Scale-Invariant Model of Boltzmann Statistical Mechanics and Generalized Thermodynamics

Modèle invariant d'échelles en mécanique statistique de Boltzmann et thermodynamique généralisée

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ABSTRACT. Some implications of a scale-invariant model of Boltzmann statistical mechanics to the laws of generalized thermodynamics are investigated. Through definition of stochastic Planck and Boltzmann universal constants, dimension of Kelvin absolute temperature T (degrees kelvin) is identified as a length (meters) associated with Wien wavelength $T_\beta = \lambda_{w\beta}$ of particle thermal oscillations. Hence, thermodynamic temperature and atomic mass of the field $\mathbb{F}_\beta$ at scale β provide *internal* measures of (extension, duration) of background space $\mathbb{S}_{\beta+1} = \mathbb{F}_\beta$ needed to define *external* space and time coordinates and atomic-mass-unit of $\mathbb{F}_{\beta+1}$. Introduction of invariant internal thermodynamic spacetime and Boltzmann factor are in harmony with modern concepts of quantum gravity as deterministic dissipative dynamic system [73]. The connections between de Pretto number 8338 and Joule-Mayer mechanical equivalent of heat $J_c = 4.169 \, \text{kJ} / \text{kcal}$ and universal gas constant $R^o = 8338 \, \text{J} / \text{kcal.m}$ are identified leading to the modified mechanical equivalent of heat $J = 2J_c = 8338 \, \text{J} / \text{kcal}$. It is shown that with work defined as Helmholtz free energy $W = F = U - TS$, Helmholtz decomposition of total thermal energy into *free heat U* and *latent heat pV* results in modified form of the first law of thermodynamics $Q = H = U - W = U + pV$. Finally, by application of Boltzmann combinatorics method, entropy of ideal gas is expressed in terms of the number of Heisenberg-Kramers virtual oscillators as $S = 4Nk$ in exact agreement with predicted entropy of black hole by Major and Setter [159].

KEYWORDS. Generalized thermodynamics, Entropy, Absolute temperature scale, Spacetime, Quantum nature of space and time, Quantum gravity, T.O.E.

1. Introduction

The laws of generalized thermodynamics and the underlying Boltzmann statistical mechanics appear to possess sufficient degree of universality to describe physical phenomena from extremely large cosmological scale to the minute scale of quantum optics. From cosmological observations, it is known that a galaxy such as the Milky Way (MW) that constitutes an "atom" of cosmic field is not at all visible at cosmological scales. However, as one approaches the MW, one first sees a small bright point appearing as a star, and only on a much closer approach one sees the enormous size of the entire MW galaxy extending about 100,000 light-years across. Similar descriptions apply to our solar system when viewed at galactic scales.

It is also known that a factor of approximately 10^{17} separates the scales of five major statistical fields in our universe starting with tachyon (graviton)-dynamics at exceedingly small Planck scale 10^{-35} m, followed by electrodynamics 10^{-18} m, hydrodynamics 10^0 m, astrophysics 10^{18} m, and finally galactic-dynamics (cosmology) 10^{35} m. Each statistical field has its characteristic "atomic" particle namely, graviton, electron, fluid-element, star, and galaxy. However, except their common similarities to turbulent phenomena, no global theoretical framework with sufficient degrees of freedom has yet been identified which allows for description of physical processes over diverse spectrum of spacetime. Clearly, systematic description of all motions of particles, both their momentum and energy, in such statistical fields require application of Boltzmann statistical mechanics. Since universality of Boltzmann statistical mechanics extends to the laws of generalized thermodynamics that govern

macroscopic behavior of statistical fields at all scales, thermodynamics becomes a natural candidate for theory of everything (T.O.E.).

A scale-invariant model of Boltzmann statistical mechanics is schematically shown in Figure 1 based on similarities between stochastic quantum fields [1-17] on the one hand, and turbulent hydrodynamic fields [18-30] on the other hand. Recently, some of the implications of the model (Figure 1) to the fields of fluid mechanics [31, 32], relativistic and classical thermodynamics [33,34], and quantum mechanics [35. 36] were examined. In addition, the gap between hydrodynamics and quantum mechanics was closed by derivation of invariant Schrödinger equation from invariant Bernoulli equation for incompressible potential flow [37].

In this study, implications of invariant Boltzmann statistical mechanics to the laws of generalized thermodynamics are further investigated. Since application of the model to relativistic [33] and classical [34] thermodynamics have been initiated, the study focuses on further clarification of concepts in generalized thermodynamics associated with hierarchies of embedded statistical fields. Also, in order to better reveal the nature of new paradigms and fundamental concepts such as temperature, heat, work, systems at thermodynamics equilibrium are considered that, once the "atom" of the field is defined, will be composed of a spectrum of atomic clusters that are stochastically stationary [33, 34, 36]. However, more complex models involving statistical fields at intermediate scales as well as non-equilibrium systems with non-local thermodynamics [38, 39, 40, 41, 42] are not addressed since the present study is focused on simple equilibrium systems.

In Section 2 the concept of *statistical field* $\mathbb{F}_{\beta}$ and *background space* $\mathbb{S}_{\beta+1}$ and their interconnections $\mathbb{S}_{\beta+1} = \mathbb{F}_{\beta}$ are discussed. It is shown that given the temperature and atomic mass of $\mathbb{F}_{\beta}$ is sufficient to define *internal spacetime* of the background "space" $\mathbb{S}_{\beta+1}$ hence *external* independent *space* and *time* coordinates as well as the atomic-mass-unit of $\mathbb{F}_{\beta+1}$ (Figure 1). Next, in Sections 3-6, new perspectives concerning the zeroth, first, second, and third law of generalized thermodynamics are presented to compliment the earlier investigations [33-37].

In the course of the present study, it is necessary to discuss possible implications of the model to the diverse fields of cosmology, astrophysics, hydrodynamics, molecular dynamics, electrodynamics, and tachyon(graviton)-dynamics. Clearly, in order to achieve a universal perspective of the physical laws over a wide range of spacetime, considerations of at least the fundamentals of different branches of physical sciences become necessary. Obviously, the discussion of all these various fields of scientific disciplines does not imply the proficiency of the author in any one of them.

The question may be raised as to why such comprehensive global modification of our physical concepts pertaining to the laws of nature are needed if these laws have been derived from the systematic application of scientific method during the evolution of the history of science. It is noted however, that although the tree of knowledge has grown to such magnificent proportions, bearing such variety of useful fruits, the very roots of this tree were formed at a time of relative scientific ignorance. As a result, our understanding of some of the most fundamental scientific concepts such as *space*, *time*, *mass*, *temperature*, etc. lack exact scientific precision. Of course, the great stability of the tree of knowledge throughout history can be attributed to the empirical nature of many of its roots. In fact, it is the strict requirement of consistency with the empirical observations that distinguishes scientific method from philosophical speculations. Therefore, the new paradigms must not only conform to all empirical observations, but also provide for a more unitary approach to their descriptions, resort to minimum number of postulates while making new predictions and extending the horizon for future scientific developments.

2. Boltzmann and Scale-Invariant Model of Statistical Mechanics

Following classical methods [19, 43-47], invariant density ρ_β, and velocity of *atom* $\mathbf{u}_\beta$, *element* $\mathbf{v}_\beta$, and *system* $\mathbf{w}_\beta$ (Figure 1) at the scale β are given as [36]

$$\rho_\beta = n_\beta m_\beta = m_\beta \int f_\beta du_\beta \quad , \quad \mathbf{u}_\beta = \mathbf{v}_{mp\beta-1} \tag{1}$$

$$\mathbf{v}_\beta = \rho_\beta^{-1} m_\beta \int \mathbf{u}_\beta f_\beta d\mathbf{u}_\beta \quad , \quad \mathbf{w}_\beta = \mathbf{v}_{mp\beta+1} \tag{2}$$

leading to peculiar and diffusion velocities

$$\mathbf{V}_\beta' = \mathbf{u}_\beta - \mathbf{v}_\beta \quad , \quad \mathbf{V}_\beta = \mathbf{v}_\beta - \mathbf{w}_\beta \quad , \quad \mathbf{V}_\beta = \mathbf{V}_{\beta+1}' \tag{3}$$

The statistical fields of *equilibrium* galactic-, planetary-, hydro-system-, fluid element-, eddy-, cluster-, molecular-, atomic-, subatomic-, kromo-, and tachyon-dynamics at the scale β = g, p, h, f, e, c, m, a, s, k, and t and the corresponding *nonequilibrium* laminar flow fields are schematically shown on the left- and right-hand sides of Figure 1. Each statistical field is governed by distribution function $f_\beta(\mathbf{x}_\beta, \mathbf{u}_\beta, t_\beta)$ and is constructed from its "atom" viewed as point-mass and a part of the background field of lower scale. Next, ensembles of "atoms" form finite-sized composite entities called "*elements*" of the field. Finally, collection of all "*elements*" is defined as the statistical "*system*" at that particular scale. The most-probable element of scale β defines the "atom" of the next higher scale $\beta+1$ such that.

$\mathbf{v}_{mp\beta} = \mathbf{u}_{\beta+1}$. Also, the most-probable element of scale β defines the system of the next lower scale $\beta-1$, $\mathbf{v}_{mp\beta} = \mathbf{w}_{\beta-1}$ thus resulting in hierarchy of statistical fields schematically shown in Figure 1. As was discussed in [36], in view of the estimated scale factor of approximately 10^{-26} between mass of graviton (tachyon) versus photon, it is reasonable to anticipate another statistical field called *equilibrium-neutrino-dynamics* END β = n, not shown in Figure 1, at an intermediate scale separating EKD β = k from ETD β = t field.

