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Sharp bounds for Steklov eigenvalues on star-shaped domains

Sur les valeurs propres de Sketlov pour certains *-forme domaines

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ABSTRACT. In this article, we consider Steklov eigenvalue problem on star-shaped bounded domain Ω in hypersurface of revolution and paraboloid, $P = \{(x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2\}$. A sharp lower bound is derived for all Steklov eigenvalues of Ω in terms of the Steklov eigenvalues of the largest geodesic ball contained in Ω with the same center as Ω. This work is a generalization of a result given by Kuttler and Sigillito (SIAM Rev $10{:}368-370$, 1968) on a star-shaped bounded domain in $\mathbb{R}^2$.
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1. Introduction

Let Ω be a bounded domain in a compact connected Riemannian manifold with smooth boundary $\partial\Omega$. The Steklov eigenvalue problem is to find all real numbers μ for which there exists a nontrivial function $\varphi \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that

$$\begin{aligned} \Delta\varphi &= 0 \quad \text{in } \Omega, \\ \frac{\partial\varphi}{\partial\nu} &= \mu\varphi \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where ν is the outward unit normal to the boundary $\partial\Omega$. This problem was introduced by Steklov [13] for bounded domains in the plane in 1902. Its importance lies in the fact that the set of eigenvalues of the Steklov problem is same as the set of eigenvalues of the well-known Dirichlet-Neumann map. This map associates to each function defined on $\partial\Omega$, the normal derivative of its harmonic extension on Ω. The eigenvalues of the Steklov problem are discrete and form an increasing sequence $0 = \mu_1 < \mu_2 \leq \mu_3 \leq \cdots \nearrow \infty$. The variational characterization of μ_l, $1 \leq l < \infty$ is given by

$$\mu_l(\Omega) = \sup_{E} \inf_{0\neq\varphi\in E^\perp} \frac{\int_\Omega \|\nabla\varphi\|^2 \, dv}{\int_{\partial\Omega} \varphi^2 \, ds}, \tag{2}$$

where E is a set of $l-1$ functions $\phi_1, \phi_2, \ldots, \phi_{l-1}$ such that $\phi_i \in H^1(\Omega)$, $1 \leq i \leq l-1$ and $E^\perp = \{\varphi \in H^1(\Omega) : \int_{\partial\Omega} \varphi\phi_i ds = 0, 1 \leq i \leq l-1\}$. For background on this problem, see [9].

The first upper bound for μ_2 was given by Weinstock [15] in 1954. He proved that among all simply connected planar domains with analytic boundary of fixed perimeter, the circle maximizes μ_2. Later F. Brock [3] obtained similar sharp upper bound for μ_2 by fixing the volume of the domain. Many results in the direction of estimating Steklov Eigenvalues have appeared in [1, 2, 5, 6, 4, 8, 11, 16]. In comparison with upper bounds of Steklov eigenvalues, less lower bounds are known in the literature. The aim of this article is to find sharp lower bound for all Steklov eigenvalues on star-shaped domains.

Throughout this article, let Ω denote a star-shaped bounded domain with smooth boundary $\partial\Omega$ and center p. Define $R_m := \min\{d(p,x)|x \in \partial\Omega\}$, $R_M := \max\{d(p,x)|x \in \partial\Omega\}$ and $h_m := \min\{\langle x, \nu\rangle | x \in \partial\Omega\}$, where ν is the outward unit normal to $\partial\Omega$.

With the above notations and for $\Omega \subset \mathbb{R}^n$, Bramble and Payne [2] showed that

$$\mu_2(\Omega) \geq \frac{{R_m}^{n-1}}{{R_M}^{n+1}} h_m.$$

Equality holds when Ω is a ball.

Kuttler and Sigillito [10] derived a lower bound for all Steklov eigenvalues on a star-shaped bounded domain in $\mathbb{R}^2$ and this bound is given in the following theorem.

Theorem 1.1 ([10]). *Let $\Omega \subset \mathbb{R}^2$ be a star-shaped bounded domain, centered at the origin. Then, for $1 \leq k < \infty$,*

$$\mu_{2k+1}(\Omega) \geq \mu_{2k}(\Omega) \geq \frac{k\left[1 - 2\Big/\left(1 + \sqrt{1 + 4\,\min\left(R(\theta)/R'(\theta)\right)^2}\right)\right]}{\max\sqrt{R^2(\theta) + {R'}^2(\theta)}},$$

where $R(\theta) = \max\left\{|x| : x \in \Omega, x = |x|e^{i\theta}\right\}$ and equality holds for a disc.

