}_0^\epsilon,\bar{\rho}_0^\epsilon)$ replaced by $(\bar{u}_0,\bar{\rho}_0)$. As mentioned before in Theorem 4.1, for each fixed $t>0$, except for a countable number of points of y, we have a unique solution to the minimization problem $\min\limits_{z\in R}\theta(y,t,z)$. This is standard from the works of Hopf

[9] and Lax [14] and can be found in Evans [6]. At these points, we can apply the Laplace asymptotic formula as given on page 205 in Evans [6] to get the formula (6.6) for the limit

$$\lim_{\epsilon\to 0}(\bar{u}^\epsilon(y,t), R^\epsilon(y,t)) = \left(\bar{u}_0(z(y,t)), \int_0^{z(y,t)} \bar{\rho}_0(s)ds\right).$$

Now $\mathbb{R}^\epsilon$ is bounded independent of ϵ and converges pointwise almost everywhere to $\int_0^{z(y,t)} \bar{\rho}_0(s)ds$ as ϵ goes to 0. Since $\bar{\rho}^\epsilon$ is the derivative of R^ϵ, we have

$$\lim_{\epsilon\to 0}\bar{\rho}^\epsilon(y,t) = \partial_y\left(\int_0^{z(y,t)} \bar{\rho}_0(s)ds\right)$$

in the distributional sense. □

Remarks : The vanishing viscosity limit in Theorem 6.1 coincides with the limit obtained using adhesion approximation in Theorem 4.1. In the concluding remarks we will show that vanishing viscosity limits u, ρ satisfy the equation in the region of smoothness and the Rankine-Hugoniot conditions on the surface of discontinuity.

Riemann type Initial Data:

When the initial conditions are of Riemann type

$$u(x,0) = \begin{cases} u_L, & \xi\cdot x < 0 \\ u_R, & \xi\cdot x > 0 \end{cases}$$
$$\rho(x,0) = \begin{cases} \rho_L, & \xi\cdot x < 0 \\ \rho_R, & \xi\cdot x > 0 \end{cases} \tag{6.8}$$

the explicit solution for u^ϵ is given by Theorem 6.1 and the limit as ϵ goes to 0 does not follow directly from this theorem. Here we compute the limit as was the case of adhesion approximation. We need to consider the density component ρ only.

First we note that the expression for R^ϵ can be simplified as

$$R^\epsilon(y,t) = \left(\frac{\rho_L.A_L^\epsilon + \rho_R.A_R^\epsilon}{A_L^\epsilon + A_R^\epsilon} + \frac{\rho_L.\sigma_L.A_L^\epsilon + \rho_R.\sigma_R.A_R^\epsilon}{A_L^\epsilon + A_R^\epsilon} + \frac{t\epsilon.e^{-\frac{y^2}{2t\epsilon}}}{A_L^\epsilon + A_R^\epsilon}\right)(y,t). \tag{6.9}$$

Once we have this, we can follow the proof of Theorem 5.1 to get the following limit for R^ϵ:

Case 1. $u_L = u_R = u_0$:

$$\lim_{\epsilon\to 0} R^\epsilon(y,t) = \begin{cases} \rho_L(y-\sigma_0 t), & y < \sigma_0 t \\ \rho_R(y-\sigma_0 t), & y > \sigma_0 t \end{cases} \tag{6.10}$$

Case 2. $\sigma_L < \sigma_R$:

$$\lim_{\epsilon\to 0} R^\epsilon(y,t) = \begin{cases} \rho_L(y-\sigma_L t), & y<\sigma_L t \\ 0, & \sigma_L t < y < \sigma_R t \\ \rho_R(y-\sigma_R t), & y > \sigma_R t \end{cases} \tag{6.11}$$

Case 3. $\sigma_L > \sigma_R$:

$$\lim_{\epsilon\to 0} R^\epsilon(y,t) = \begin{cases} \rho_L(y-\sigma_L t), & y< st \\ \rho_R(y-\sigma_R t), & y > st \end{cases} \tag{6.12}$$

where $s = \frac{\sigma_L+\sigma_R}{2}$.