Beginning with the scale of laminar molecular dynamics LMD, molecules, clusters of molecules, and cluster of clusters of molecules (eddy) are "atom", "element", and "system" of LMD field with the velocities $(\mathbf{u}_m, \mathbf{v}_m, \mathbf{w}_m)$. Similarly, the fields of laminar cluster-dynamics LCD and eddy-dynamics LED will have the velocities $(\mathbf{u}_c, \mathbf{v}_c, \mathbf{w}_c)$ and $(\mathbf{u}_e, \mathbf{v}_e, \mathbf{w}_e)$ by Equations (1)-(2).
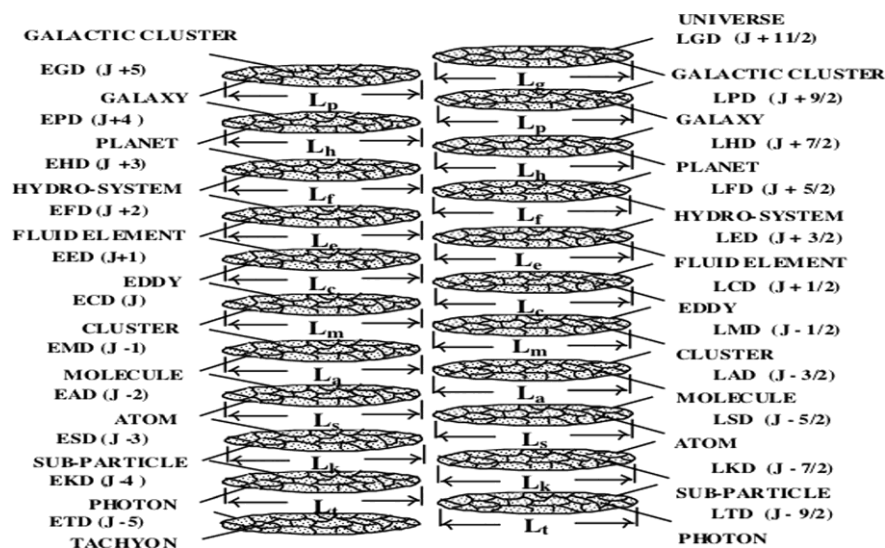

Figure 1. *Scale-invariant model of statistical mechanics. Equilibrium-β-dynamics of the left-hand side and nonequilibrium laminar-β-dynamics on the right-hand side for scales β = g, p, h, f, e, c, m, a, s, k, and t as defined in Sec. 2. Lengths of (system, element, atom) are $(L_\beta, \lambda_\beta, \ell_\beta)$ and λ_β is the mean-free-path [36]*

Typical (atom, element, system) lengths at (EED, ECD, EMD) scales are

EED $(\ell_e, \lambda_e, L_e) = (10^{-5}, 10^{-3}, 10^{-1})$ m

ECD $(\ell_c, \lambda_c, L_c) = (10^{-7}, 10^{-5}, 10^{-3})$ m (4)

EMD $(\ell_m, \lambda_m, L_m) = (10^{-9}, 10^{-7}, 10^{-5})$ m

The relative system sizes of these statistical fields are schematically shown in Figure 2.

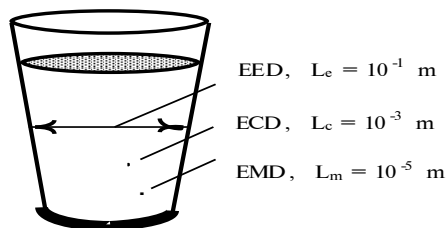**Figure 2.** *Lengths of (system, element, atom) are* $(L_\beta, \lambda_\beta, \ell_\beta)$ *and* λ_β *is the mean-free-path [36]*

Applying the same $(\ell_\beta, \lambda_\beta, L_\beta)$ as in Equation (4) and relation $\ell_\beta = \lambda_{\beta-1} = L_{\beta-2}$ between scales to Figure 1 results in the hierarchy of overlapping statistical fields schematically shown in Figure 3. In Figure 3, starting from *Planck* length $(\hbar G / c^3)^{1/2} \approx 10^{-35}$ m where G is the gravitational constant, seven generations of statistical fields or a factor of about 10^{17} separates the five major statistical fields of tachyon(graviton)-dynamic 10^{-35}, electro-dynamic 10^{-18}, hydrodynamics 10^0, astrophysics 10^{18}, and finally galactic-dynamics or cosmology 10^{35} m.
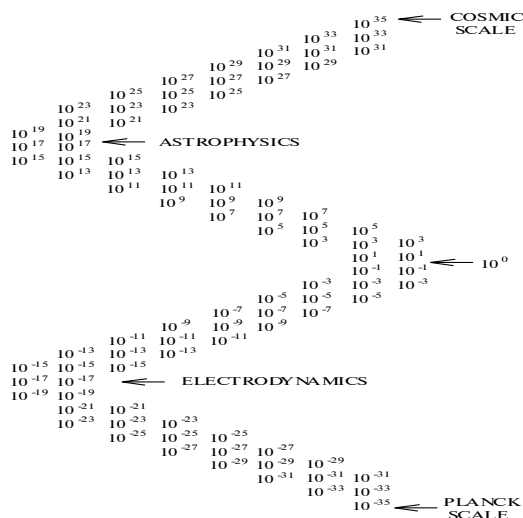

Figure 3. *Hierarchy of statistical fields with chracteristic (atom, element, system)lenngth scales* $(\ell_\beta, \lambda_\beta, L_\beta)$ *from cosmic to Planck scales [36]*

Since invariant Schrödinger equation was derived from invariant Bernoulli equation [37], all statistical fields in Figure 1 are governed by quantum mechanics. Due to the absence of any physical or mathematical reasons to the contrary, the hierarchy in Figure 3 could continue to larger (many universe) and smaller (sub-tachyonic) scales ad infinitum. Also, notwithstanding the often-quoted statement: "God does not play dice" by Einstein, according to Figure 3, the question is not whether Almighty plays dice but rather, "How does God play dice?" [48]. Interestingly, with a transparent material such as Lucite (Plexiglass), one can construct a simple system of finite embedded dice as schematically shown in Figure 4.
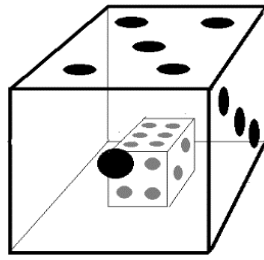
Figure 4. *Hierarchy in Figures 1 and 3 like embedded dice show that chance is the essence of all existence from cosmic to sub-photonic scales*

Also, *universal* nature of Schrödinger equation [37] leads to universal probabilistic nature of the laws of nature in view of probabilistic interpretation of wave function [49]. This not only ensures *stochastic law of justice* in nature but also saves the grace because in an absolutely deterministic Laplacian world, an absolutely benevolent God could not justify the presence of any evil, whereas in a probabilistic world all evil could be attributed to chance.

The universality of invariant Schrödinger equation suggests that the laws of nature over the entire spectrum of statistical fields shown in Figures 1 and 3 are governed by quantum mechanics and hence are probabilistic as is to be expected. Of course, this does not mean that deterministic systems of classical mechanics do not play their central role in modelling physical phenomena. The new paradigm of physical foundation of quantum mechanics discussed in [37] is in harmony with the perceptions of *Heisenberg* [50] in regards to the domain of applicability of classical versus quantum mechanics,

> *"We no longer say "Newtonian mechanics is false and must be replaced by quantum mechanics, which is correct." Instead, we adopt the formula "Classical mechanics is a consistent self–enclosed scientific theory. It is strictly correct description of nature wherever its concepts can be applied"*

Due to its statistical foundation, quantum mechanics concerns behavior of Heisenberg-Kramers [51] *virtual oscillators* or de Broglie [3] wave packets in *statistical fields* such as cosmology, astrophysics, hydrodynamics, molecular dynamics, electrodynamics or optics [37]. Classical Newtonian mechanics on the other hand is mainly concerned with *motion of a few bodies* such as the classical Poincarè [52] three-body-problem of earth-moon-sun that resulted in his discovery of chaos theory.

When particles are assumed to act as composite bosons [53, 54] made of "cooper pairs" of atoms and follow corrected Boltzmann statistics [55,56], velocity, energy, and speed of particles at the state of thermodynamic equilibrium will be respectively governed by invariant Gaussian (Maxwell), Planck, and Maxwell–Boltzmann distribution functions [35,36]

$$f_\beta(v_\beta) = (\frac{m_\beta}{2\pi k T_\beta})^{3/2} \ e^{-m_\beta u_\beta^2 / 2k T_\beta} \tag{5a}$$

$$\frac{\varepsilon_\beta dN_\beta}{V} = \frac{8\pi h}{u_\beta^3} \ \frac{v_\beta^3}{e^{hv_\beta / kT} - 1} \ dv_\beta \tag{5b}$$

$$\frac{dN_{u\beta}}{N} = 4\pi \ (\frac{m_\beta}{2\pi k T_\beta})^{3/2} u_\beta^2 \ e^{-m_\beta u_\beta^2 / 2k T_\beta} du_\beta \tag{5c}$$

For example, the invariant Maxwell-Boltzmann distribution in Equation (5c) is applied to calculate the speed distributions at three consecutive scales of equilibrium (cluster, molecular, atomic) dynamics (ECD, EMD, EAD) at a constant temperature $T = 300$ m as shown in Figure 5.
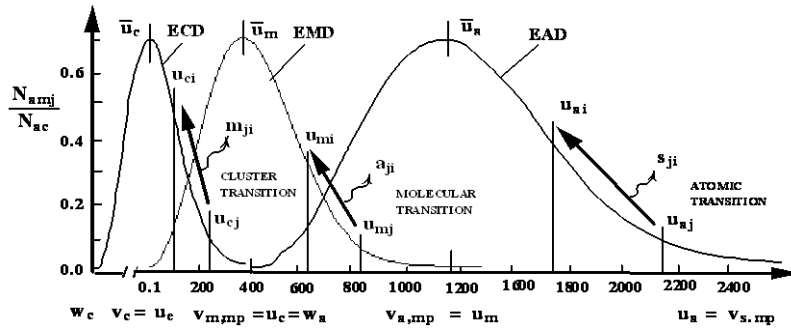
Figure 5. *Maxwell-Boltzmann distribution of (ECD, EMD, EAD) at T = 300 m [36]*

Starting with an atom $\mathbf{u}_\beta$ that is stabilized by an external pressure known as *Poincaré stress* [35], spectrum of atomic clusters with velocity $\mathbf{v}_{j\beta}$ form an *equilibrium statitical field* when absolute thermodynamic temeprature defined as kientic energy of the most probable cluster $kT_\beta = m_\beta v_{mp\beta}^2$ becomes stochastically stationary (Figure 5).

As discussed in [36], the conventional field of fluid dynamics does not correspond to laminar molecular dynamics LMD $\beta = m$ but rather to the next higher scale of laminar cluster-dynamics LCD $\beta = c$ becuase at EMD scale, the root-mean-square speed of molecules is the speed of sound [37,57] that does not correspond to local velocity in hydrodynamics. Hence, in stationary fluid at ECD scale, Maxwell-Boltzmann distribution function governs *stochastically stationary* cluster sizes. Random motion of clusters accounts for the Brownian motion of small suspensions that is known to be a *non-dissipative stationary phenomenon* [35]. Therefore, Maxwell *demon paradox* discussed by Poincaré [58] is resolved since Brownian particles will be in equilibrium with clusters in ECD field (Figures 1 and 2) that are undergoing Brownian motions themselves. Such a scale-invariant model of "Brownian motions" is in accordance with perceptions of Gouy [59] and harmonious with classical theories of Brownian motions [60–70].