A similar bound for the first nonzero Steklov eigenvalue on a star-shaped domain in $\mathbb{R}^n$ and $\mathbb{S}^n$ is obtained in [7] and [12], respectively. For any point $q \in \partial\Omega$, there exists α such that $0 \leq \theta(q) \leq \alpha < \frac{\pi}{2}$, where $\cos(\theta(q)) = \langle \nu(q), \partial_r(q)\rangle$. Define $a := \tan^2\alpha$.

Theorem 1.2 ([7]). *Let $\Omega \subset \mathbb{R}^n$. Then the first nonzero eigenvalue of the Steklov problem $\mu_2(\Omega)$ satisfies*

$$\mu_2(\Omega) \geq \frac{(R_m)^{n-2}}{(R_M)^{n-1}} \frac{\left\{2 + a - \sqrt{a^2 + 4\,a}\right\}}{2\sqrt{a+1}}.$$

Theorem 1.3 ([12]). *Let $\Omega \subset \mathbb{S}^n \backslash \{-p\}$. Then the first nonzero Steklov eigenvalue $\mu_2(\Omega)$ satisfies*

$$\mu_2(\Omega) \geq \left(\frac{R_m}{R_M}\right)\left(\frac{(2+a) - \sqrt{a^2 + 4\,a}}{2\,\sqrt{1+a}}\right)\frac{\sin^{n-1}(R_m)}{\sin^{n-1}(R_M)}\,\mu_2\left(B\left(R_m\right)\right).$$

In the present article, we generalize above results for star-shaped domain in a hypersurface of revolution and paraboloid in $\mathbb{R}^3$. The main results of this article are Theorem 2.3 and 3.1 which are proved in Section 2 and 3, respectively. The tool used to prove these results is the construction of suitable test function for the variational characterization of the corresponding eigenvalues.

2. Eigenvalues on hypersurface of revolution

Let M be a hypersurface of revolution with metric $g = dr^2 + h^2(r) g_{\mathbb{S}^{n-1}}$, where $g_{\mathbb{S}^{n-1}}$ is the usual metric on $\mathbb{S}^{n-1}$ and $r \in [0, L]$ for some $L \in \mathbb{R}^+$. Moreover, We assume that h satisfies $h(0) = 0$, $h'(0) = 1$. Let

$\Omega \subset M$ be a star-shaped bounded domain with respect to the pole p of M. Since Ω is star-shaped with respect to the point p and have smooth boundary, then for every point $q \in \partial\Omega$, there exists a unique unit vector $u \in T_pM$ and $R_u > 0$ such that $q = \exp_p(R_u\, u)$. We also assume that $R_u \leq \mathrm{inj}_p M$, injectivity radius of M at p. Observe that in geodesic polar coordinates, Ω and $\partial\Omega$ can be written as

$$\partial\Omega = \{(R_u, u) : u \in T_p\mathbb{S}^n, \|u\| = 1\} \text{ and}$$
$$\Omega \backslash \{p\} = \{(r, u) : u \in T_p\mathbb{S}^n, \|u\| = 1, 0 < r < R_u\}.$$

Define $R_m = \min R_u,\ R_M = \max R_u$.

Let ∂_r be the radial vector field starting at p, the center of Ω and ν be the unit outward normal to $\partial\Omega$. Since Ω is a star-shaped bounded domain, for any point $q \in \partial\Omega$, $\cos(\theta(q)) = \langle \nu(q), \partial_r(q)\rangle > 0$. Therefore $\theta(q) < \frac{\pi}{2}$ for all $q \in \partial\Omega$. By compactness of $\partial\Omega$, there exists a constant α such that $0 \leq \theta(q) \leq \alpha < \frac{\pi}{2}$ for all $q \in \partial\Omega$. Note that for any point $q \in \partial\Omega$, $\tan^2(\theta(q)) = \frac{\|\overline{\nabla} R_u\|^2}{h^2(R_u)}$. Additionally, assume that h also satisfies the following conditions:

(i) $\frac{h(r)}{r}$ is a decreasing function of r on $[0, R_M]$,

(ii) $h(r)$ is an increasing function of r on $[0, R_M]$.

Lemma 2.1. *Let $h(r)$ be a function defined on $[0, R]$ such that $\frac{h(r)}{r}$ is a decreasing function. Then $h(r)$ satisfies the following properties:*

(i) If $0 \leq t \leq 1$, then $h(tr) \geq th(r)$.