7. Concluding remarks

Suppose (u,ρ) is a generalized solution which is smooth except along a discontinuity surface $\Gamma_t = \{x : S(x,t)=0\}$ and has the form

$$\begin{aligned} u(x,t) &= u_+(x,t)H(S(x,t)>0) + u_-(x,t)H(S(x,t)<0), \\ \rho(x,t) &= \rho_+(x,t)H(S(x,t)>0) + \rho_-(x,t)H(S(x,t)<0) + \hat{e}(t)\delta_{S(x,t)=0} \end{aligned}$$

where (u_+,ρ_+) and (u_-,ρ_-) are smooth functions defined on the domains $\Omega_t^+ := \{x : S(x,t)>0\}$ and $\Omega_t^- := \{x : S(x,t)<0\}$ respectively with traces on Γ_t from either sides of $\Omega_t^\pm$, namely $(u_\pm,\rho_\pm)$. Here we assume that Γ_t is smooth so that at each point of this surface, a normal exists. Albeverio and Shelkovich [1] proved short time existence of such a generalized solution starting with initial data of the same form as the weak asymptotic solution of (1.1). Further they proved that it satisfies the equation in the classical sense in the region of smoothness and along the surface of discontinuity, the following Rankine-Hugoinot condition is satisfied.

$$\begin{aligned} &\{S_t + u_\delta.\nabla_x S\}\,|_{\Gamma_t} = 0 \\ &\frac{\delta\hat{e}}{\delta t} + div_{\Gamma_t}(\hat{e}u_\delta) = ([\rho u]-[\rho]u_\delta)\cdot\nabla_x S|_{\Gamma_t} \end{aligned} \tag{7.1}$$

where $u_\delta = \frac{u_+ + u_-}{2}\Big|_{\Gamma_t}$ and $\frac{\delta\hat{e}}{\delta t} = \frac{\partial e}{\partial t} - \frac{S_t}{S_y}.\frac{\partial e}{\partial y}$. In this section, we show that in the region of approximate continuity, the equation is satisfied in the classical sense and along the surface of discontinuity, the Rankine-Hugoniot condition is satisfied.

First recall that $y = \xi\cdot x$, $u(x,t) = \bar{u}(y,t)$, $\rho(x,t) = \bar{\rho}(y,t)$ and $\sigma(y,t) = \xi\cdot u(x,t)$. The function ϕ defined by

$$\phi(y,t) := \min_{-\infty<z<\infty}\left\{\int_0^z \xi\cdot\bar{u}_0(y)dy + \frac{(y-z)^2}{2t}\right\} \tag{7.2}$$

is a Lipschitz continuous function of y and satisfies the Hamilton-Jacobi equation

$$\phi_t + \frac{\phi_y^2}{2} = 0.$$

Taking derivative of this w.r.t y and using the relation $\sigma = \phi_y$, we get

$$\sigma_t + \left(\frac{\sigma^2}{2}\right)_y = 0.$$

We need to show that $u = \nabla_x \phi$ satisfies

$$u_t + (u \cdot \nabla_x)u = 0\ ,\ \rho_t + \nabla_x.(\rho u) = 0$$

in the region of smoothness. By Chain Rule, we have $\frac{\partial \phi}{\partial x_i} = \xi_i . \frac{\partial \phi}{\partial y}$ for each i, so that

$$\left(\phi_t + \frac{|\nabla_x \phi|^2}{2}\right)(x,t) = 0.$$

Taking gradient w.r.t. x, we verify the first equation for u. For verifying the second equation for ρ, we use the analysis of the previous sections and have

$$\sigma(y,t) = \frac{y - z(y,t)}{t},$$

where $z(y,t)$ minimizes (7.2). Then $z(y,t)$ is given by

$$z(y,t) = y - t \cdot \sigma(y,t)$$

and hence,

$$z_t(y,t) = -t \cdot \sigma_t(y,t) - \sigma(y,t),$$
$$z_y(y,t) = 1 - t \cdot \sigma_y(y,t)$$

This implies that

$$z_t + \sigma z_y = -t \cdot (\sigma_t + \sigma.\sigma_y) = 0.$$

So if (y,t) is a point of approximate continuity and $R(y',t') = \int_0^{z(y',t')} \bar{\rho}_0(s)ds$, we have

$$(R_t + \sigma R_y)(y,t) = \bar{\rho}_0(z(y,t))[z_t + \sigma z_y](y,t) = 0.$$

Taking derivative w.r.t y, we get

$$\rho_t + (\sigma\rho)_y = \rho_0(z(y,t))[z_t + \sigma z_y] = 0.$$

Now $y = \xi \cdot x$ and thus $(\rho u_i)_{x_i} = \xi_i(\bar{\rho}\sigma)_y$ for each i, so that $\nabla_x.(\rho u) = (\bar{\rho}\sigma)_y$. This gives

$$\rho_t + \nabla_x.(\rho u) = 0$$

Next, we verify the Rankine-Hugoniot Condition. Consider a solution $(\bar{\sigma}(y,t), \bar{\rho}(y,t))$ of the system

