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On some nonlocal elliptic systems with multiple parameters

Sur certains systèmes elliptiques non locaux à paramètres multiples

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ABSTRACT. In this paper, using sub-supersolution argument, we prove some existence results on positive solution for a class of nonlocal elliptic systems with multiple parameters in bounded domains.
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1. Introduction

In this paper, we are interested in a class of nonlinear elliptic systems of the form

$$\begin{cases} -M_1\left(\int_\Omega |\nabla u|^p\,dx\right)\Delta_p u = \lambda_1 a(x)f(v)+\mu_1 c(x)h(u) & \text{in } \Omega, \\ -M_2\left(\int_\Omega |\nabla v|^p\,dx\right)\Delta_p v = \lambda_2 b(x)g(u)+\mu_2 d(x)\tau(v) & \text{in } \Omega, \\ u=v=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $M_i : \mathbb{R}^+ \to \mathbb{R}^+$, $i = 1, 2$, are continuous functions, $1 < p < N$, λ_1, λ_2, μ_1 and μ_2 are positive parameters and a, b, c, d are C^1 sign-changing weight functions, that maybe negative near the boundary. f, g, h, τ are C^1 non-decreasing non-negative functions on $(0, \infty)$.

Since the first equation in (1.1) contains an integral over Ω, it is no longer a pointwise identity; therefore it is often called nonlocal problem. This problem models several physical and biological systems, where u describes a process which depends on the average of itself, such as the population density, see [5]. Moreover, problem (1.1) is related to the stationary version of the Kirchhoff equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

presented by Kirchhoff in 1883, see [11]. This equation is an extension of the classical d'Alembert's wave equation by considering the effects of the changes in the length of the string during the vibrations. The parameters in (1.2) have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension.

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In recent years, problems involving Kirchhoff type operators have been studied in many papers, we refer to [3, 4, 6, 7, 12, 14, 15], in which the authors have used different methods to get the existence of solutions for (1.2). In the papers [13, 16], Zhang and Perera studied the existence of nontrivial solutions and sign-changing solutions for (1.1).

In this paper, we are interested in finding positive solutions for system (1.1) via sub-supersolution method. Our paper is motivated by the recent results in [1, 2, 3, 4, 10]. In the paper [4], the authors studied the existence of a positive solution for the nonlocal problem of the form

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2\,dx\right)\Delta u = |u|^{p-1}u + \lambda f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$ and $p > 1$, i.e. the nonlinear term is superlinear at infinity, f is a sign-changing function. Using the sub-supersolution method combining a comparison principle introduced in [3], the authors established the existence of a positive solution for (1.3) when the parameter $\lambda > 0$ is small enough. It was also prove in [4] that problem (1.3) has no positive when λ large enough. In [2] Afrouzi *et al.* investigated the existence and nonexistence of positive solution to the following elliptic system:

$$\begin{cases} -M_1\left(\int_\Omega |\nabla u|^2\,dx\right)\Delta_u = \lambda f(v) & \text{in } \Omega, \\ -M_2\left(\int_\Omega |\nabla v|^2\,dx\right)\Delta_v = \lambda g(u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

In the present paper, we consider system (1.1) in the case when the nonlinearities are sublinear at infinity. We are inspired by the ideas in the interesting paper [10], in which the authors considered (1.1) in the case $M_1(t) = M_2(t) \equiv 1$. Here we focus on further extending the study in [2] to the system (1.1) which features multiple parameters, weight functions and stronger coupling. More precisely, under suitable conditions on f, g, h, τ we shall show that system (1.1) has a positive solution for λ_i, μ_i, $i = 1, 2$ sufficiently large.

Proposition 1.1. *Assume that $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and increasing function satisfying*

$$M(t) \geq m_0 \quad \textit{for all } t \in \mathbb{R}^+, \tag{1.5}$$

where m_0 is a positive constant. Assume that u, v are two non-negative functions such that

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^p\,dx\right)\Delta_p u \geq -M\left(\int_\Omega |\nabla v|^p\,dx\right)\Delta_p v & \textit{in } \Omega, \\ u = v = 0 & \textit{on } \partial\Omega. \end{cases} \tag{1.6}$$

Then $u \geq v$ a.e. in Ω.

Proof. Our proof is based on the arguments presented in [8, 9]. Define the functional $\Phi : W_0^{1,p}(\Omega) \to \mathbb{R}$ by the formula

$$\Phi(u) := \frac{1}{p}\widehat{M}\left(\int_\Omega |\nabla u|^p\,dx\right), \qquad u \in W_0^{1,p}(\Omega).$$

It is obvious that the functional Φ is a continuously Gâteaux differentiable whose Gâteaux derivative at the point $u \in W_0^{1,p}(\Omega)$ is the functional $\Phi' \in W_0^{-1,p}(\Omega)$, given by

$$\Phi'(u)(\varphi) = M\left(\int_\Omega |\nabla u|^p\,dx\right)\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\,dx, \qquad \varphi \in W_0^{1,p}(\Omega).$$