(ii) If $t \geq 1$, then $h(tr) \leq th(r)$.

Proof. Since $\frac{h(r)}{r}$ is a decreasing function of r,

$$\text{for } 0 \leq t \leq 1, \text{ we have } 0 \leq tr \leq r, \text{ this gives } \frac{h(r)}{r} \leq \frac{h(tr)}{tr},$$
$$\text{and for } t \geq 1, \text{ we have } tr \geq r, \text{ this gives } \frac{h(r)}{r} \geq \frac{h(tr)}{tr}.$$

This completes the proof. □

Next we give a proposition which is essential to derive the main result. A similar result is already proved in [12] but for the sake of completeness, we are providing it here.

Proposition 2.2. *Let f be a continuously differentiable real valued function defined on $\overline{\Omega}$. Then with the above notations, we have*

$$\frac{\int_\Omega \|\nabla f\|^2\, dv}{\int_{\partial\Omega} f^2\, ds} \geq \left(\frac{R_m}{R_M}\right) \left(\frac{(2+a) - \sqrt{a^2 + 4\,a}}{2}\right) \frac{h^{n-1}(R_m)}{\sec(\alpha)\, h^{n-1}(R_M)} \frac{\int_{B(R_m)} \|\nabla f\|^2\, dv}{\int_{S(R_m)} f^2\, ds}. \tag{3}$$

Proof. For a continuously differential real valued function f defined on $\overline{\Omega}$, we first find a lower bound for $\int_\Omega \|\nabla f\|^2\, dv$ and then an upper bound for $\int_{\partial\Omega} f^2\, ds$ to find a lower bound for $\frac{\int_\Omega \|\nabla f\|^2\, dv}{\int_{\partial\Omega} f^2\, ds}$.

Note that

$$\int_{\Omega} \|\nabla f\|^2 \, dv = \int_{U_p\Omega} \int_0^{R_u} \left[\left(\frac{\partial f}{\partial r} \right)^2 + \frac{1}{h^2(r)} \|\overline{\nabla} f\|^2 \right] h^{n-1}(r) \, dr \, du.$$

Let $u' = u$, $\rho = \frac{r\,R_m}{R_u}$. Then $\overline{\nabla} f = \overline{\nabla}_{u'} f - \frac{\rho}{R_{u'}} \frac{\partial f}{\partial \rho} \overline{\nabla}_{u'} R_{u'}$. By abuse of notations, we denote u' by u and $\overline{\nabla}_{u'}$ by $\overline{\nabla}$. Then the above integral can be written as

$$\int_{\Omega} \|\nabla f\|^2 \, dv = \int_{U_p\Omega} \int_0^{R_m} \left[\left(\frac{R_m}{R_u} \right)^2 \left(\frac{\partial f}{\partial \rho} \right)^2 + \frac{1}{h^2\left(\frac{\rho\,R_u}{R_m}\right)} \left\{ \|\overline{\nabla} R_u\|^2 \left(\frac{\rho}{R_u} \frac{\partial f}{\partial \rho} \right)^2 \right.\right.$$
$$\left.\left. + \|\overline{\nabla} f\|^2 - \frac{2\,\rho}{R_u} \frac{\partial f}{\partial \rho} \langle \overline{\nabla} f, \overline{\nabla} R_u \rangle \right\} \right] h^{n-1}\left(\frac{\rho\,R_u}{R_m} \right) \left(\frac{R_u}{R_m} \right) \, d\rho \, du.$$

Next we estimate $\langle \overline{\nabla} f, \overline{\nabla} R_u \rangle$. For any function β^2 on $\overline{\Omega}$, Cauchy-Schwarz inequality gives

$$-\frac{2\,\rho}{R_m\,h^2\left(\frac{\rho\,R_u}{R_m}\right)} \left(\frac{\partial f}{\partial \rho} \right) \langle \overline{\nabla} f, \overline{\nabla} R_u \rangle \geq -\frac{1}{\beta^2} \frac{\|\overline{\nabla} R_u\|^2}{R_u\,R_m} \left(\frac{\rho}{h\left(\frac{\rho\,R_u}{R_m}\right)} \right)^2 \left(\frac{\partial f}{\partial \rho} \right)^2$$
$$- \frac{\beta^2\,R_u}{R_m\,h^2\left(\frac{\rho\,R_u}{R_m}\right)} \|\overline{\nabla} f\|^2.$$