Although atoms are under constant transitions between different clusters, also called *energy-levels* as shown in Figure 5, clusters remain *stochastically stationary* due to the principle of detailed balance [36]. For example, in isotropic stationary homogeneous turbulence corresponding to EED, transition of a cluster from a small fast-oscillating "eddy-j" (high energy-level-j) to a large slowly-oscillating "eddy-i" (low energy-level-i) results in emission of a "sub-particle" that is a molecule to carry away the excess energy in harmony with Bohr [71] frequency formula $\Delta\varepsilon_{ji\beta} = h(v_{j\beta} - v_{i\beta})$ as schematically shown in Figure 6a.
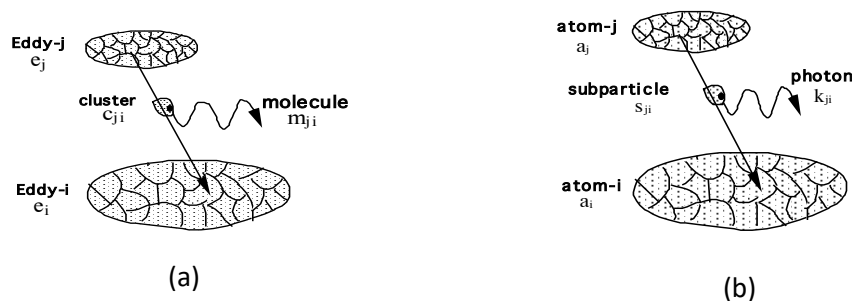

(a)

(b)

Figure 6. *(a) Transition of cluster from eddy-j to eddy-i leading to emission of molecule. (b) Transitions of Electron from atom-j to atom-i leading to emission of photon [37]*

Similarly, in equilibrium-electro-dynamics ESD $\beta = s$, i.e. stochastic electrodynamics (SED), transition of an electron from a small fast-oscillating "atom-j" (high energy-level-j) to a large slow-

oscillating "atom-i" (low energy-level-i) leads to emission of a sub-particle photon γ_{ji} as shown in Figure 6b. According to the model in Figure 6, any particle of cluster-j could go through *any trajectory* to cluster-i emitting energy $\Delta\varepsilon_{ji\beta} = h(\nu_{j\beta} - \nu_{i\beta})$ with the same final state. By Equation (90) of Section 5, Boltzmann factor governs *transition probabilities* hence particle trajectories between different clusters (energy levels) thus accounting for the success of Feynman *integral over all paths*.

The new paradigm of physical foundation of quantum mechanics and transitions between particle clusters, de Broglie wave-packets (Figure 6), is in harmony with quantum transitions between different "cells" in recent *cellular automaton interpretation* of quantum mechanics [72]. The central role of equilibrium thermodynamics in quantum theory of gravity has been emphasized by 't Hooft [73]

> *"Models describing only a small number of quantum states, such as all the above, do not show very clearly the most salient difficulty encountered when one attempts to construct realistic models. This is the fact that our universe is known to be thermodynamically stable. A system in thrermodynamic equilibrium is governed by Boltzmann factor $e^{-\beta E_i}$, where β is the inverse temperature."*

In Figure 6, atomic transitions between clusters (energy-levels) $(i \to j)$, $(j \to k)$ are governed by *transition enthalpies* $H_{ij} = 4kT(N_i - N_j)$, $H_{jk} = 4kT(N_j - N_k)$ that occur in Boltzmann factors $p_{ij} = e^{-H_{ij}/kT}$ and $p_{jk} = e^{-H_{jk}/kT}$ by Equation (90), such that $p_{ij} \cdot p_{jk} = p_{ik}$ in harmony with Heisenberg [74] matrix mechanics. At the scale of Casimir [75] vacuum or *physical space* $\beta = k$ (Figure 1), since $p_{ij} \neq p_{ji}$ follows Heisenberg [51,74] *matrix mechanics*, physical space will be governed by *noncommutative geometry* discussed by Connes [76].

According to the invariant model of Boltzmann statistical mechanics (Figure 1), the statistical field $\mathbb{F}_\beta$ at scale β resides within and is in thermodynamics equilibrium $T_\beta = T_{\beta-1}$ with the background "space" $\mathbb{S}_\beta$ that is identified as the statsitical field at the next lower scale $\mathbb{S}_\beta = \mathbb{F}_{\beta-1}$. For example, given $T_{\beta-1}$ hence internal *spacetime* of $\mathbb{F}_{\beta-1}$ in astrophysics $\beta - 1 = s$, one can define external *space* and *time* coordimates $(x_\beta, t_\beta) = (N_x \lambda_{w\beta-1}, N_t \tau_{w\beta-1})$ at scale $\beta = g$ of cosmology. Hence, every *point* of background "space" $\mathbb{S}_\beta = \mathbb{F}_{\beta-1}$ in cosmology is occupied by either an atom i.e., a galaxy, or else by Casimir [75] vacuum as discussed in [77]. Also, since in classical physics, *atomic masses* are defined in terms of *amu* that is identified as *electromagnetic mass* of photon in Equation (17) of the following Section, given $\mathbb{F}_{\beta-1}(T_{\beta-1}, m_{\beta-1})$, *atomic-mass-unit* of scale β is defined as the *electromagnetic mass* of atom of lower scale β−1

$$amu_\beta = 3kT_{\beta-1} = 3m_{\beta-1}v_{w\beta-1}^2 = m_{\beta-1}v_{r\beta-1}^2 = m_{\beta-1}c_{s\beta-1}^2 = \hat{u}_{\beta-1} \tag{6}$$

where $c_{s\beta-1}$ is the root-mean-square speed or speed of "sound" [37, 57]. In general, mass at scale β is defined as $M_\beta = N_\beta(amu_\beta)$. Thus, $\mathbb{F}_{\beta-1}(T_{\beta-1}, m_{\beta-1})$ represents the "space" $\mathbb{S}_\beta = \mathbb{F}_{\beta-1}$ wihtin which one can define (length, time, mass) = $(x_\beta, t_\beta, M_\beta)$, for example (m, s, kg), of a new statsitical field $\mathbb{F}_\beta(T_\beta, m_\beta)$ leading to the hierarchy of embedded fields

$$\mathbb{F}_\beta[\mathbb{F}_{\beta-1}(T_{\beta-1}, m_{\beta-1})[\mathbb{F}_{\beta-2}(T_{\beta-2}, m_{\beta-2})[\ldots]] \tag{7}$$

As a *metaphor,* the three independent variables *(space, time, mass)* $= (x_\beta, t_\beta, M_\beta)$, like the three primary colors (*red, yellow, blue*), are all that is needed to paint the entire tapestry of creation from cosmic to photonic scales (Figures 1 and 3).

The universal nature of quantum fields in Equation (7) is in harmony with perceptions of Rovelli as described in Figure 7.8 of his book [78] stating that *covariant quantum fields* are all that is needed to define (space, time, particle) in theory of quantum gravity. The role of thermodynamics as one of the three primary ingredients of quantum gravity as universal theory of nature has been emphasized by Rovelli [78]

> *"All of this, I believe, indicates that in order to grasp the basic grammar of the world, we need to merge three basic ingredients, not just two: not just general relativity and quantum mechanics, but also the theory of heat, that is, statistical mechanics and thermodynamics, which we can also describe as "information theory". But the thermodynamics of general relativity—that is to say, the statistical mechanics of quanta of space—is as yet only in its early infancy. Everything is still confused, and there is a very great deal that remains to be understood."*

Invariant model of Boltzmann statistical mechanics (Figures 1, 3) and the corresponding concepts regarding relativistic and classical thermodynamics [33,34] and quantum mechanics [35,36,37] are harmonious with modern theoretical concepts of quantum gravity [73,76,78,79,80,81,82] As discussed above, when space or Casimir [75] vacuum is identified as a compressible tachyon fluid, a new paradigm for physical foundation of quantum gravity [83, 84] becomes possible. Because of the definition of *internal atomic time* [77], quantum theory of gravity will have wave function Ψ_β that instead of Wheeler-DeWitt equation [81,82] satisfies the invariant Schrödinger equation [37]. The resurrection of time in Wheeler-DeWitt equation becomes possible because the new "*atomic*" *time* arises from *internal degrees of freedom* associated with thermodynamic temperature of cosmic field [77].

Given $\mathbb{F}_\beta(T_\beta, m_\beta)$ and hence independent space and time coordinates as well as atomic mass, following the classical methods [19, 43-47], the scale-invariant forms of mass, thermal energy, linear and angular momentum conservation equations for reactive fields at scale β are given as [36]

$$\frac{\partial \rho_{i\beta}}{\partial t_\beta} + \nabla.(\rho_{i\beta} \mathbf{v}_\beta) = \mathfrak{R}_{i\beta} \tag{8}$$

$$\frac{\partial \varepsilon_{i\beta}}{\partial t_\beta} + \nabla.(\varepsilon_{i\beta} \mathbf{v}_\beta) = 0 \tag{9}$$

$$\frac{\partial \mathbf{p}_{i\beta}}{\partial t_\beta} + \nabla.(\mathbf{p}_{i\beta} \mathbf{v}_\beta) = -\nabla.\mathbf{P}_{ij\beta} \tag{10}$$

$$\frac{\partial \boldsymbol{\pi}_{i\beta}}{\partial t_\beta} + \nabla.(\boldsymbol{\pi}_{i\beta} \mathbf{v}_\beta) = \rho_{i\beta} \boldsymbol{\omega}_\beta.\nabla \mathbf{v}_\beta \tag{11}$$

with *volumetric density* of thermal energy $\varepsilon_{i\beta} = \rho_{i\beta} \tilde{h}_{i\beta}$, linear momentum $\mathbf{p}_k = \rho_{i\beta} \mathbf{v}_{i\beta}$, and angular momentum $\boldsymbol{\pi}_{i\beta} = \rho_{i\beta} \boldsymbol{\omega}_{i\beta}$ (since $\mathbf{r}_{a\beta-1} = 1$). Also, $\mathfrak{R}_{i\beta}$ is the chemical reaction rate and $\tilde{h}_{i\beta} = \hat{h}_{i\beta} / m_\beta$ is absolute enthalpy [36]. It is noted that the time coordinates in Equations (8-11) also have a scale subscript β [77].

3. Kelvin Absolute Temperature Scale and Zeroth Law of Thermodynamics

As discussed in [33,34], a fundamental shift of paradigm concerns the change of dimension of Kelvin absolute thermodynamic temperature T from (degrees kelvin) to length (meter). Clearly, changing the dimension of temperature will have a global impact on all branches of natural sciences. Such a fundamental change of scientific paradigm is mandated by the new definitions of *stochastic* Planck h and Boltzmann k universal constants [33, 34]. Finally, because temperature is a *measured property*, its dimension must be ultimately based on the theory that defines it as emphasized by Planck [85].

> *"Every measurement first acquires its meaning for physical science through the significance which a theory gives it"*

Because dimensions of all other quantities in physics are based on three fundamental dimensions namely (length, time, mass) that define $\mathbb{F}_\beta(x_\beta, t_\beta, M_\beta)$, the name degrees kelvin does not reveal the true dimension of absolute temperature since "degree kelvin" is not a fundamental physical *dimension* (m, s, kg). Also, in view of atomic energy $kT_\beta = \hat{\epsilon}_\beta$, dimensions of both T and Boltzmann constant k should be simultaneously modified.