It is obvious that Φ' is continuous and bounded since the function M is continuous. We will show that Φ' is strictly monotone in $W_0^{1,p}(\Omega)$. Indeed, for any $u, v \in W_0^{1,p}(\Omega)$, $u \neq v$, without loss of generality, we may assume that

$$\int_\Omega |\nabla u|^p \, dx \geq \int_\Omega |\nabla v|^p \, dx$$

(otherwise, changing the role of u and v in the following proof). Therefore, we have

$$M\left(\int_\Omega |\nabla u|^p \, dx\right) \geq M\left(\int_\Omega |\nabla v|^p \, dx\right), \tag{1.7}$$

since $M(t)$ is a monotone function. Using Cauchy's inequality, we have

$$\nabla u \cdot \nabla v \leq |\nabla u||\nabla v| \leq \frac{1}{2}(|\nabla u|^2 + |\nabla v|^2). \tag{1.8}$$

Using (1.8) we get

$$\int_\Omega |\nabla u|^p \, dx - \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx \geq \frac{1}{2}\int_\Omega |\nabla u|^{p-2}(|\nabla u|^2 - |\nabla v|^2) \, dx, \tag{1.9}$$

and

$$\int_\Omega |\nabla v|^p \, dx - \int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla u \, dx \geq \frac{1}{2}\int_\Omega |\nabla v|^{p-2}(|\nabla v|^2 - |\nabla u|^2) \, dx. \tag{1.10}$$

If $|\nabla u| \geq |\nabla v|$, using (1.7)-(1.10) we have

$$\begin{aligned}
I_1 &:= \Phi'(u)(u) - \Phi'(u)(v) - \Phi'(v)(u) + \Phi'(v)(v) \\
&= M\left(\int_\Omega |\nabla u|^p \, dx\right)\left(\int_\Omega |\nabla u|^p \, dx - \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx\right) \\
&\quad - M\left(\int_\Omega |\nabla v|^p \, dx\right)\left(\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla u \, dx - \int_\Omega |\nabla v|^p \, dx\right) \\
&\geq \frac{1}{2}M\left(\int_\Omega |\nabla u|^p \, dx\right)\int_\Omega |\nabla u|^{p-2}(|\nabla u|^2 - |\nabla v|^2) \, dx \\
&\quad - \frac{1}{2}M\left(\int_\Omega |\nabla v|^p \, dx\right)\int_\Omega |\nabla u|^{p-2}(|\nabla u|^2 - |\nabla v|^2) \, dx \\
&= \frac{1}{2}M\left(\int_\Omega |\nabla v|^p \, dx\right)\int_\Omega (|\nabla u|^{p-2} - |\nabla v|^{p-2})(|\nabla u|^2 - |\nabla v|^2) \, dx \\
&\geq \frac{M_0}{2}\int_\Omega (|\nabla u|^{p-2} - |\nabla v|^{p-2})(|\nabla u|^2 - |\nabla v|^2) \, dx.
\end{aligned} \tag{1.11}$$

If $|\nabla v| \geq |\nabla u|$, changing the role of u and v in (1.7)-(1.11) we have

$$\begin{aligned}
I_2 &:= \Phi'(v)(v) - \Phi'(v)(u) - \Phi'(u)(v) + \Phi'(u)(u) \\
&= M\left(\int_\Omega |\nabla v|^p\,dx\right)\left(\int_\Omega |\nabla v|^p\,dx - \int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla u\,dx\right) \\
&\quad - M\left(\int_\Omega |\nabla u|^p\,dx\right)\left(\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\,dx - \int_\Omega |\nabla u|^p\,dx\right) \\
&\geq \frac{1}{2}M\left(\int_\Omega |\nabla v|^p\,dx\right)\int_\Omega |\nabla v|^{p-2}(|\nabla v|^2 - |\nabla u|^2)\,dx \\
&\quad - \frac{1}{2}M\left(\int_\Omega |\nabla u|^p\,dx\right)\int_\Omega |\nabla u|^{p-2}(|\nabla v|^2 - |\nabla u|^2)\,dx \\
&= \frac{1}{2}M\left(\int_\Omega |\nabla v|^p\,dx\right)\int_\Omega (|\nabla v|^{p-2} - |\nabla u|^{p-2})(|\nabla v|^2 - |\nabla u|^2)\,dx \\
&\geq \frac{M_0}{2}\int_\Omega (|\nabla v|^{p-2} - |\nabla u|^{p-2})(|\nabla v|^2 - |\nabla u|^2)\,dx.
\end{aligned} \tag{1.12}$$

From (1.11) and (1.12) we have

$$\Big(\Phi'(u) - \Phi'(v)\Big)(u-v) = I_1 = I_2 \geq 0, \qquad \forall u, v \in W_0^{1,p}(\Omega). \tag{1.13}$$

Moreover, if $u \neq v$ and $\Big(\Phi'(u) - \Phi'(v)\Big)(u-v) = 0$, then we have

$$\int_\Omega (|\nabla u|^{p-2} - |\nabla v|^{p-2})(|\nabla u|^2 - |\nabla v|^2)\,dx = 0,$$

so $|\nabla u| = |\nabla v|$ in Ω. Thus, we deduce that

$$\begin{aligned}
\Big(\Phi'(u) - \Phi'(v)\Big)(u-v) &= \Phi'(u)(u-v) - \Phi'(v)(u-v) \\
&= M\left(\int_\Omega |\nabla u|^p\,dx\right)\int_\Omega |\nabla u|^{p-2}|\nabla u - \nabla v|^2\,dx \\
&= 0,
\end{aligned} \tag{1.14}$$

i.e., $u - v$ is a constant. In view of $u = v = 0$ on $\partial\Omega$ we have $u \equiv v$ which is contrary with $u \neq v$. Therefore $\Big(\Phi'(u) - \Phi'(v)\Big)(u-v) > 0$ and Φ' is strictly monotone in $W_0^{1,p}(\Omega)$.