Thus

$$\int_{\Omega} \|\nabla f\|^2 \, dv \geq \int_{U_p\Omega} \int_0^{R_m} \left[\left\{ \left(\frac{R_m}{R_u} \right) - \left(\frac{1}{\beta^2} - 1 \right) \frac{\|\overline{\nabla} R_u\|^2}{R\,R_m} \left(\frac{\rho}{h\left(\frac{\rho\,R_u}{R_m}\right)} \right)^2 \right\} \left(\frac{\partial f}{\partial \rho} \right)^2 \right.$$
$$\left. + \frac{R_u\,(1-\beta^2)}{R_m\,h^2\left(\frac{\rho\,R_u}{R_m}\right)} \|\overline{\nabla} f\|^2 \right] h^{n-1}\left(\frac{\rho\,R_u}{R_m} \right) \, d\rho \, du. \tag{4}$$

Note that $0 \leq \frac{\rho}{R_m} \leq 1 \leq \frac{R_u}{R_m}$ and $0 \leq \rho \leq \frac{\rho\,R_u}{R_m} \leq R_u$. Using Lemma 2.1 and the fact that h is an increasing function, it follows that

$$\begin{aligned} \frac{\rho}{R_m} h(R_u) \leq h\left(\frac{\rho\,R_u}{R_m} \right) &\leq \frac{R_u}{R_m} h(\rho), \\ 0 \leq h^{n-1}(\rho) &\leq h^{n-1}\left(\frac{\rho\,R_u}{R_m} \right). \end{aligned} \tag{5}$$

We assume $\beta^2 < 1$ and by substituting above inequalities in (4), we get

$$\int_{\Omega} \|\nabla f\|^2 \, dv \geq \int_{U_p\Omega} \int_0^{R_m} \left[\left\{ \left(\frac{R_m}{R_u} \right) - \left(\frac{1}{\beta^2} - 1 \right) \frac{\|\overline{\nabla} R_u\|^2}{R_u\,R_m} \left(\frac{R_m}{h(R_u)} \right)^2 \right\} \left(\frac{\partial f}{\partial \rho} \right)^2 \right.$$
$$\left. + \frac{R_u\,(1-\beta^2)}{R_m} \left(\frac{R_m}{R_u\,h(\rho)} \right)^2 \|\overline{\nabla} f\|^2 \right] h^{n-1}(\rho) \, d\rho \, du$$
$$\geq \left(\frac{R_m}{R_M} \right) \int_{U_p\Omega} \int_0^{R_m} \left[\left\{ 1 - \left(\frac{1}{\beta^2} - 1 \right) a \right\} \left(\frac{\partial f}{\partial \rho} \right)^2 + \frac{(1-\beta^2)}{h^2(\rho)} \|\overline{\nabla} f\|^2 \right] h^{n-1}(\rho) \, d\rho \, du.$$

By solving the equation $1-\left(\frac{1}{\beta^2}-1\right)a = 1-\beta^2$ for β^2, we see that

$$1-\left(\frac{1}{\beta^2}-1\right)a = 1-\beta^2 = \frac{(2+a)-\sqrt{a^2+4\,a}}{2} > 0.$$

From this it follows that

$$\begin{aligned}\int\limits_{\Omega} \|\nabla f\|^2\,dv \geq{}& \left(\frac{R_m}{R_M}\right)\left(\frac{(2+a)-\sqrt{a^2+4\,a}}{2}\right)\int\limits_{U_p\Omega}\int\limits_{0}^{R_m}\left[\left(\frac{\partial f}{\partial \rho}\right)^2\right.\\ &\left.+\frac{1}{h^2(\rho)}\|\overline{\nabla} f\|^2\right] h^{n-1}(\rho)\,d\rho\,du\\ ={}& \left(\frac{R_m}{R_M}\right)\left(\frac{(2+a)-\sqrt{a^2+4\,a}}{2}\right)\int\limits_{B(R_m)}\|\nabla f\|^2\,dv. \end{aligned}\tag{6}$$

Next we find an upper bound for $\int_{\partial\Omega} f^2\,ds$.