Since the mean particle velocity must vanish $<\mathbf{u}_k>= 0$ at thermodynamic equilibrium, particle energy $\epsilon_\beta = m_\beta < u_\beta^2 >= \overline{p}_\beta < \lambda_\beta^2 >^{1/2} < v_\beta^2 >^{1/2}$ can be expressed in terms of either frequency $\epsilon_\beta = h_\beta < v_\beta^2 >^{1/2}$ or wavelength $\epsilon_\beta = k_\beta < \lambda_\beta^2 >^{1/2}$ when stochastic (Planck, Boltzmann) factors are defined as ($h_\beta = \overline{p}_\beta < \lambda_\beta^2 >^{1/2}, k_\beta = \overline{p}_\beta < v_\beta^2 >^{1/2}$) and $\overline{p}_\beta = m_\beta < u_\beta^2 >^{1/2}$ is the *root-mean-square momentum* of particle [33,34]. As discussed in Section 4, particles are considered to have harmonic translational motions in two directions such that $\epsilon_{t\beta} = m_\beta < u_{x+\beta}^2 > /2 + m_\beta < u_{x-\beta}^2 > /2 = m_\beta < u_{x+\beta}^2 >$. At EKD scale $\beta = k$ (Figure 1), the system is identified as *physical space* or *Casimir vacuum* [75] that may be also called Aristotle *fifth element*, de Broglie *hidden thermostat* [3], or Dirac *stochastic ether* [86]. Naturally, Planck and Boltzmann factors are identified as Planck and Boltzmann universal constants associated with spatial $< \lambda_\beta^2 >^{1/2}$ and temporal $< v_\beta^2 >^{1/2}$ aspects of Casimir [75] *vacuum fluctuations* [33,34]

$$h = h_k = \overline{p}_k \langle \lambda_k^2 \rangle^{1/2} = m_k c \lambda_{rmsk} = m_k c \lambda_{rk} = 6.626 \times 10^{-34} \ [J-s] \tag{12a}$$

$$k = k_k = \overline{p}_k \langle v_k^2 \rangle^{1/2} = m_k c v_{rmsk} = m_k c v_{rk} = 1.381 \times 10^{-23} = [J/m] \tag{12b}$$

By definition $\overline{p}_k = \overline{p}_k = mv_{rk} = mc$, Planck and Boltzmann constants (h, k) are respectively defined in terms of root-mean-square wavelength and frequency $(\lambda_{rmsk}, v_{rmsk}) = (\lambda_{rk}, v_{rk})$ where $\lambda_{rk} v_{rk} = c$ and c is the speed of light in vacuum [34].

Parallel to *Heisenberg spatial uncertainty principle* from Equation (12a),

$$\Delta\lambda_\beta \Delta\overline{p}_\beta \geq \hbar \tag{13a}$$

that limits the resolution of spatial measurements, Equation (12b) leads to *temporal uncertainty principle* [35]

$$\Delta v_\beta \Delta\overline{p}_\beta \geq k \tag{13b}$$

that limits the resolution of time measurements. Implication of generalized uncertainty principle [87] to temporal uncertainty principle in Equation (13b) will be interesting.

By Equation (12), photon internal or electromagnetic energy in Planck equilibrium radiation is expressed as

$$\varepsilon_{rk} = h\nu_{rk} = m_k v_{rk}^2 = m_k c^2 = 3kT \tag{14}$$

$$\varepsilon_{rk} = k\lambda_{rk} = m_k v_{rk}^2 = m_k c^2 = 3kT \tag{15}$$

Hence, propagation speed of light is identified as *root-mean-square speed* of photons in Casimir [75] vacuum at EKD scale $\beta = k$. This is in accordance with perceptions of Huygens [88] who believed that propagation of light waves in compressible ether is analogous to propagation of sound waves in compressible air. Because of the fundamental significance of this analogy to the gap between Planck equilibrium radiation theory and Boltzmann kinetic theory of ideal gas [36], the complete statement is quoted from Huygens book [88]

> *"As regards the different modes in which I have said the movement of Sound and of Light are communicated, one may sufficiently comprehend how this occurs in the case of Sound if one considers that the air is of such nature that it can be compressed and reduced to a much smaller space than that which it ordinarily occupies.*
>
> *Now in applying this kind of movement to that which produces Light there is nothing to hinder us from estimating the particles of the ether to be of a substance as nearly approaching to perfect hardness and pocessing a springiness as prompt as we choose. It is not necessary to examine here the causes of this hardness, or of that springiness, the consideration of which would lead us too far from our subject. I will say, however, in passing that we may conceive that the particles of the ether, notwithstanding their smallness, are in turn composed of the other parts and that their springiness consists in the very rapid movement of a subtle matter which penetrates them from every side and constrains their structure to assume such disposition as to give to this fluid matter the most overt and easy passage possible. This accords with the explanation which Mr. Des Cartes gives for the spring, though I do not, like him, suppose the pores to be in the form of round hollow canals. And it must not be thought that in this there is anything absurd or impossible, it being on the contrary quite credible that it is this infinite series of different sizes of corpuscles, having different degrees of velocity, of which Nature makes use to produce so many marvelous effects,"*

It is most interesting that because of his deep appreciation of mathematics, Huygens kept an open mind concerning potential divisibility of atoms of ether.

At LKD (Figure 1) scale, Casimir [75] vacuum is considered to be a compressible tachyon fluid, Planck compressible ether [89] as discussed in [84]. Therefore, the most fundamental significance of Huygens [88] analogy between propagation of light and sound in the above quotation is the compressibility of physical space or Casimir [75] vacuum. Lorentz perceptions about the medium of space as Aristotle or Huygens ether [84] is described in the following quotation by Verhulst [90] from Lorentz 1915 lecture at the Royal Academy of Sciences in Amsterdam:

> *"Why can we not speak of the ether instead of vacuum? Space and time are not symmetric; a material point can at different times be at different spots, but not in different places at the same time"*

Therefore, like sound waves, light waves have a longitudinal component or polarization associated with their gravitational mass through Higgs mechanism [35]. Space compressibility was shown [91] to account for *causal* nature of relativistic effects such as Lorentz-FitzGerald contraction when Michelson number $Mi = v/c$ approaches unity thus leading to Poincaré-Lorentz *dynamic theory of relativity* as

opposed to Einstein *kinematic theory of relativity* [91] in harmony with ideas of Darrigol [92] and Galison [93].

Although velocity of light is nearly constant, it is expected that it actually decreases extremely slowly, at cosmological time scale i.e., eons, with temperature of Casimir [75] vacuum due to expansion of the universe. Because of closure of the gap between radiation and gas theory [36], one could view Planck energy spectrum of equilibrium radiation as energy spectrum *of* Sackur "*clusters*" or Planck [94] *"quantum sphere of action"* as described by Darrigol [95-96] and in harmony with perceptions of de Broglie [36,97,98]

> *"Existence of conglomerations of atoms of light whose movements are not independent but coherent"*

Hence, just like ideal gas, photon gas in Planck equilibrium radiation has granular structure and composed of ensembles of photon *clusters* with stationary energy, hence freuency (color), distribution.

The definition of Planck and Boltzmann universal constants in Equation (12) result in prediction of electromagnetic mass of photon, atomic-mass-unit *amu*, Avogadro-Loschmidt number, and universal gas constant as [34]

$$m_k = (hk/c^3)^{1/2} \approx 1.84278 \times 10^{-41} \ kg \tag{16}$$

$$amu = m_k c^2 = (hkc)^{1/2} \approx 1.6563 \times 10^{-27} \ kg \tag{17}$$

$$N^o = 1/(m_k c^2) = 1/(hkc)^{1/2} \approx 6.0376 \times 10^{26} \ kmol^{-1} \tag{18}$$

$$R^o = N^o k = (k/hc)^{1/2} \approx 8.338 \ kJ/kmol\text{-}m \tag{19}$$

Because it is known that all baryonic matter is made of atoms, by Equation (17) all matter in the universe is composed of photon or *electromagnetic mass* in harmony with perceptions of Newton [99], Lorentz [100], and Poincaré [101-103].

By equation (15), Kelvin absolute temperature T is identified as the most-probable or Wien wavelength $\lambda_{mp\beta} = \lambda_{w\beta}$ of particle thermal oscillations that is related to its de Broglie wavelength $\lambda_{r\beta} = \lambda_{d\beta}$ as [33, 34]

$$T_\beta = \langle \lambda^2_{mp\beta} \rangle^{1/2} = \lambda_{w\beta} = \lambda_{d\beta}/3 \tag{20}$$

Hence, zeroth law of thermodynamics of Fowler stating that equalities $T_A = T_B$ and $T_A = T_C$ require equality $T_B = T_C$, by Equation (20), gives $< \lambda^2_{wB} >^{1/2} =< \lambda^2_{wC} >^{1/2}$. Also, two kinds of "atomic" volume based on either Wien or de Borglie wavelengths [33]

$$\hat{v}_{w\beta} = T_\beta^3 = \lambda^3_{w\beta} \qquad , \qquad \hat{v}_{d\beta} = \lambda^3_{d\beta} \tag{21}$$

lead to corresponding total volumes

$$V_{w\beta} = N_\beta \hat{v}_{w\beta} \qquad , \qquad V_{d\beta} = N_\beta \hat{v}_{d\beta} \tag{22}$$

where $V_d = 27 V_w$. According to Equation (22), system volume increases either by increaseing temperature hence atomic volume in Equation (21) or by increasing the number of particles N_β.

Increase of system volume with temerature in Equation (22) is consistent with observed operation of thermometers since most substances expand upon heating, except anomaly for water in the temperatures range $0 < T < 4\,^{\circ}C$. Also, since like photon gas, pressure and internal energy of ideal gas are related by $p_\beta = \tilde{u}_\beta / 3$ [34], multiplication of Equation (22) with pressure results in ideal gas law

$$p_\beta V_\beta = p_\beta N_\beta \hat{v} = \frac{\tilde{u}_\beta \hat{v}}{3} N_\beta = \frac{\hat{u}_\beta}{3} N_\beta = N_\beta k T_\beta \tag{23}$$

As described in [33, 34], because harmonic motion of oscillators are assumed to occur in two directions (x^+, x^-), the classical and modified definitions of temperature and pressure are related by

$$T' = 2T \quad , \qquad p' = 2p \tag{24}$$

The factor of 2 in Equation (24) leads to a modified expression for the speed of sound in standard atmosphere [57]

$$c_s = v_{rm} = \sqrt{3kT/m} = \sqrt{3kT'/(2m)} \approx 343 \text{ m/s} \tag{25}$$

in closer agreement with observed speed of sound than the classical Newton formula $c_s' = \sqrt{p'/\rho}$. The result in Equation (25) is also in close agreement with Laplace's formula $c_s'' = \sqrt{\gamma k T'}$ with $\gamma = c_p / c_v$ denoting Poisson polytropic index. However, more recent investigation concerning physical foundation of Schrödinger equation [37] suggests that, as anticipated by Huygens [88] and described in the eloquent quotation above, analogous to the speed of light, the speed of sound is root-mean-square speed of molecules.