Let u, v be two functions such that (1.6) is verified. Taking $\varphi = (u-v)^+$, the positive part of $u - v$, as a test function of (1.6), we have

$$
\begin{aligned}
(\Phi'(u)-\Phi'(v))(\varphi) = {} & M\left(\int_\Omega |\nabla u|^p\,dx\right)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,dx \\
& - M\left(\int_\Omega |\nabla v|^p\,dx\right)\int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla\varphi\,dx \\
& \leq 0.
\end{aligned}
\tag{1.15}
$$

Relations (1.14) and (1.15) mean that $u \leq v$. □

2. Existence result

In this section, we shall state and prove the main result of this paper. Let λ_p be the first eigenvalue of $-\Delta_p$ with Dirichlet boundary conditions and ϕ the corresponding positive eigenfunction with $\|\phi\|_\infty = 1$. Then there exist positive constants m_0, δ, r such that $|\nabla\phi|^p - \lambda_p\phi^p \geq m_0$ on $\overline{\Omega}_\delta := \{x \in \Omega : \ d(x,\partial\Omega) \leq \delta\}$ and $\phi \geq r$ on $\Omega \setminus \overline{\Omega}_\delta$.

We assume that the weight functions a, b, c, d take negative values in $\overline{\Omega}_\delta$ but require a, b, c, d to be strictly positive in $\Omega\backslash\Omega_\delta$. To be precise we assume that there exist positive constants a_0, b_0, c_0, d_0 and a_1, b_1, c_1, d_1 such that $a(x) \geq -a_0$, $b(x) \geq -b_0$, $c(x) \geq -c_0$, $d(x) \geq -d_0$ on Ω_δ and $a(x) \geq a_1$, $b(x) \geq b_1$, $c(x) \geq c_1$, $d(x) \geq d_1$ on $\Omega\backslash\Omega_\delta$.

In what follows, if f_1, f_2, g_1, g_2 are real-valued functions, we write $(f_1, f_2) \leq (g_1, g_2)$ if and only if $f_1 \leq g_1$ and $f_2 \leq g_2$.
In such a case, we write $(u, v) \in [(f_1, f_2), (g_1, g_2)]$ if and only if $f_1 \leq u \leq g_1$ and $f_2 \leq v \leq g_2$.

Let us assume the following assumptions:

(H_1) Assume that $M_1, M_2 : \mathbb{R}^+ \to \mathbb{R}^+$ are two continuous and increasing functions and there exists $m_1, m_2 > 0$ such that

$$
M_1(t) \geq m_1, \quad M_2(t) \geq m_2 \qquad \text{for all } t \in \mathbb{R}^+.
$$

(H_2) $f, g : [0, +\infty) \to \mathbb{R}$ are C^1 increasing functions such that

$$
\lim_{t\to+\infty} f(t) = \lim_{t\to+\infty} g(t) = +\infty.
$$

(H_3) It holds that

$$
\lim_{t\to+\infty} \frac{f\left(K\left(g(t)^{\frac{1}{p-1}}\right)\right)}{t^{p-1}} = 0,
$$

for every constant $K > 0$.

(H_4) $h, \tau \in C^1([0,\infty))$ are non-negative, non-decreasing functions such that

$$
\lim_{u\to+\infty} \frac{h(u)}{u^{p-1}} = 0, \qquad \lim_{u\to+\infty} \frac{\tau(u)}{u^{p-1}} = 0, \qquad \lim_{u\to+\infty} h(u) = \lim_{u\to+\infty} \tau(u) = +\infty\,.
$$

(H_5) There exists $\gamma, s > 0$ such that

$$\min\left\{\Big(\frac{a_0 f((\gamma+s)^{\frac{1}{p-1}})}{\gamma}, \frac{c_0 h((\gamma+s)^{\frac{1}{p-1}})}{s}\Big), \Big(\frac{b_0 g((\gamma+s)^{\frac{1}{p-1}})}{\gamma}, \frac{d_0 \tau((\gamma+s)^{\frac{1}{p-1}})}{s}\Big)\right\}$$

$$< \min\left\{\Big(\frac{a_1 f((\gamma+s)^{\frac{1}{p-1}} \frac{p-1}{p} r^{\frac{p}{p-1}})}{s\lambda_p}, \frac{c_1 h((\gamma+s)^{\frac{1}{p-1}} \frac{p-1}{p} r^{\frac{p}{p-1}})}{\gamma\lambda_p}\Big),\right.$$

$$\left.\Big(\frac{b_1 g((\gamma+s)^{\frac{1}{p-1}} \frac{p-1}{p} r^{\frac{p}{p-1}})}{s\lambda_p}, \frac{d_1 h((\gamma+s)^{\frac{1}{p-1}} \frac{p-1}{p} r^{\frac{p}{p-1}})}{\gamma\lambda_p}\Big)\right\},$$