Recall that the Riemannian volume element on $\partial\Omega$, denoted by ds, is given by $ds = \sec(\theta)\,h^{n-1}(R_u)\,du$ (see [14]). Then

$$\int\limits_{\partial\Omega} f^2\,ds = \int\limits_{U_p\Omega} f^2\,\sec(\theta)\,h^{n-1}(R_u)\,du.$$

By using the fact that $h^{n-1}(R_m) \leq h^{n-1}(R_u) \leq h^{n-1}(R_M)$ and substituting $r = \frac{\rho\,R_u}{R_m}$, this integral becomes

$$\int\limits_{\partial\Omega} f^2\,ds \leq \frac{\sec(\alpha)\,h^{n-1}(R_M)}{h^{n-1}(R_m)}\int\limits_{S(R_m)} f^2\,ds. \tag{7}$$

Combining inequalities (6) and (7), we get the desired inequality. ☐

Now we state and prove the main result of this section.

Theorem 2.3. *Let $\Omega\subset M$, ν, α, R_m and R_M be as the above. Let $a=\tan^2(\alpha)$. Then $\mu_l(\Omega)$, $1\leq l<\infty$ satisfies the following inequality.*

$$\mu_l(\Omega) \geq \left(\frac{R_m}{R_M}\right)\left(\frac{(2+a)-\sqrt{a^2+4\,a}}{2\sqrt{1+a}}\right)\frac{h^{n-1}(R_m)}{h^{n-1}(R_M)}\,\mu_l\left(B\left(R_m\right)\right), \tag{8}$$

where $B(R_m)\subset M$ is the geodesic ball of radius R_m centered at p. Further, if Ω is a geodesic ball, then equality occurs. Conversely, if equality holds for some l, then Ω is a geodesic ball of radius R_m.

Proof. We construct some specific test functions for the variational characterization of $\mu_l(\Omega)$.

We choose the functions ϕ_i, $1\leq i<\infty$ such that $\phi_i h^{n-2}(R_u)\sqrt{h^2(R_u)+\|\overline{\nabla}R_u\|^2}$ is the ith Steklov eigenfunction of $B(R_m)$. Let φ be an arbitrary function which satisfies

$$\int\limits_{\partial B(R_m)} \varphi\phi_i h^{n-2}(R_u)\sqrt{h^2(R_u)+\|\overline{\nabla}R_u\|^2}\,ds = 0.$$

Note that

$$\int\limits_{\partial\Omega} \varphi\phi_i ds = \int\limits_{U_p\Omega} \varphi\phi_i \, \frac{\sqrt{h^2(R_u) + \|\overline{\nabla} R_u\|^2}}{h(R_u)} \, h^{n-1}\left(R_u\right) \, du.$$

By substituting $r = \frac{\rho R_u}{R_m}$, the above integral becomes

$$\int\limits_{\partial\Omega} \varphi\phi_i ds = \frac{1}{h^{n-1}\left(R_m\right)} \int\limits_{\partial B(R_m)} \varphi\phi_i \sqrt{h^2(R_u) + \|\overline{\nabla} R_u\|^2} \, h^{n-2}\left(R_u\right) \, ds = 0.$$

Fix $E = \{\phi_1, \phi_2, \ldots, \phi_{l-1}\}$ in (2). Then it follows from (2) that

$$\begin{aligned} \mu_l(\Omega) &\geq \inf_{\substack{\varphi\neq 0 \\ \int_{\partial\Omega} \varphi\phi_i ds = 0, \\ 1\leq i\leq l-1}} \frac{\int_\Omega \|\nabla\varphi\|^2 \, dv}{\int_{\partial\Omega} \varphi^2 \, ds} \\ &\geq \left(\frac{R_m}{R_M}\right) \left(\frac{(2+a) - \sqrt{a^2 + 4\,a}}{2\sqrt{1+a}}\right) \frac{h^{n-1}\left(R_m\right)}{h^{n-1}\left(R_M\right)} \\ &\quad \left(\inf_{\substack{\varphi\neq 0 \\ \int_{\partial B(R_m)} \varphi\phi_i h^{n-2}(R_u)\sqrt{h^2(R_u) + \|\overline{\nabla} R_u\|^2} ds = 0, \\ 1\leq i\leq l-1}} \frac{\int_{B(R_m)} \|\nabla\varphi\|^2 \, dv}{\int_{\partial B(R_m)} \varphi^2 \, ds} \right). \end{aligned} \tag{9}$$