The results in Equation (15) regarding mass-energy equivalence and pioneering studies of Poincaré [101], de Pretto [104], Hasenöhrl [105], and Einstein [106] were discussed in an earlier study [34]. In the following we further examine the interesting but unfortunately little known study of Olinto de Pretto [104] who wrote:

> *"The formula mv^2 gives us $mv^2 / 8338$ the living force and the formula gives us, expressed in calories, such energy.*
>
> *Given that m = 1 and v equals three hundred thousand kilometers per second, that is 300 million meters, which would be the speed of light, also allowed for the ethere, everyone will see that you get an amount of calories represented by 10794 followed by 9 zeros and that is over ten million million."*

The formula of de Pretto in the above quotation written in the present notation is

$$mc^2 \ \text{kcal} = mc^2 / 8338 \ \text{Joule} \tag{26}$$

De Pretto number 8338 in Equation (26) was recently [107] identified as universal gas constant in Equation (19) expressed in MKS units

$$R^o = kN^o = 8338 \ \text{Joule/kcal-m} \tag{27}$$

Therefore, de Pretto Equation (26) shows that mechanical equivalent of heat should be numerically equal to $R^o = 8338$ Joule / kcal - m.

Accounting for the factor of 2 in definition of temperature $T'_\beta = 2T_\beta$ from Equation (24), one finds the modified value of Joule-Mayer mechanical equivalent of heat

$$J = 2J_c = 2 \times 4169 = 8338 \text{ Joule/kcal} \tag{28}$$

where the value $J_c = 4.169 \approx 4.17 \text{ kJ/kcal}$ is the average of the two values $J_c = (4.15, 4.19)$ reported by Pauli [108]. Although R^o in Equation (27) and J in Equation (28) are numerically equal, since their dimensions are different, this equality does not fully explain the nature of de Pretto Equation (26). This is because the same factor of 2 also occurs in the modified definition of pressure in Equation (24) making mechanical work on the left-hand-side of $-pV = -NkT$ to counterbalance the factor of 2 in temperature on its right-hand-side.

To reveal the true nature of mechanical equivalent of heat in de Pretto Equation (26), it is first noted that the universal gas constant may be expressed in terms of de Broglie wavelength from Equation (20) as

$$R^o = kN^o = \frac{k}{m_k c^2} = \frac{m_k \nu_{rk} c}{m_k c^2} = \frac{1}{\lambda_{rk}} = \frac{1}{\lambda_{dk}} \quad m^{-1} \tag{29}$$

Next, for ideal gas and by Equation (20), equal amounts of energy are expressed in terms of *atomic heat and work* as [34]

$$\hat{q} = Q/N = \hat{c}_v T = 3kT = \hat{u} = m_k c^2 \quad \text{kcal} \tag{30a}$$

$$\hat{w} = W/N = 3p\hat{v} = 3kT = 3k\lambda_w = k\lambda_d = m_k c^2 \text{ Joule} \tag{30b}$$

Equations (29, 30a, 30b) give de Pretto Equation (26)

$$\hat{q} = m_k c^2 \text{ kcal} = \frac{k\lambda_d}{R^o(\lambda_d = 1)} = \frac{\hat{w}}{J} = \frac{m_k c^2}{8338} \quad \text{Joule} \tag{31}$$

Thus, the *definition of kcal* as the unit of thermal energy is identified in

$$J = R^o \text{ (Joule/kcal.m).}(\lambda_d = 1 \text{ m}) = 8338 \text{ Joule/kcal} \tag{32}$$

as energy per unit de Broglie temperature $T_d = \lambda_d = 3\lambda_w = 3T = 1 \text{ m}$ corresponding to $p\hat{v} = kT = k/3$, $\hat{u} = 3kT = k$, and $\hat{h} = 4kT = 4k/3$. Multiplying Equation (31) by total number of particles N gives the classical Joule-Mayer mechanical equivalent of heat

$$W = JQ \tag{33}$$

It is now clear that calculations based on the classical definition of temperature $T'_\beta = 2T_\beta$ with the extra factor of 2 *fortuitously* compensated for the factor of ½ in the classical Joule-Mayer mechanical equivalent of heat $J_c = J/2 = 4.169 \text{ kJ/kcal}$ thus resulting in agreement with empirical observations.

Since by Equation (20) temperature is identified as a metric of space, in a recent study [77] the concepts of internal *spacetime* versus external *space* and *time* were introduced. Assuming that a statistical field at scale β is in thermodynamic equilibrium with the physical space at scale (β −1) within which it resides, both fields will have a homogenous constant temperature $T_\beta = T_{\beta-1}$ defined as

$$m_\beta u_\beta^2 = m_{\beta-1} v_{w\beta-1}^2 = kT_{\beta-1} = k\lambda_{w\beta-1} \tag{34}$$

Hence, by definition of most-probable or Wien speed $v_{ws} = \lambda_{ws}\nu_{ws} = \lambda_{ws}/\tau_{ws}$, one can associate constant internal measures of (extension, duration) [77]

$$\begin{aligned} \lambda_{ws} \quad &\text{Internal measure of extension} \\ \tau_{ws} \quad &\text{Internal measure of duration} \end{aligned} \tag{35}$$

with every "point" or "atom" of space at constant temperature $T_\beta = T_{\beta-1}$. Next, external space and time that are independent of each other are defined in terms of the internal spacetime as

$$(x_\beta, y_\beta, z_\beta) = (N_{x\beta}, N_{y\beta}, N_{z\beta})\lambda_{w\beta-1} \qquad , \qquad t_\beta = N_{t\beta}\tau_{w\beta-1} \tag{36}$$

with the four numbers $(N_{x\beta}, N_{y\beta}, N_{z\beta}, N_{t\beta})$ being independent numbers. For example, at cosmic scale $\beta = g$, the *internal* (ruler, clock) of astrophysics $\beta = s$ are employed to define *external* independent space and time coordinates by Equation (36) [77]. Therefore, internal spacetime in Equation (35) provides *local* structure of spacetime, whereas the external space and time in Equation (36) describe *global* dynamics of the system and are irreversible. According to Equations (35-36) both internal and external space and times are quantized. Irreversibility of space coordinates corresponds to non-commutative geometry [76] discussed in previous Section. The broad spectrum of physical time scales has been described in an excellent recent book [109]. The four dimensions $(x_\beta, y_\beta, z_\beta, t_\beta)$ with three real space and one imaginary time coordinates represent Poincaré [103] and Minkowski [110] four-dimensional spacetime manifold.

It was emphasized by Pauli [108] that time dimension does not occur in classical thermodynamics. However, according to Rovelli [111], thermodynamics plays a central role in the definition of *physical time*

> *"The very concept of time is meaningful only in the thermodynamic context"*

The definition of internal spacetime in Equation (35) is based on thermodynamic equilibrium since the objective is to define what the variable called *physical time* represents as discussed by Rovelli [111]

> *"However, we are not concern here with versus of the time flow: we are concerned with definition of a variable that represents time"*

Therefore, the *external time* quantitatively defined in Equation (36) is called *Rovelli thermal time*. It is ironic that the definition of physical time i.e. *Rovelli thermal time* must be based on systems that are at thermodynamic equilibrium hence *stochastically stationary* thus *timeless*. This is in harmony with the perceptions of both Aristotle [112] and St Augustine [113] who noted that *the essence of time appears to be in what it is not*. Of course, in all real processes dissipations lead to entropy generation resulting in change in temperature hence internal measures of *spacetime* in Equation (35). For example, in cosmology, the internal measure of spacetime change extremely slowly (eons) due to changes in Penzias-Wilson [114] cosmic background microwave radiation temperature of Casimir [75] vacuum $T_{\text{CMB}} \approx 2.73$ m caused by dissipations as well as expansion of the universe.

Since according to Equation (36), external (ruler, clock) = (x_β, t_β) at scale β are always defined in terms of internal spacetime = $(\lambda_{w\beta-1}, \tau_{w\beta-1})$ at the next lower scale, the definition of (extension, duration)

= (space, time) can always be relegated to lower scales ad infinitum. This is because infinite divisibility of both extension and duration must follow the philosophy of Anaxagoras disccuessed by Bell [115]

> *"Neither is there a smallest part of what is small, but there is always a smaller, for it is impossible that what is should ever cease to be"*

Hence, *absolute zero* and *absolute infinite* (extension, duration) are singularities and concieved as ideal Aristotle *potential limits* that strictly speaking are never actualized.

If one does not wish to allow for infinite divisibility of time, then following *Leibniz* [84] one must introduce the *temporal monad* just like the *spatial monad* to represent the absolute smallest "atom" of time or *Chronon*. The potentially infinite divisibility of time is related to the paradox of the present moment of time, instanton [116] or "now" that separates two *non-existent worlds*, a past already lived and a future yet to come. Poincaré gives an eloquent description of the mystery of life through "now" at the very end of his wonderful book [117]

> *"And yet--strange contradictions for those who believe in time--geologic history shows us that life is only a short episode between two eternities of death, and that, even in this episode, conscious thought has lasted and will last only a moment. Thought is only a gleam in the midst of a long night. But it is this gleam which is everything."*

It is interesting that an *instant of time* in cosmology was estimated [77] as 100,000 years, namely the time it takes to cross the Milky Way travelling at the speed of light.

Because of change of units m/cm = 100 required by Wien displacement law $\lambda_w T = 0.29 \text{ cm.K} = 0.0029 \text{ m}^2$, the classical formula for temperature conversion $T[\text{K}] = {}^o\text{C} + 273.1$ is modiefied as

$$T(\text{m}) = {}^o C(\text{m}) + 2.731(\text{m}) \tag{37}$$

It is important to note that the constant 2.731 m in Equation (37) is close to Penzias-Wilson [114] cosmic microwave background radiation temperature $T_{\text{CMB}} \approx 2.73 \text{ m}$.

It is also interesting to examine implication of change of dimension of temperature to the physical meaning of Planck [118] temperature

$$T_p = (hc^5 / Gk^2)^{1/2} \approx 3.55 \times 10^{32} \text{ m} \tag{38}$$

as cosmic temperature at the initial explosion of Lemaître [119] "*primordial atom*" of our universe. Because by Equation (20) absolute temerature is the most-probable or Wien wavelength of thermal oscillations $T_\beta = \lambda_{w\beta}$, Planck temperature in Equation (38) is puzzling since it is much larger than the size of our universe $L_g \approx 10^{26} \text{ m}$ reported in the literature. Based on known approximate length scales of our uinverse, and the scale facor of 10^{17} in Figure 3, one may consider the (atom, element, system) = $(\ell_\beta, \lambda_\beta, L_\beta)$ lengths,

$$\text{EGD} \quad (\ell_g, \lambda_g, L_g) = (10^{18}, 10^{26}, 10^{35}) \text{ m} \tag{39a}$$

$$\text{EUD} \quad (\ell_u, \lambda_u, L_u) = (10^{26}, 10^{35}, 10^{43}) \text{ m} \tag{39b}$$

repectively corresponding to cosmology or equilibrium galactic-dynamics EGD at scale $\beta = g$ (Figure 1) and the next major statistical field called *equilibrium-universal-dynamics* EUD $\beta = u$ (not shown in Figure 1), in harmony with many-uinverse concept of Everett [120]. Hence, what is believed to be

the present size of our universe appears to correspond to the *visible horizon* associated with the *most-probable galactic-cluster* in a much larger ultra- cosmic field required in order to accommodate photons with mean-free-paths as long as $T_g = T_s = \lambda_{wg}$ in Equation (38). Clearly, this result enhances the probability of existence of *multiverse* in harmony with the hierarchy shown in Figure 1, the perceptions of Everett [120], as well as inflationary models of cosmology [121-124].