$$\Big(\frac{1}{\|b\|_\infty}, \frac{1}{\|d\|_\infty}\Big) <$$

$$\min\left\{\Big(\frac{a_1 f((\gamma+s)^{\frac{1}{p-1}} \frac{p-1}{p} r^{\frac{p}{p-1}})}{s\lambda_p}, \frac{c_1 h((\gamma+s)^{\frac{1}{p-1}} \frac{p-1}{p} r^{\frac{p}{p-1}})}{\gamma\lambda_p}\Big),\right.$$

$$\left.\Big(\frac{b_1 g((\gamma+s)^{\frac{1}{p-1}} \frac{p-1}{p} r^{\frac{p}{p-1}})}{s\lambda_p}, \frac{d_1 h((\gamma+s)^{\frac{1}{p-1}} \frac{p-1}{p} r^{\frac{p}{p-1}})}{\gamma\lambda_p}\Big)\right\}.$$

It should be noticed that the nonlinearities $\alpha_i t^{q_i} - \beta_i$, where $t \in \mathbb{R}^+$, $\alpha_i, \beta_i > 0$, $q_i \in (0, p-1)$, $1 \leq i \leq 4$, for the functions f, g, h and τ, satisfy the condition (H_5).

Our main result in this paper is given by the following theorem.

Theorem 2.1. *Assume that the conditions (H_1)-(H_5) hold. Then there exists $[A, B]$ such that problem (1.1) has a positive solution (u, v) for every $(\lambda_i, \mu_i) \in [A, B]$, $i = 1, 2$.*

Proof. Define $f(t) = 0$ and $g(t) = 0$ for all $t < 0$. We shall establish Theorem 2.1 by constructing a positive weak subsolution $(\underline{u}, \underline{v}) \in W_0^{1,p}(\Omega, \mathbb{R}^2)$ and a positive supersolution $(\overline{u}, \overline{v}) \in W_0^{1,p}(\Omega, \mathbb{R}^2)$ of problem (1.1) such that $\underline{u} \leq \overline{u}$ and $\underline{v} \leq \overline{v}$ in Ω. That is, $\underline{u}$, $\overline{u}$, $\underline{v}$, $\overline{v}$ satisfies

$$M_1\left(\int_\Omega |\nabla \underline{u}|^p\,dx\right)\int_\Omega |\nabla \underline{u}|^{p-2}\nabla \underline{u}\cdot\nabla\varphi\,dx \leq \int_\Omega [\lambda_1 a(x) f(\underline{v}) + \mu_1 c(x) h(\underline{u})]\varphi\,dx$$

$$M_2\left(\int_\Omega |\nabla \underline{v}|^p\,dx\right)\int_\Omega |\nabla \underline{v}|^{p-2}\nabla \underline{v}\cdot\nabla\varphi\,dx \leq \int_\Omega [\lambda_2 b(x) g(\underline{v}) + \mu_2 d(x) \tau(\underline{u})]\varphi\,dx$$

and

$$M_1\left(\int_\Omega |\nabla \overline{u}|^p\,dx\right)\int_\Omega |\nabla \overline{u}|^{p-2}\nabla \underline{u}\cdot\nabla\varphi\,dx \geq \int_\Omega [\lambda_1 a(x) f(\overline{v}) + \mu_1 c(x) h(\overline{u})]\varphi\,dx$$

$$M_2\left(\int_\Omega |\nabla \overline{v}|^p\,dx\right)\int_\Omega |\nabla \overline{v}|^{p-2}\nabla \underline{v}\cdot\nabla\varphi\,dx \geq \int_\Omega [\lambda_2 b(x) g(\overline{v}) + \mu_2 d(x) \tau(\overline{u})]\varphi\,dx$$

for all $\varphi \in W_0^{1,p}(\Omega)$ with $\varphi \geq 0$.

We define

$$A = \max\left\{\Big(\frac{s\lambda_p}{b_1 g((\gamma+s)^{\frac{1}{p-1}} \frac{p-1}{p} r^{\frac{p}{p-1}})}, \frac{\gamma\lambda_p}{d_1 h((\gamma+s)^{\frac{1}{p-1}} \frac{p-1}{p} r^{\frac{p}{p-1}})}\Big),\right.$$

$$\Big(\frac{s\lambda_p}{a_1 f((\gamma+s)^{\frac{1}{p-1}}\frac{p-1}{p}r^{\frac{p}{p-1}})}, \frac{\gamma\lambda_p}{c_1 h((\gamma+s)^{\frac{1}{p-1}}\frac{p-1}{p}r^{\frac{p}{p-1}})}\Big)\Big\},$$

$$B=\min\Big\{\Big(\frac{s}{b_0 g((\gamma+s)^{\frac{1}{p-1}})}, \frac{\gamma}{d_0 h((\gamma+s)^{\frac{1}{p-1}})}\Big),$$

$$\Big(\frac{s}{a_0 f((\gamma+s)^{\frac{1}{p-1}})}, \frac{\gamma}{c_0 h((\gamma+s)^{\frac{1}{p-1}})}\Big), \Big(\frac{1}{\|b\|_\infty}, \frac{1}{\|d\|_\infty}\Big)\Big\}.$$