Since $\phi_i h^{n-2}(R_u)\sqrt{h^2(R_u) + \|\overline{\nabla} R_u\|^2}$ is the ith Steklov eigenfunction of $B(R_m)$, we have

$$\inf_{\substack{0\neq\varphi \\ \int_{\partial B(R_m)} \varphi\phi_i h^{n-2}(R_u)\sqrt{h^2(R_u) + \|\overline{\nabla} R_u\|^2} ds = 0, \\ 1\leq i\leq l-1}} \frac{\int_{B(R_m)} \|\nabla\varphi\|^2 \, dv}{\int_{\partial B(R_m)} \varphi^2 \, ds} = \mu_l\left(B\left(R_m\right)\right).$$

By substituting the above value in (9), we get (8). If Ω is a geodesic ball, then $R_m = R_M$ and $a = 0$, hence equality holds in (8). Next if equality holds in (8) for some l, then equality holds in (5) and in Cauchy-Schwarz inequality. This gives $R_u = R_m$. Hence Ω is a geodesic ball. This completes proof of the theorem. $\square$

Remark 2.4. In [7] and [12], authors obtained a lower bound for the first nonzero Steklov eigenvalue on a star-shaped bounded domain in $\mathbb{R}^n$ and $\mathbb{S}^n$, respectively. Using the above idea, a similar bound can be obtained for all nonzero Steklov eigenvalues on a star-shaped bounded domain in $\mathbb{R}^n$ and $\mathbb{S}^n$.

3. Eigenvalues on a paraboloid in $\mathbb{R}^3$

In this section, we state and prove the result for a star-shaped bounded domain in a paraboloid $P = \left\{(x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2\right\}$. We first fix some notations which will be used to state the main result of this section.

We use the parametrization $\left(r\cos\theta, r\sin\theta, r^2\right)$ for paraboloid P, where $\theta \in [0, 2\pi)$ and $r \geq 0$. Then the line element ds^2 and the area element dA on P is given by $ds^2 = \left(1 + 4r^2\right) dr^2 + r^2 \, d\theta^2$ and

$dA = r\sqrt{1+4r^2}\,dr\,d\theta$, respectively. Let $\Omega \subset P$. Then there exists a function $R : [0, 2\pi) \longrightarrow \mathbb{R}^+$ such that

$$\partial\Omega = \{(R(\theta), \theta) : \theta \in [0, 2\pi)\} \text{ and}$$
$$\Omega \backslash \{0\} = \{(r, \theta) : \theta \in [0, 2\pi), 0 < r < R(\theta)\}.$$

Hereafter, we denote $R(\theta)$ by R_θ. Let $R_m = \min\{R_\theta : \theta \in [0, 2\pi)\}$ and $R_M = \max\{R_\theta : \theta \in [0, 2\pi)\}$. Define $B(R_m) = \{(R_m, \theta) : \theta \in [0, 2\pi)\}$. Let ν be the outward unit normal to $\partial\Omega$. Let $a = \max\left\{\left(1+4R_\theta^2\right)\left(\frac{R'_\theta}{R_\theta}\right)^2 : \theta \in [0, 2\pi)\right\}$. With these notations, we prove the following theorem.

Theorem 3.1. *Let Ω, ν, a, R_m and R_M be as the above. Then $\mu_l(\Omega)$, $1 \leq l < \infty$ satisfies*

$$\mu_l(\Omega) \geq \left(\frac{R_m}{R_M}\right)^3 \left(\frac{(2+a) - \sqrt{a^2+4\,a}}{2\sqrt{1+a}}\right) \mu_l\left(B\left(R_m\right)\right). \tag{10}$$

Furthermore, if equality holds for some l then Ω is a geodesic ball of radius R_m and if Ω is a geodesic ball then equality holds in (10).

Proof. Let f be a continuously differentiable real valued function defined on $\overline{\Omega}$. We first obtain a lower bound for $\int_\Omega \|\nabla f\|^2\,dA$.

$$\begin{aligned}
\int\limits_\Omega \|\nabla f\|^2\,dA &= \int\limits_0^{2\pi}\int\limits_0^{R_\theta}\left[\frac{1}{1+4r^2}\left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2\right] r\sqrt{1+4r^2}\,dr\,d\theta \\
&= \int\limits_0^{2\pi}\int\limits_0^{R_\theta}\left[\frac{r}{\sqrt{1+4r^2}}\left(\frac{\partial f}{\partial r}\right)^2 + \frac{\sqrt{1+4r^2}}{r}\left(\frac{\partial f}{\partial \theta}\right)^2\right] dr\,d\theta
\end{aligned}$$