4. Helmholtz Decomposition of Energy and the First Law of Thermodynamics

The empirical foundation of principle of conservation of energy or the first law of thermodynamics was based on experimental studies by Mayer and Joule focused on mechanical equivalent of heat and performance of heat engines compared to ideal Carnot cycle. Therefore, classical form of the first law of thermodynamics [108,125]

$$dE = \delta Q - \delta W \tag{40}$$

expresses the change of system energy as a result of heat added to the system and work done by the system

$$\text{Total energy inside system} = \text{Heat input} - \text{Work output}$$

However, it was emphasized by Warner [126] that the form in Equation (40) is only applicable to reversible processes and some authors have instead used an alternative formulation

$$dE = \delta Q + \delta W' \tag{41}$$

where $\delta W'$ is the work done on the system by external forces that are not derivable from a potential, and dE and δQ retain the same meaning as in Equation (40).

On the other hand, theoretical foundation of the first law of thermodynamics was first introduced by Helmholtz [127] and Clausius [128, 129] through application of virial theorem to weakly interacting particles such as ideal gas. Helmholtz in his pioneering 1847 paper concerning conservation of energy argued that the *quantity of heat* or thermal energy of the system may be decomposed into two parts respectively associated with the *kinetic energy* or *vis viva* of particles and *potential energy* due to the stress or pressure field caused by their mutual interactions [127]

> *"That which has been heretofore named the quantity of heat, according to this, be the expression, first, of the quantity of vis viva of the caloric motion, and secondly, of the quantity of those tensions between the atoms, which, by changing the arrangement of the latter, such a motion can develop. The first portion would correspond to that which has been heretofore called free heat, the second with that which has been named latent heat".*

Following Helmholtz [127], the first law of thermodynamics is expressed as [33,34],

$$\text{System total thermal energy} = \text{Free heat} + \text{Laten heat} \tag{42}$$

When *free heat* and *latent heat* are respectively identified as internal energy U and potential energy pV, such that total thermal energy of the system becomes enthalpy $H = Q$, and work is identified as Helmholtz free energy $W = F = U - TS = -pV$, by Equation (42), the modified form of the first law of thermodynamics becomes [33, 34]

$$dQ = dH = d(TS) = dU - dF = dU - dW = dU + d(pV) \tag{43}$$

with *reversible heat* and *reversible work* defined as

$$Q = H = TS \qquad \text{Reversible heat} \tag{44}$$

$$W = F = -pV \qquad \text{Reversible work} \tag{45}$$

In Equation (45), the choice of negative sign of reversible work or Helmholtz free energy $W = F = U - TS = -pV$ ensures that *work done on the system* by compression thus reducing its volume $V_2 < V_1$ will be positive $W = -p(V_2 - V_1) > 0$ as required. The first law of thermodynamics in Equations (43-45) that is based on perceptions of Helmholtz [127] requires new scientific paradigms involving a modified *mechanical theory of heat* as well as a new definition of *potential energy* that are described in the following.

As discussed in [33,34], according to Equations (1,3), particle translational velocity is the sum of the mean or "cluster" velocity and its peculiar velocity

$$\mathbf{u}_{mj} = \mathbf{v}_{mj} + \mathbf{V}'_{mj} \tag{46}$$

The inclusion of peculiar velocity is of central importance since it defines what is called *potential energy* of particle. For example, in cosmology, peculiar velocity of a galaxy is the difference between its velocity and the velocity of the cluster of galaxies to which it belongs. In ideal gas at thermodynamic *equilibrium,* all three velocities in Equation (46) will be random such that Equation (46) after being squared, averaged, and multiplied by particle mass leads to kinetic and potential energies [34]

$$\hat{u}_t = \frac{1}{2} m \overline{v^2_{wmx+}} + \frac{1}{2} m \overline{v^2_{wmx-}} = \frac{1}{2} m \overline{u^2_{cx+}} + \frac{1}{2} m \overline{u^2_{cx-}} = m \overline{v^2_{wmx+}} \tag{47}$$

$$\hat{\varepsilon}_p = \frac{1}{2} m \overline{V'^2_{wmx+}} + \frac{1}{2} m \overline{V'^2_{wmx-}} = m \overline{V'^2_{wmx+}} = \frac{1}{3} m \overline{V'^2_{wm+}} = p\hat{v} \tag{48}$$

with pressure defined as $p = nm \overline{V'^2_{wm+}} / 3$. Therefore, the total energy due to harmonic translational motion of particle called *atomic translational emthalpy* becomes

$$\hat{h}_t = \hat{u}_t + \hat{\varepsilon}_p = \hat{u}_t + p\hat{v} = 2kT \tag{49}$$

As discussed in [33, 34], the conventional practice of attributing identical translational kinetic energy $(\hat{u}_{tx}, \hat{u}_{ty}, \hat{u}_{tz})$ to all three coordinate directions (x, y, z) is not appropriate because a particle cannot *simultaneously* move along three *independent coordinate directions*. In addition, as was emphasized by Clausius [128], rotational and vibrational energy of particles cannot be properly neglected

> *"In liquids, therefore, an oscillatory, a rotatory, and a translator motion of the molecules take place, but in such a manner that these molecules are not themselves separated from each other, but even in the absence of external forces, remain within a certain volume"*

Following Clausius [128,129], harmonic translational kinetic energy in (y^+, y^-) and (z^+, z^-) directions are instead attributed to kinetic energies due to particles harmonic rotational (θ^+, θ^-) and vibrational (r^+, r^-) motions in two coordinate directions [34]

$$\hat{u}_r = \varepsilon_r = \frac{1}{2} I \overline{\omega^2_{m\theta+}} + \frac{1}{2} I \overline{\omega^2_{m\theta-}} = I \overline{\omega^2_{m\theta+}} \tag{50}$$

$$\hat{u}_v = \varepsilon_v = \frac{1}{2}\chi\overline{r_{m+}^2} + \frac{1}{2}\chi\overline{r_{m-}^2} = \chi\overline{r_{m+}^2} \tag{51}$$

with (I, χ) respectively denoting particle moment of inertia and spring constant.

Next, total atomic internal energy is defined as the sum of translational, rotational, and vibrational atomic internal energies from Equations (47), (50), and (51) as

$$\hat{u} = \hat{u}_t + \hat{u}_r + \hat{u}_v = m\overline{v_{x+}^2} + I\overline{\omega_{x+}^2} + \chi r_{x+}^2 \tag{52}$$

Since Boltzmann principle of equipartition of energy requires all four degrees of freedom to have the same energy

$$\hat{u}_t = \hat{u}_r = \hat{u}_v = p\hat{v} = kT \tag{53}$$

atomic enthaply becomes

$$\hat{h} = \hat{u} + p\hat{v} = 4kT \tag{54}$$

Multiplication of Equation (54) with number of atoms N results in what Sommerfeld [125] called *total heat* or enthalpy

$$H = U + pV \tag{55}$$

As emphasized by Pauli [108],

> *"In thermodynamics no considerations concerning the nature of heat are made. This problem is first dealt with in the kinetic theory of gases."*

The fact that a satisfactory mechanical theory of heat in thermodynamics was not developed since the pioneering work of Helmholtz [127] could be in part due to the neglect of *peculiar velocity* of particles in Equation (46). Since peculiar velocity represents an *independent translational degree of freedom*, associated with particle *potential energy* in Equation (48) hence Poincaré stress [35, 37], without its inclusion in the kinetic theory of gases a satisfactory mechanical theory of heat cannot be realized. Equations (53)-(55) constitute a "*derivation*" of the first law of thermodynamics and the state function H, named *enthalpy* by Onnes and called "*heat function*" or "*total heat*" by Sommerfeld [125], can be identified as the *mechanical theory of heat in thermodynamics* that was called for in the above quotation from Pauli [108].

The first law of thermodynamics in Equation (43) can be directly derived from classical results [125] by addition of Gibbs equation

$$TdS = dU + pdV - \sum_i \mu_i dN_i \tag{56a}$$

and Gibbs-Duhem equation

$$SdT = Vdp - \sum_i N_i d\mu_i \tag{56b}$$

hence

$$d(ST) = dU + d(pV) - \sum_i d(N_i g_i) \tag{57}$$

In non-reactive systems $dN_i = 0$, Equation (57) reduces to Equation (43) since Gibbs chemical potential $\mu_i = g_i$ of ideal gas is zero [33,34]

$$\mu_i = g_i = (\partial U / \partial N_i)_{S,N,j\neq i} = 0 \tag{58}$$

Modified form of the first law of thermodynamics in Equation (43) is to be compared with the classical form [108, 125]

$$dU = \delta Q - \delta W \tag{59}$$

with heat δQ and work δW that are *path-dependent* hence not state functions. Since both reversible heat and work in Equations, (44)-(45) are state functions, the paradox of exact cancellation of *path-dependence* of heat and work postulated in classical thermodynamics [108,125]

$$\oiint (\delta Q - \delta W) = 0 \tag{60}$$

in order to obtain an exact differential dU in Equation (59) is resolved.

To reveal the origin of path-dependence one notes that both reversible heat $Q(T,S) = TS$ and reversible work $W(p,V) = -pV$ depend on *two independent variables* hence the total differentials

$$dQ = TdS + SdT \quad , \quad dW = -pdV - Vdp \tag{61}$$

Because *reversible* heat and work transfer are achieved respectively under *isothermal SdT = 0* and *isobaric Vdp = 0* conditions, the last terms in both parts of Equation (61) become identically zero. If, however, such *reversibility conditions* are not satisfied but the last terms in Equation (61) are nevertheless neglected, then differentials δQ_{irr} and δW_{irr} will no longer be exact and become path-dependent.

For an ideal gas, enthalpy $H = 4NkT = ST$, internal energy $U = 3NkT = (3/4)TS$, and work or potential energy $W = -PV = -NkT = -TS/4$ are all state functions involving Boltzmann constant k that always occurs with absolute temperature due to the definition of atomic energy $kT_\beta = \hat{\varepsilon}_\beta$. However, the universal constant k, first calculated by Planck and named Boltzmann constant, also occurs in Boltzmann formula for entropy $S = k \ln W$. Therefore, to follow the notation of classical thermodynamics, in the definitions of (*H*, *U*, *W*) above, one groups k with *4N* in the definition of entropy $S = 4kN$ [33, 34] rather than with *T* in the definition of "atomic" energy $kT = \hat{\varepsilon}$.