Let us define

$$\underline{u}=(\gamma+s)^{\frac{1}{p-1}}\frac{p-1}{p}\Big(\frac{1}{m_1 m_0}\Big)^{\frac{1}{p-1}}\phi^{\frac{p}{p-1}}, \qquad \underline{v}=(\gamma+s)^{\frac{1}{p-1}}\frac{p-1}{p}\Big(\frac{1}{m_2 m_0}\Big)^{\frac{1}{p-1}}\phi^{\frac{p}{p-1}},$$

where m_1, m_2 are given by the condition (H_1). We shall verify that $(\underline{u}, \underline{v})$ is a subsolution of problem (1.1) for $(\lambda_i, \mu_i) \in [A, B]$, $i = 1, 2$. Indeed, let $\varphi \in W_0^{1,p}(\Omega)$ with $\varphi \geq 0$ in Ω. By (H_1), a simple calculation shows that

$$\begin{aligned}
& M_1\left(\int_\Omega |\nabla \underline{u}|^p\,dx\right)\int_\Omega |\nabla \underline{u}|^{p-2}\nabla \underline{u}\cdot\nabla\varphi\,dx \\
&= M_1\left(\int_\Omega |\nabla \underline{u}|^p\,dx\right)\frac{(\gamma+s)}{m_1 m_0}\int_\Omega \phi|\nabla\phi|^{p-2}\nabla\phi\cdot\nabla\varphi\,dx \\
&= \frac{(\gamma+s)}{m_1 m_0}M_1\left(\int_\Omega |\nabla \underline{u}|^p\,dx\right)\left\{\int_\Omega |\nabla\phi|^{p-2}\nabla\phi\cdot[\nabla(\phi\varphi)-\varphi\nabla\phi]\,dx\right\} \\
&= \frac{(\gamma+s)}{m_1 m_0}M_1\left(\int_\Omega |\nabla \underline{u}|^p dx\right)\left\{\int_\Omega |\nabla\phi|^{p-2}\nabla\phi\cdot\nabla(\phi\varphi)\,dx\right\} \\
&\quad - \frac{(\gamma+s)}{m_1 m_0}M_1\left(\int_\Omega |\nabla \underline{u}|^p\,dx\right)\left\{\int_\Omega |\nabla\phi|^p\varphi\,dx\right\} \qquad (2.1)\\
&= \frac{(\gamma+s)}{m_1 m_0}M_1\left(\int_\Omega |\nabla \underline{u}|^p\,dx\right)\left\{\int_\Omega \lambda_p|\phi|^{p-2}\phi\cdot(\phi\varphi)dx\right\} \\
&\quad - \frac{(\gamma+s)}{m_1 m_0}M_1\left(\int_\Omega |\nabla \underline{u}|^p\,dx\right)\left\{\int_\Omega |\nabla\phi|^p\varphi\,dx\right\} \\
&= \frac{(\gamma+s)}{m_1 m_0}M_1\left(\int_\Omega |\nabla \underline{u}|^p\,dx\right)\int_\Omega (\lambda_p\phi^p-|\nabla\phi|^p)\varphi\,dx \\
&\le \frac{(\gamma+s)}{m_0}\int_\Omega (\lambda_p\phi^p-|\nabla\phi|^p)\varphi\,dx.
\end{aligned}$$

On $\overline{\Omega}_\delta$, we have $|\nabla\phi|^p - \lambda_p\phi^p \ge m_0$, which implies that

$$\begin{aligned}\frac{(\gamma+s)}{m_0}(\lambda_p\phi^p - |\nabla\phi|^p) &\le -\gamma - s\\ &\le -\lambda_1 a_0 f((\gamma+s)^{\frac{1}{p-1}}) - \mu_1 c_0 h((\gamma+s)^{\frac{1}{p-1}})\\ &\le -\lambda_1 a_0 f((\gamma+s)^{\frac{1}{p-1}}\frac{p-1}{p}\Big(\frac{1}{m_2m_0}\Big)^{\frac{1}{p-1}}\phi^{\frac{p}{p-1}})\\ &\quad - \mu_1 c_0 h((\gamma+s)^{\frac{1}{p-1}}\frac{p-1}{p}\Big(\frac{1}{m_1m_0}\Big)^{\frac{1}{p-1}}\phi^{\frac{p}{p-1}})\\ &\le \lambda_1 a(x) f(\underline{v}) + \mu_1 c(x) h(\underline{u}).\end{aligned} \tag{2.2}$$

Next, on $\Omega\backslash\overline{\Omega}_\delta$ we have $\phi \ge r$ for some $r > 0$, and therefore by the condition (H_2) and the definition of $\underline{u}$, $\underline{v}$, it follows that

$$\begin{aligned}(\gamma+s)(\lambda_p\phi^p - |\nabla\phi|^p) &\le (\gamma+s)\lambda_p\\ &\le \lambda_1 a_1 f((\gamma+s)^{\frac{1}{p-1}}\frac{p-1}{p}\Big(\frac{1}{m_2m_0}\Big)^{\frac{1}{p-1}} r^{\frac{p}{p-1}})\\ &\quad + \mu_1 c_1 h((\gamma+s)^{\frac{1}{p-1}}\frac{p-1}{p}\Big(\frac{1}{m_1m_0}\Big)^{\frac{1}{p-1}} r^{\frac{p}{p-1}})\\ &\le \lambda_1 a(x) f(\underline{v}) + \mu_1 c(x) h(\underline{u}),\end{aligned} \tag{2.3}$$

for $(\lambda_1, \mu_1) \in [A, B]$.