Let $\phi = \theta$, $\rho = \frac{r\,R_m}{R_\theta}$. Since $\rho = \frac{r\,R_m}{R_\theta} \leq r$, we have $\sqrt{1+4r^2} \geq \sqrt{1+4\rho^2}$ and $\frac{r}{\sqrt{1+4r^2}} \geq \frac{\rho}{\sqrt{1+4\rho^2}}$. Thus the above integral can be written as

$$\begin{aligned}
\int\limits_\Omega \|\nabla f\|^2\,dA &\geq \int\limits_0^{2\pi}\int\limits_0^{R_m}\left[\frac{\rho}{\sqrt{1+4\rho^2}}\left(\frac{R_m}{R_\phi}\frac{\partial f}{\partial \rho}\right)^2 + \frac{R_m\sqrt{1+4\rho^2}}{\rho R_\phi}\left(\frac{\partial f}{\partial \phi} - \frac{\rho R'_\phi}{R_\phi}\frac{\partial f}{\partial \rho}\right)^2\right] \\
&\qquad \frac{R_\phi}{R_m}\,d\rho\,d\phi \\
&= \int\limits_0^{2\pi}\int\limits_0^{R_m}\left[\frac{\rho}{\sqrt{1+4\rho^2}}\left(\frac{\partial f}{\partial \rho}\right)^2 + \frac{R_\phi\sqrt{1+4\rho^2}}{\rho R_m}\left(\frac{\partial f}{\partial \phi} - \frac{\rho R'_\phi}{R_\phi}\frac{\partial f}{\partial \rho}\right)^2\right] \\
&\qquad \frac{R_m}{R_\phi}d\rho\,d\phi \\
&\geq \int\limits_0^{2\pi}\int\limits_0^{R_m}\left[\frac{\rho}{\sqrt{1+4\rho^2}}\left(\frac{\partial f}{\partial \rho}\right)^2 + \frac{\sqrt{1+4\rho^2}}{\rho}\left\{\left(\frac{\partial f}{\partial \phi}\right)^2 + \left(\frac{\rho R'_\phi}{R_\phi}\frac{\partial f}{\partial \rho}\right)^2\right.\right. \\
&\qquad \left.\left. -2\frac{\rho R'_\phi}{R_\phi}\frac{\partial f}{\partial \rho}\frac{\partial f}{\partial \phi}\right\}\right]\frac{R_m}{R_\phi}d\rho\,d\phi.
\end{aligned}$$

For any function β^2 on $\overline{\Omega}$, Cauchy-Schwarz inequality gives

$$-2\frac{\rho R'_\phi}{R_\phi}\frac{\partial f}{\partial \rho}\frac{\partial f}{\partial \phi} \geq -\frac{1}{\beta^2}\left(\frac{\rho R'_\phi}{R_\phi}\right)^2\left(\frac{\partial f}{\partial \rho}\right)^2 - \beta^2\left(\frac{\partial f}{\partial \phi}\right)^2.$$

As a consequence, we have

$$\begin{aligned}\int\limits_{\Omega}\|\nabla f\|^2\,dA &\geq \int\limits_0^{2\pi}\int\limits_0^{R_m}\left[\frac{\rho}{\sqrt{1+4\rho^2}}\left(\frac{\partial f}{\partial \rho}\right)^2 + \frac{\sqrt{1+4\rho^2}}{\rho}\left\{\left(1-\beta^2\right)\left(\frac{\partial f}{\partial \phi}\right)^2\right.\right.\\ &\qquad \left.\left. -\left(\frac{1}{\beta^2}-1\right)\left(\frac{\rho R'_\phi}{R_\phi}\frac{\partial f}{\partial \rho}\right)^2\right\}\right]\frac{R_m}{R_\phi}d\rho\,d\phi\\ &= \int\limits_0^{2\pi}\int\limits_0^{R_m}\left[\left\{1-\left(1+4\rho^2\right)\left(\frac{1}{\beta^2}-1\right)\left(\frac{R'_\phi}{R_\phi}\right)^2\right\}\frac{\rho}{\sqrt{1+4\rho^2}}\left(\frac{\partial f}{\partial \rho}\right)^2\right.\\ &\qquad \left. +\left(1-\beta^2\right)\frac{\sqrt{1+4\rho^2}}{\rho}\left(\frac{\partial f}{\partial \phi}\right)^2\right]\frac{R_m}{R_\phi}d\rho\,d\phi.\end{aligned}$$