$$\bar{\sigma}_t + \left(\frac{\bar{\sigma}^2}{2}\right)_y = 0\ ,\ \bar{\rho}_t + (\bar{\sigma}\bar{\rho})_y = 0$$

with a discontinuity on the surface $S(y,t) = 0$, with $S(y,t) = y - s(t)$. In a neighborhood of this surface, we assume that (σ, ρ) has the form

$$\sigma(y,t) = \bar{\sigma}(y,t)\ ,\ \rho(y,t) = \bar{\rho}(y,t) + \hat{e}(t)\delta_{y=s(t)}$$

where $\bar{\sigma}, \bar{\rho}$ are smooth except on the surface $y = s(t)$, and $\hat{e}(t)$ is a differentiable function of t. The Rankine-Hugoniot condition for (σ, ρ) takes the form

$$\frac{ds(t)}{dt} = \frac{\sigma_+ + \sigma_-}{2}\ ,\ \frac{d\hat{e}}{dt} = [\bar{\rho}\sigma] - [\bar{\rho}]\frac{ds}{dt}. \tag{7.3}$$

We show that the distributions u, ρ satisfy the Rankine-Hugoniot conditions given by (7.1). Since $S(x,t) = S(y,t) = y - s(t)$, with $y = \xi \cdot x$, we have

$$S_t = -\frac{ds}{dt}\,, \nabla_x S = \xi.$$

The first equation of (7.1) becomes

$$(S_t + u_\delta \cdot \nabla_x S)\big|_{\Gamma_t} = -\frac{ds}{dt} + \xi \cdot \frac{u_+ + u_-}{2} = -\frac{ds}{dt} + \frac{\sigma_+ + \sigma_-}{2} = 0,$$

where we have used the first equation of (7.3). To verify the second equation in (7.1), we compute each term separately. We know from [1] that

$$\nabla_{\Gamma_t}.(\hat{e}u_\delta) = -2KG\hat{e},$$

where K is the mean curvature of the surface of discontinuity and $G = -\frac{S_t}{S_y}$. In our case $K = 0$ and thus we get

$$\nabla_{\Gamma_t}.(\hat{e}u_\delta) = 0.$$

Also an easy calculation shows that $\frac{\delta\hat{e}}{\delta t} = \frac{d\hat{e}}{dt}$ and

$$([\rho u] - [\rho]u_\delta]).\nabla_x S|_{\Gamma_t} = ([\bar{\rho}u] - [\bar{\rho}]u_\delta) \cdot \xi = [\bar{\rho}\sigma] - [\bar{\rho}]\sigma_\delta.$$

Using all these, we get

$$\frac{\delta\hat{e}}{\delta t} + \nabla_{\Gamma_t}.(\hat{e}u_\delta) - ([\rho u] - [\rho]u_\delta).\nabla_x S|_{\Gamma_t} = \frac{d\hat{e}}{dt} - \left([\bar{\rho}\sigma] - [\bar{\rho}]\frac{ds}{dt}\right) = 0,$$

where in the last equality we have used the second equation of (7.3).

For the Riemann type plane wave solution, the calculations are more simple. In this case, we have

$$\begin{aligned} S(y,t) &= y - st\,,\ y = \xi \cdot x, \\ \Gamma_t &= \{z : S(z,t) = 0\}, \\ u_\delta &= s, \\ \hat{e}(y,t) &= \frac{1}{2}(\sigma_L - \sigma_R).(\rho_L + \rho_R).t \text{ on } \Gamma_t. \end{aligned}$$

On Γ_t, we calculate

$$\begin{aligned} ([\rho u]) &= \rho_L.\sigma_L - \rho_R.\sigma_R, \\ [\rho] \cdot u_\delta &= (\rho_L - \rho_R) \cdot s \end{aligned}$$

and hence,

$$([\rho u] - [\rho]u_\delta) \cdot \nabla S(y,t) = \frac{1}{2}(\sigma_L - \sigma_R).(\rho_L + \rho_R).$$

Since $\hat{e}$ is independent of y, we have

$$div_{\Gamma_t}(\hat{e}u_\delta) = 0$$

and hence,

$$\frac{\delta\hat{e}}{\delta t} + div_{\Gamma_t}(\hat{e}u_\delta) = \frac{\delta\hat{e}}{\delta t} = \frac{1}{2}(\sigma_L - \sigma_R).(\rho_L + \rho_R).$$

We also calculate

$$S_t(y,t) = -s,$$
$$u_\delta \cdot \nabla S(y,t) = s \cdot 1 = s.$$

Therefore, we have

$$\{S_t + u_\delta \cdot \nabla S\}\,|_{\Gamma_t} = 0.$$

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