Relations (2.2) and (2.3) implies that

$$M_1\left(\int\limits_{\Omega\backslash\overline{\Omega}_\delta} |\nabla\underline{u}|^p\,dx\right)\int\limits_{\Omega\backslash\overline{\Omega}_\delta} |\nabla\underline{u}|^{p-2}\nabla\underline{u}\cdot\nabla\varphi\,dx \le \int\limits_{\Omega\backslash\overline{\Omega}_\delta}[\lambda_1 a(x) f(\underline{v}) + \mu_1 c(x) h(\underline{u})]\varphi\,dx, \tag{2.4}$$

for any $\varphi \in W_0^{1,p}(\Omega)$ with $\varphi \ge 0$ in Ω. Similarly,

$$M_1\left(\int\limits_{\Omega\backslash\overline{\Omega}_\delta} |\nabla\underline{v}|^p\,dx\right)\int\limits_{\Omega\backslash\overline{\Omega}_\delta} |\nabla\underline{v}|^{p-2}\nabla\underline{v}\cdot\nabla\varphi\,dx \le \int\limits_{\Omega\backslash\overline{\Omega}_\delta}[\lambda_2 b(x) g(\underline{v}) + \mu_2 d(x) \tau(\underline{u})]\varphi\,dx, \tag{2.5}$$

for any $\varphi \in W_0^{1,p}(\Omega)$ with $\varphi \ge 0$ in Ω. From (2.4) and (2.5), $(\underline{u}, \underline{v})$ is a subsolution of problem (1.1). Moreover, we have $\underline{u} > 0$ and $\underline{v} > 0$ in Ω.

Next, we shall construct a supersolution of problem (1.1). For this purpose, let ϕ_0 be the solution of the following problem

$$\begin{cases} -\Delta_p\varphi = 1 & \text{in } \Omega,\\ \varphi = 0, & \text{on } \partial\Omega.\end{cases} \tag{2.6}$$

Let

$$\overline{u} := \frac{C}{R}\phi_0, \qquad \overline{v} := \frac{1}{m_2^{\frac{1}{p-1}}}\Big[2g(C)\Big]^{\frac{1}{p-1}}\phi_0,$$

where $R = \|\phi_0\| > 0$ and $C > 0$ is a large positive real number to be chosen later. We shall verify that $(\overline{u}, \overline{v})$ is a supersolution of problem (1.1). To this end, let $\varphi \in W_0^{1,p}(\Omega)$ with $\varphi \geq 0$ in Ω, from (H_1), we have

$$\begin{aligned}
&M_1\left(\int\limits_\Omega |\nabla \overline{u}|^p\,dx\right)\int\limits_\Omega |\nabla \overline{u}|^{p-2}\nabla \overline{u}\cdot\nabla\varphi\,dx \\
&= M_1\left(\int\limits_\Omega |\nabla \overline{u}|^p\,dx\right)\left(\frac{C}{R}\right)^{p-1}\int\limits_\Omega |\nabla \phi_0|^{p-2}\nabla\phi_0\cdot\nabla\varphi\,dx \\
&= \left(\frac{C}{R}\right)^{p-1} M_1\left(\int\limits_\Omega |\nabla \overline{u}|^p\,dx\right)\int\limits_\Omega \varphi\,dx \\
&\geq m_1\left(\frac{C}{R}\right)^{p-1}\int\limits_\Omega \varphi\,dx.
\end{aligned} \tag{2.7}$$

Using the condition (H_3), we can choose the number $C > 0$ large enough so that

$$C^{p-1} \geq \frac{R^{p-1}}{m_1}\left(\lambda_1\|a\|_\infty f\Big(\frac{1}{m_2^{\frac{1}{p-1}}}\Big[2g(C)\Big]^{\frac{1}{p-1}}\phi_0\Big) + \mu_1\|c\|_\infty h\Big(\frac{C}{R}\|\phi_0\|_\infty\Big)\right),$$

and therefore

$$\begin{aligned}
&M_1\left(\int\limits_\Omega |\nabla \overline{u}|^p\,dx\right)\int\limits_\Omega |\nabla \overline{u}|^{p-2}\nabla \overline{u}\cdot\nabla\varphi\,dx \\
&\geq m_1\left(\frac{C}{R}\right)^{p-1}\int\limits_\Omega \varphi\,dx \\
&\geq \int\limits_\Omega \left(\lambda_1\|a\|_\infty f(\frac{1}{m_2^{\frac{1}{p-1}}}\Big[2g(C)\Big]^{\frac{1}{p-1}}\phi_0) + \mu_1\|c\|_\infty h(\frac{C}{R}\|\phi_0\|_\infty)\right)\varphi\,dx \\
&\geq \int\limits_\Omega [\lambda_1 a(x) f(\overline{v}) + \mu_1 c(x) h(\overline{u})]\varphi\,dx.
\end{aligned} \tag{2.8}$$