Note that $\left(1+4\rho^2\right)\left(\frac{R'_\phi}{R_\phi}\right)^2 \leq \left(1+4R_\phi^2\right)\left(\frac{R'_\phi}{R_\phi}\right)^2 \leq a$ and $\frac{R_m}{R_\phi} \geq \frac{R_m}{R_M}$. Let's assume $\beta^2 < 1$, then the above integral becomes

$$\begin{aligned}\int\limits_{\Omega}\|\nabla f\|^2\,dA &\geq \left(\frac{R_m}{R_M}\right)\int\limits_0^{2\pi}\int\limits_0^{R_m}\left[\left\{1-\left(\frac{1}{\beta^2}-1\right)a\right\}\frac{\rho}{\sqrt{1+4\rho^2}}\left(\frac{\partial f}{\partial \rho}\right)^2\right.\\ &\qquad \left. +\left(1-\beta^2\right)\frac{\sqrt{1+4\rho^2}}{\rho}\left(\frac{\partial f}{\partial \phi}\right)^2\right]d\rho\,d\phi.\end{aligned}$$

Solving the equation $1-\left(\frac{1}{\beta^2}-1\right)a = 1-\beta^2$ for β^2, we obtain

$$1-\left(\frac{1}{\beta^2}-1\right)a = 1-\beta^2 = \frac{(2+a)-\sqrt{a^2+4\,a}}{2} > 0.$$

By substituting these values, we have

$$\begin{aligned}\int\limits_{\Omega}\|\nabla f\|^2\,dA &\geq \left(\frac{R_m}{R_M}\right)\frac{(2+a)-\sqrt{a^2+4\,a}}{2}\int\limits_0^{2\pi}\int\limits_0^{R_m}\left[\frac{\rho}{\sqrt{1+4\rho^2}}\left(\frac{\partial f}{\partial \rho}\right)^2\right.\\ &\qquad \left. +\frac{\sqrt{1+4\rho^2}}{\rho}\left(\frac{\partial f}{\partial \phi}\right)^2\right]d\rho\,d\phi\\ &= \left(\frac{R_m}{R_M}\right)\frac{(2+a)-\sqrt{a^2+4\,a}}{2}\int\limits_0^{2\pi}\int\limits_0^{R_m}\left[\frac{1}{1+4\rho^2}\left(\frac{\partial f}{\partial \rho}\right)^2+\frac{1}{\rho^2}\right.\\ &\qquad \left.\left(\frac{\partial f}{\partial \phi}\right)^2\right]\rho\sqrt{1+4\rho^2}\,d\rho\,d\phi\\ &= \left(\frac{R_m}{R_M}\right)\frac{(2+a)-\sqrt{a^2+4\,a}}{2}\int\limits_{B(R_m)}\|\nabla f\|^2\,dA. \qquad (11)\end{aligned}$$

Now we give a lower bound for $\int_{\partial\Omega} f^2\,ds$.

$$\int\limits_{\partial\Omega} f^2\,ds = \int\limits_0^{2\pi} f^2 \sqrt{1 + (1 + 4R_\theta^2)\left(\frac{R'_\theta}{R_\theta}\right)^2}\, R_\theta\, d\theta$$
$$\leq \sqrt{1+a} \int\limits_0^{2\pi} f^2\, R_\theta\, d\theta.$$

By substituting $\phi = \theta$, $\rho = \frac{r\,R_m}{R_\theta}$ and using the fact that $R_\theta \leq R_M$, we get

$$\int\limits_{\partial\Omega} f^2\,ds \leq \frac{R_M\sqrt{1+a}}{R_m} \int\limits_0^{2\pi} f^2\, R_m\, d\phi = \frac{R_M\sqrt{1+a}}{R_m} \int\limits_{\partial B(R_m)} f^2\,ds. \tag{12}$$

Hence for a continuously differentiable real valued function f defined on $\overline{\Omega}$, it follows from (11) and (12) that

$$\frac{\int_\Omega \|\nabla f\|^2\,dA}{\int_{\partial\Omega} f^2\,ds} \geq \left(\frac{R_m}{R_M}\right)^2 \frac{(2+a) - \sqrt{a^2+4\,a}}{2\sqrt{1+a}} \frac{\int_{B(R_m)} \|\nabla f\|^2\,dA}{\int_{\partial B(R_m)} f^2\,ds}.$$

Now using the same argument as in Theorem 2.3, we get the desired result. $\square$

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