On a related subject matter concerning exact differentials, because of the modified definition of velocity potential $\mathbf{v} = -\nabla\Phi$ [35, 37], the *complex velocity potential* becomes

$$Z(z) = \tilde{\Psi}(x,y) + i\Phi(x,y) \tag{62}$$

and velocity potential Φ rather than stream function $\tilde{\Psi}$ becomes the imaginary part of Z(z) such that with complex variable $z = x + iy$, Cauchy-Riemann conditions for analyticity of Z(z) become

$$\Phi_x = -\tilde{\Psi}_y \quad , \quad \Phi_y = \tilde{\Psi}_x \tag{63}$$

This modification is important because according to invaraint Schrödinger equation [37], quantum mechanics wave function Ψ is related to the *velocity potential of particle peculier motion Φ'*. Thus, in harmony with perceptions of de Broglie [3], the complex wave function Ψ guides the motion of particle that is a singularity, i.e., wave-packet at lower scale $\beta-1$, on this wave [37]. Since Φ', $\tilde{\Psi}$, and Z satisfy Cauchy-Riemann conditions in Equation (63), they are all *harmonic functions* that satisfy

Laplace equation and hence wave equations. Stationary waves of T, ρ, and p lead to waves of energy (H, U, W) associated with *stationary states* corresponding to solutions of time-independent Schrödinger equation as described in [37].

Since according to Gibbs's phase rule, given any state function H, U, … the state of a pure substance is defined by the remaing two thermdynamic variables, one introduces the complex variable $z = T + iS$ with the imaginary axis corresponding to entropy. Because of *virtual* nature of Heisenbery-Kramers [51] oscillators, associating *imaginary numbers* with S rather than T is reasonable. Thus, with vertical axis corresponding to complex part of $z = T + iS$, Carnot cycle is shown as (*S-T*) diagram in Figure 6.
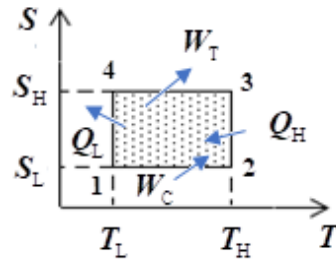

Figure 6. *Carnot cycle.*

For an ideal gas, the total system thermal energy $H = ST = 4\mathrm{k}NT$ can change by either changing the number of oscillators $S = 4\mathrm{k}N$, or by changing the energy per oscillator $\hat{\varepsilon} = \mathrm{k}T$ hence T, or both. In Figure 6, heating $2 \rightarrow 3$ and cooling $4 \rightarrow 1$ processes in boiler and condenser occur *isothermally* through increase and decrease of the number of oscillators $\Delta Q = (4\mathrm{k}T)\Delta N$ such as occur by evaporation and condensation during change of phase. On the other hand, work transfer in turbine $3 \rightarrow 4$ and compressor $1 \rightarrow 2$ occur at constant entropy (isentropic) by changing temperature $\Delta W = -\mathrm{k}N\Delta T$ due to cooling and heating of the working fluid by expansion and compression. It is noted that isentropic change of volume $V = N\hat{v}$ during expansion $3 \rightarrow 4$ or compression $1 \rightarrow 2$ occur through changes of temperature hence *atomic volume* $\hat{v} = T^3 = \lambda_w^3$ [33] rather than S hence number of virtual oscillators N.

Closure of the gap between Planck theory of equilibrium photon gas and Boltzmann kinetic theory of ideal gas [36], requires the total energy of a system composed of either gas be the enthalpy H rather than the internal energy U as reported in some literature. To show this, one notes that for equilibrium radiation, the number of photons in volume V of Casimir [75] vacuum is given as [36]

$$N = \frac{8\pi^5 V}{45}\left(\frac{\mathrm{k}T}{\mathrm{hc}}\right)^3 \tag{64}$$

Integration of Planck energy distribution in Equation (6) for equilibrium photon gas over the total frequency range $v = (0 \rightarrow \infty)$ gives

$$<E> = T'S = \frac{8\pi^5 V}{90}\frac{\mathrm{k}^4 T'^4}{\mathrm{h}^3 \mathrm{c}^3} \tag{65}$$

Substitutions for $T' = 2T$ and N from Equations (24) and (64) in Equation (65) give

$$H = TS = \frac{4}{3}\left\{3\left[\frac{8\pi^5 V}{45}\left(\frac{\mathrm{k}T}{\mathrm{hc}}\right)^3\right]\mathrm{k}T\right\} = \frac{4}{3}(3N\mathrm{k}T) = \frac{4}{3}U = 4N\mathrm{k}T \tag{66}$$

The result in Equation (66) is in agreement with the relation between total energy and electromagnetic energy of photon predicted by Hasenöhrl [105,130]

$$H_\beta = \mathrm{M_t c^2} = \frac{4}{3}\mathrm{M_{EM}c^2} \tag{67}$$

and relates photon total mass and electro-magnetic mass [37] as

$$\mathrm{M_t} = \frac{4}{3}\mathrm{M_{EM}} \tag{68}$$

Since the prediction of Hasenöhrl [105,130] in Equation (67) is for equilibrium radiation in enclosures, photons will be under *stationary-state* and must oscillate at the most probable or Wien speed

$$v^2 = v^2_{kw} = v^2_{kw+} + v^2_{kw-} = 2v^2_{kw+} \tag{69}$$

Substituting for v^2 from Equation (69) in Lorentz [100] relativistic mass formula

$$m_r = m_o / \sqrt{1 - v^2/c^2} \tag{70}$$

results in

$$m_{rk} = m_{ok}(1 + v^2_{kw}/2c^2) = (1 + v^2_{kw+}/c^2) = m_{ok}(1 + 1/3) = (4/3)m_k \tag{71}$$

in accordance with Equations (67, 68).

The above results resolve the classical problem of total radiant energy under Planck distribution function being internal energy $<E> = U$ instead of the enthalpy $<E> = H$. Interestingly, the correct relationship between enthalpy $H = TS$ and internal energy U in Equation (67) also appear in the books by both Planck [118] as well as Sommerfeld [125]. However, classical expressions for the number of photons in vacuum instead of the result in Equation (64) do not lead to the correct total energy H under Planck distribution curve given in Equation (66).

The result in Equation (71) is of fundamental significance because exactly ¾ and ¼ of the total thermal energy under Planck [94] distribution curve fall on $v > v_w$ and $v < v_w$ sides of Wien frequency v_w [77] as shown in Figure 7.
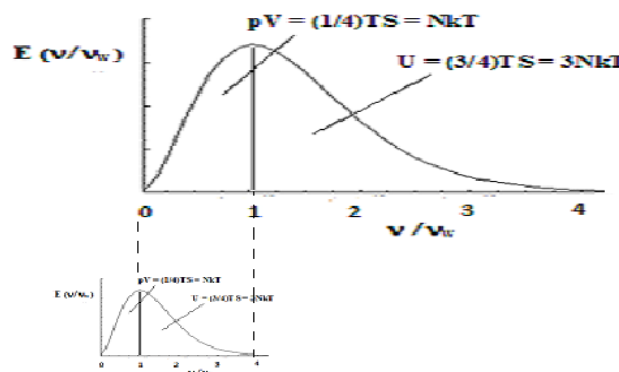

Figure 7. *Re-normalized Planck energy distribution as a function of* v/v_w *[77]*

For an ideal gas the potential and internal energy are related by [33, 34]

$$p_\beta V_\beta = \frac{U_\beta}{3} = \frac{3N\mathrm{k}T_\beta}{3} = N\mathrm{k}T_\beta \tag{72}$$

From Equation (55), enthalpy and internal energy are related by

$$H_\beta = U_\beta + \frac{U_\beta}{3} = U_\beta + \frac{H_\beta}{4} = U_\beta + \frac{1}{4}(U_\beta + \frac{U_\beta}{3}) = \ \dots = U_\beta (1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \dots) = \frac{4}{3} U_\beta \tag{73}$$

Since according to Figure 7, potential energy of scale β is identified as the total thermal energy or enthalpy of the lower scale $\beta-1$ [33]

$$DM_\beta = p_\beta V_\beta = \frac{H_\beta}{4} = H_{\beta-1} \tag{74}$$

total system energy in Equation (73) may be expressed as

$$H_\beta = U_\beta + \frac{H_\beta}{4} = U_\beta + H_{\beta-1} = U_\beta + U_{\beta-1} + \frac{H_{\beta-1}}{4} = \dots = \sum_{n=0} U_{\beta-n} \tag{75}$$

Applying Helmholtz decomposition at cosmic scale $\beta = g$, of the total thermal energy H of the universe, ¾ is *free heat* or *dark-energy* and is associated with *electromagnetic mass*, and ¼ is *laten heat* or *dark-matter* and is associated with *gravitational mass* [33, 131-135]. The ¾ and ¼ division of energy in Equations (67) and (74) is also in accordance with recent observations [136-139] and general theory of relativity [140-141] as described by Pauli [141]

> "*The energy of a spatially finite universe is three-quarters electromagnetic and one-quarter gravitational in origin*"

Scale invariance of the model suggests that dark matter in Equation (74) may be expressed as [33]

$$DM_{\beta+1} = H_\beta = U_\beta + p_\beta V_\beta = DE_\beta + DM_\beta = DE_\beta + DE_{\beta-1} + DM_{\beta-2} = \dots = \sum_{\beta}^{\beta-\infty} DE_\beta \tag{76}$$

By Equations (18,74,76), all baryonic matter in the universe is made of *dark energy* or electromagnetic mass as anticipated by Newton [99], Lorentz [100] and Poincaré [101-103].

5. Clausius, Boltzmann, and the Second Law of Thermodynamics

In classical thermodynamics, definition of entropy introduced by Clausius is based on reversible heat as

$$dS = \frac{dQ_{rev}}{T} \tag{77}$$

In view of Equation (43) and reversible heat in Equation (44), entropy of ideal gas is directly related to enthalpy by

$$S = \frac{Q_{rev}}{T} = \frac{H}{T} \tag{78}$$

Hence, entropy of an ideal gas by Equations (77) and (78) becomes [33, 34]

$$S = 4kN \tag{79}$$

The direct connections of entropy S and total heat or enthalpy H hence the number of Heisenberg-Kramers [51] virtual oscillators reveal its *objective* rather than *subjective* nature as discussed in [142] in connection to Shannon information theory.

For isolated systems, constant total system energy gives

$$dQ = dH = d(TS) = 0 \tag{80}$$

leading directly to the second law of thermodynamics

$$TdS = -SdT \geq 0 \tag{81}$$

Since by Clausius statement of second law, heat must always spontaneously flow from high to low temperature $T_H \to T_L$, one must have $dT \leq 0$ leading to Equation (81). Therefore, in isolated systems ordered (correlated) motions *transform*, hence the name entropy chosen by Clausius [129], into disordered (un-correlated or random) motions leading to increase of system entropy $dS = 4kdN \geq 0$ by Equation (81). As emphasized by Boltzmann [143], the reversal of the process hence de-randomization of microscopic thermal motions back into macroscopic ordered motions will be extremely unlikely bordering impossibility. It is well known that Eddington arrow of time arises in all real hence none-equilibrium processes and is directly connected to irreversibility associated with increase of entropy by dissipative processes [35,36,77,144]. The Zemerlo-Boltzmann debates concerning reversibility and Poincaré recurrence theorem were discussed in [36].