Next, from the definition of $\overline{v}$, the conditions (H_1), (H_5), (H_4) and the fact that g is increasing, we also deduce that

$$
\begin{aligned}
&M_2\left(\int_\Omega |\nabla\overline{v}|^p\,dx\right)\int_\Omega |\nabla\overline{v}|^{p-2}\nabla\overline{v}\cdot\nabla\varphi\,dx\\
&= M_2\left(\int_\Omega |\nabla\overline{v}|^p\,dx\right)\frac{1}{m_2}2g(C)\int_\Omega |\nabla\phi_0|^{p-2}\nabla\phi_0\cdot\nabla\varphi\,dx\\
&= M_2\left(\int_\Omega |\nabla\overline{v}|^p\,dx\right)\frac{1}{m_2}2g(C)\int_\Omega \varphi\,dx\\
&\geq 2g(C)\int_\Omega \varphi\,dx\\
&\geq \int_\Omega 2g\Big(\frac{C}{R}\phi_0\Big)\varphi\,dx\\
&= \int_\Omega \left[g\Big(\frac{C}{R}\phi_0\Big)+g\Big(\frac{C}{R}\phi_0\Big)\right]\varphi\,dx\\
&\geq \int_\Omega [\lambda_2\|b\|_\infty g(\overline{u})+\tau(\overline{v})]\varphi\,dx\\
&\geq \int_\Omega [\lambda_2\|b\|_\infty g(\overline{u})+\mu_2\|d\|_\infty\tau(\overline{v})]\varphi\,dx\\
&\geq \int_\Omega [\lambda_2 b(x)g(\overline{u})+\mu_2 d(x)\tau(\overline{v})]\varphi\,dx.
\end{aligned}
\tag{2.9}
$$

From (2.8) and (2.9), $(\overline{u},\overline{v})$ is a supersolution of problem (1.1) with $\underline{u}\leq\overline{u}$ and $\underline{v}\leq\overline{v}$ for $C>0$ large.

In order to obtain a weak solution of problem (1.1) we shall use the arguments by Azzouz and Bensedik [4]. For this purpose, we define a sequence $\{(u_n,v_n)\}\subset W_0^{1,p}(\Omega,\mathbb{R}^2)$ as follows: $u_0:=\overline{u}$, $v_0:=\overline{v}$ and (u_n,v_n) is the unique solution of the system

$$
\begin{cases}
-M_1\left(\int_\Omega |\nabla u_n|^p\,dx\right)\Delta_p u_n=\lambda_1 a(x)f(v_{n-1})+\mu_1 c(x)h(u_{n-1}) & \text{in } \Omega,\\
-M_2\left(\int_\Omega |\nabla v_n|^p\,dx\right)\Delta_p v_n=\lambda_2 b(x)g(u_{n-1})+\mu_2 d(x)\tau(v_{n-1}) & \text{in } \Omega,\\
u_n=v_n=0 & \text{on } \partial\Omega.
\end{cases}
\tag{2.10}
$$

System (2.10) is (M_1,M_2)-linear in the sense that, if $(u_{n-1},v_{n-1})\in W_0^{1,p}(\Omega,\mathbb{R}^2)$ is given, the right hand sides of (2.10) is independent of u_n,v_n. Then we deduce from a result in [3] that system (2.10) has a unique solution $(u_n,v_n)\in W_0^{1,p}(\Omega,\mathbb{R}^2)$.

Using (2.10) and the fact that (u_0,v_0) is a supersolution of (1.1) we have

$$
-M_1\left(\int_\Omega |\nabla u_0|^p\,dx\right)\Delta_p u_0\geq\lambda_1 a(x)f(v_0)+\mu_1 c(x)h(u_0)=-M_1\left(\int_\Omega |\nabla u_1|^p\,dx\right)\Delta_p u_1,
$$

$$
-M_2\left(\int_\Omega |\nabla v_0|^p\,dx\right)\Delta_p v_0\geq\lambda_2 b(x)g(u_0)+\mu_2 d(x)\tau(v_0)=-M_2\left(\int_\Omega |\nabla v_1|^p\,dx\right)\Delta_p v_1,
$$

and by Proposition 1.1, $u_0 \geq u_1$ and $v_0 \geq v_1$. Also, since $u_0 \geq \underline{u}$, $v_0 \geq \underline{v}$, then by the monotonicity of f, h, τ, g one has

$$\begin{aligned} -M_2\left(\int\limits_\Omega |\nabla u_1|^p\,dx\right)\Delta_p u_1 &= \lambda_1 a(x) f(v_0) + \mu_1 c(x) h(u_0) \\ &\geq \lambda_1 a(x) f(\underline{v}) + \mu_1 c(x) h(\underline{u}) \\ &\geq -M_2\left(\int\limits_\Omega |\nabla \underline{u}|^p\,dx\right)\Delta_p \underline{u}, \end{aligned}$$

$$\begin{aligned} -M_2\left(\int\limits_\Omega |\nabla v_1|^p\,dx\right)\Delta_p v_1 &= \lambda_2 b(x) g(u_0) + \mu_2 d(x) \tau(v_0) \\ &\geq \lambda_2 b(x) g(\underline{u}) + \mu_2 d(x) \tau(\underline{v}) \\ &\geq -M_2\left(\int\limits_\Omega |\nabla \underline{v}|^p\,dx\right)\Delta_p \underline{v}, \end{aligned}$$

from which, according to Proposition 1.1, $u_1 \geq \underline{u}$, $v_1 \geq \underline{v}$. For u_2, v_2 we write

$$\begin{aligned} -M_1\left(\int\limits_\Omega |\nabla u_1|^p\,dx\right)\Delta_p u_1 &= \lambda_1 a(x) f(v_0) + \mu_1 c(x) h(u_0) \\ &\geq \lambda_1 a(x) f(v_1) + \mu_1 c(x) h(u_1) \\ &= -M_1\left(\int\limits_\Omega |\nabla u_2|^p\,dx\right)\Delta_p u_2, \end{aligned}$$

$$\begin{aligned} -M_2\left(\int\limits_\Omega |\nabla v_1|^p\,dx\right)\Delta_p v_1 &= \lambda_2 b(x) g(u_0) + \mu_2 d(x) \tau(v_0) \\ &\geq \lambda_2 b(x) g(u_1) + \mu_2 d(x) \tau(v_1) \\ &= -M_2\left(\int\limits_\Omega |\nabla v_2|^p\,dx\right)\Delta_p v_2, \end{aligned}$$

and then $u_1 \geq u_2$, $v_1 \geq v_2$. Similarly, $u_2 \geq \underline{u}$ and $v_2 \geq \underline{v}$ because

$$\begin{aligned} -M_1\left(\int\limits_\Omega |\nabla u_2|^p\,dx\right)\Delta_p u_2 &= \lambda_1 a(x) f(v_1) + \mu_1 c(x) h(u_1) \\ &\geq \lambda_1 a(x) f(\underline{v}) + \mu_1 c(x) h(\underline{u}) \\ &\geq -M_1\left(\int\limits_\Omega |\nabla \underline{u}|^p\,dx\right)\Delta_p \underline{u}, \end{aligned}$$

$$-M_2\left(\int\limits_\Omega |\nabla v_2|^p\,dx\right)\Delta_p v_2 = \lambda_2 b(x)g(u_1)+\mu_2 d(x)\tau(v_1)$$
$$\geq \lambda_2 b(x)g(\underline{u})+\mu_2 d(x)\tau(\underline{v})$$
$$\geq -M_2\left(\int\limits_\Omega |\nabla \underline{v}|^p\,dx\right)\Delta_p \underline{v}.$$

Repeating this argument we get a bounded monotone sequence $\{(u_n, v_n)\} \subset W_0^{1,p}(\Omega, \mathbb{R}^2)$ satisfying

$$\overline{u} = u_0 \geq u_1 \geq u_2 \geq \cdots \geq u_n \geq \cdots \geq \underline{u} > 0, \tag{2.11}$$

$$\overline{v} = v_0 \geq v_1 \geq v_2 \geq \cdots \geq v_n \geq \cdots \geq \underline{v} > 0. \tag{2.12}$$

Using the continuity of the functions f, g, h, τ, the definition of the sequences $\{u_n\}$, $\{v_n\}$, there exist four constants C_1, C_2, C_3, $C_4 > 0$ independent of n such that

$$|f(v_{n-1})| \leq C_1, \quad |g(u_{n-1})| \leq C_2, \quad |h(u_{n-1})| \leq C_3, \quad |\tau(v_{n-1})| \leq C_4 \quad \text{for all } n. \tag{2.13}$$

From (2.12), multiplying the first equation of (2.10) by u_n, integrating, using the Hölder inequality and the Sobolev embedding we can show that

$$m_1 \int\limits_\Omega |\nabla u_n|^p\,dx \leq M_1\left(\int\limits_\Omega |\nabla u_n|^p\,dx\right)\int\limits_\Omega |\nabla u_n|^p\,dx$$
$$= \int\limits_\Omega (\lambda_1 a(x) f(v_{n-1}) + \mu_1 c(x) h(u_{n-1}))u_n\,dx$$
$$\leq \int\limits_\Omega (\lambda_1 \|a\|_\infty |f(v_{n-1})| + \mu_1 \|c\|_\infty |h(u_{n-1})|)|u_n|\,dx$$
$$\leq C\int\limits_\Omega |u_n|\,dx$$
$$\leq C'\|u_n\|,$$

or

$$\|u_n\| \leq C', \qquad \forall n, \tag{2.14}$$

where $C' > 0$ is a constant independent of n. Similarly, there exists $C'' > 0$ independent of n such that

$$\|v_n\| \leq C'', \qquad \forall n. \tag{2.15}$$

From (2.14), (2.15) we infer that $\{(u_n, v_n)\}$ has a subsequence which converges in $L^p(\Omega, \mathbb{R}^2)$ to a limit (u, v) with the properties $u \geq \underline{u} > 0$ and $v \geq \underline{v} > 0$. Being monotone, $\{(u_n, v_n)\}$ converges itself to (u, v). Now, lettting $n \to \infty$ in (2.10), we deduce that (u, v) is a positive solution of system (1.1). This completes the proof. $\square$

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