The statistical nature of entropy is based on the fundamental question asked by both Boltzmann and Planck namely: given the total energy H and the total number of particles or oscillators N, what is the distribution of numbers N_j of indistinguishable particles placed in g_j distinguishable boxes or "clusters" in order to achieve *equilibrium* speed (Maxwell-Boltzmann) and energy (Planck) distributions. Of course, the particles are under constant motion by transition between clusters that are *stochastically stationary* by the principle of detailed balance. Clearly, this question directly relates to the problems of kinetic theory of ideal gas of Boltzmann [143] and theory of equilibrium radiation of photon gas of Planck [94]. Boltzmann anticipated quantum mechanics by about three decades as evidenced by the following quotation taken from his pioneering and often neglected 1872 paper [145],

> *"We wish to replace the continuous variable x by a series of discrete values $\varepsilon, 2\varepsilon, 3\varepsilon, \dots p\varepsilon$. Hence, we must assume that our molecules are not able to take up a continuous series of kinetic energy values, but rather only values that are multiples of a certain quantity ε. Otherwise, we shall treat exactly the same problem as before. We have many gas molecules in a space R. They are able to have only the following kinetic energies:*
>
> $$\varepsilon_j, 2\varepsilon_j, 3\varepsilon_j, 4\varepsilon_j, \dots p\varepsilon_j$$
>
> *No molecule may have an intermediate or greater energy. When two molecules collide, they can change their kinetic energies in many different ways. However, after the collision the kinetic energy of each molecule must always be a multiple of ε. I certainly do not need to remark that for the moment we are not concerned with a real physical problem. It would be difficult to imagine an apparatus that could regulate the collisions of two bodies in such a way that their kinetic energies after a collision are always multiples of ε. That is not a question here."*

Both Boltzmann [143, 145] and Planck [94] applied combinatorics to arrive at the number of what Boltzmann called "complexions" for distributing N_j *indistinguishable* particles among g_j *distinguishable* boxes or cells

$$W_j = \frac{(N_j + g_j - 1)!}{N_j!(g_j - 1)!} \quad , \qquad W = \prod_j W_j \tag{82}$$

When number of boxes or cells g_j is viewed as number of particle "clusters", because the smallest cluster can contain only a single particle, the number of *available* clusters or quantum states will be $(g_j - 1)$. As a result, the inverse of Planck- Boltzmann Equation (82) is exactly the *probability* of distributing N_j *indistinguishable* oscillators amongst $(g_j - 1)$ *distinguishable available* clusters [33, 34],

$$p_j = \frac{N_j!(g_j-1)!}{(N_j+g_j-1)!} = \frac{1}{W_j} \qquad , \qquad p = \prod_j p_j = \frac{1}{W} \tag{83}$$

In the realistic limit $g_j \gg N_j \gg 1$ all three statistical fields namely Boltzmann, Fermi-Dirac, and Bose-Einstein statisitcs reduce to what is called *corrected* Boltzmann statistics [55,56]

$$W_j = \frac{g_j^{N_j}}{N_j!} \tag{84}$$

Substituting for g_j in terms of the volume V_j of cluster j and its atomic volume $\hat{v}_j = T^3 = \lambda_{wj}^3$ the number of complexions in Equation (84) simplifies to

$$W_{1j} = \frac{(V_j / \lambda_{wj}^3)^{N_j}}{N_j!} = \frac{N_j^{N_j}}{N_j!} \approx e^{N_j} \qquad , \qquad W_1 = \prod_j W_{1j} \approx e^N \tag{85}$$

that is in accordance with partition function Z of an ideal gas given by Kardar [56]. Next, one accounts for four degrees of freedom associated with translational, rotational, vibrational, and potential energy of each oscillator and by (85) and Boltzmann equipartition principle the total numbers of complexions become [33, 34]

$$W_j = W_{4j} = W_{tj}W_{rj}W_{vj}W_{pj} = e^{4N_j} \qquad , \qquad W = \prod_j W_{4j} \approx e^{4N} \tag{86}$$

In view of Equation (83) Boltzmann [143, 145] formula for entropy expressed in terms of probabilities is [33],

$$S_j = -k\ln p_j \qquad , \qquad S = \sum_j S_j = -k\ln p \tag{87}$$

Following Boltzmann [143, 145], one introduces "atomic" probability as

$$\hat{p} = e^{-\hat{h}/kT} \tag{88}$$

By definition of atomic enthalpy in Equation (54) and Equations (86)-(88), entropy of "atom", cluster-j, and system become [33, 34]

$$\hat{s} = -k\ln\hat{p} = 4k \qquad , \qquad \hat{p} = e^{-\hat{h}/kT} = e^{-4} \tag{89}$$

$$S_j = \sum_j \hat{s} = -k\ln p_j = 4kN_j \qquad , \qquad p_j = \prod_j \hat{p} = e^{-H_j/kT} = e^{-4N_j} \tag{90}$$

$$S = \sum_j S_j = -k\ln p = 4kN \qquad , \qquad p = \prod_j p_j = e^{-H/kT} = e^{-4N} \tag{91}$$

for statistically independent clusters.

According to Equations (89-91), entropy of ideal gas and photon gas are related to the number of Heisenberg-Kramers [51] or Planck [94] virtual oscillators. Due to inevitable presence of dissipations in all real processes, energy is *transformed* from *macroscopic ordered motions* to *microscopic random motions* known as heat thus increasing the entropy of the system by Equations (77-78). Also, by Equation (89), atomic entropy is 4 times Boltzmann universal constant k and hence independent of particular chemical species. Furthermore, from Equations (91) and (32), molar entropy of ideal gas is equal to universal gas constant $\tilde{S} = N^o k = R^o$. Direct connection between entropy and heat according to its macroscopic definition in Equation (77) of Clausius or Equation (78) reveals its *objective* nature as opposed to its *subjective* interpretation in Shannon information theory [142]. For example, when liquid water evaporates at constant pressure and temperature, the relatively small number of large oscillators in liquid phase convert to large numbers of small oscillators in vapor phase thus increasing entropy $S = 4kN$ while keeping atomic energy $\hat{\varepsilon} = kT$ constant [33]. With oscillators acting as composite

bosons [53, 54], like photon gas, equilibrium energy spectrum of ideal gas is given by invaraint Planck [94] distribution in Equation (5b) that is weakly dependant on the number of oscillators on account of their large number, 10^{26} per one kmol that is practically infinite.

As stated in Section 3, space is a compressible tachyon fluid in harmony with perceptions of Aristotle [112], Huygens [88], Newoton [99], Euler [146], Lorentz [100], Poincaré [101-103], de Broglie [3], Casimir [75], and Dirac [86] amongst others. With Casimir [75] vacuum identified as superfluid liquid photon or Bose-Einstein condensate, it is reasonable to anticipate that when heated to a critical vaporization or *boiling temperature* T_b, space will nucleate photon gas bubbles that following Dirac [142] could be referred to as *holes* or *anti-matter* particles. Similarly, if Casimir vacuum is cooled below certain critical fusion or *melting temperature* T_m, liquid photon is expected to undergo a second phase transition and become solidified thus forming *solid-light* that was defined as black hole BH [148]. Such phase transition of physical space is harmonious with finite *zero point energy* hence *zero-point pressure* of Casimir [75] vacuum according to the modified form of the van der Waals *law of corresponding states* [149]

$$p_{rg} = p_{ra} - p_{rv} = \frac{1}{Z_c}\left[\frac{T_r}{\tilde{v}_r - 1/3} - \frac{9}{8\tilde{v}_r^2} + Z_c - \frac{3}{8}\right] \qquad (92)$$

with predictions that are closer to experimental data as shown in Figure 8.
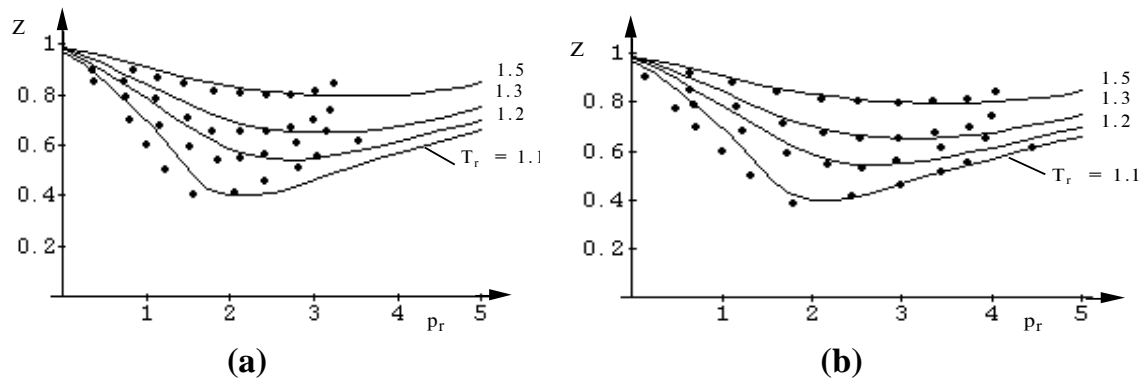

Figure 8. *Comparisons between experimental data (solid lines) and the calculations (solid circles) based on (a) van der Waals (b) modified van der Waals equation of state [149]*

Since $\tilde{v}_r \to \infty$ leads to $p_{ra} \to 0$ and for most substances $Z_c \leq Z_{cVDW} = 3/8$, Equation (92) results in positive vacuum pressure [149]

$$p_{rv} = \frac{3/8}{Z_c} - 1 \geq 0 \qquad (93)$$

Also, since pressure can be viewed as volumetric potential energy density of the field, negative pressures are expected to be non-physical [34]. It is reasonable to suggest that *Dirac Sea* be associated with anti-matter and correspond to pressures lower than that of Casimir [75] vacuum $0 \leq p_{ra} \leq p_{rv}$ thus helping to avoid negatively induced cosmological constant [150].

Therefore, pressures of matter p_m and anti-matter p_{am} will be respectively larger and smaller than pressure p_v of Casimir vacuum [149]

$$0 = p_{WH} < p_{am} < p_v < p_m < p_{BH} = \infty \qquad (94)$$

with *white hole* $p_{WH} = 0$ and *black hole* $p_{BH} = \infty$ pressures identified as two singularities of the field. A schematic diagram of physical space as a compressible tachyon fluid from infinite rarefaction (*white hole* WH) to infinite compression (*black hole* BH) is shown in Figure 9.
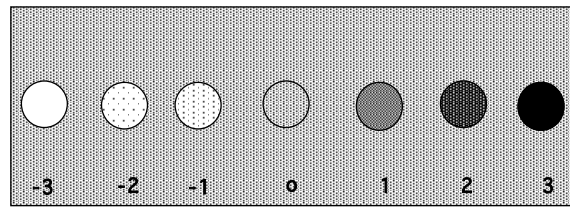
Figure 9. *Different degrees of rarefaction and compression of Casimir vacuum identified as a compressible fluid. (-3) White hole* $\rho_{WH} = 0$ *(-2, -1) Anti-matter* $\rho_{AM} < \rho_v$ *(0) Casimir vacuum* $\rho = \rho_v$ *(1, 2) Matter* $\rho_{MA} > \rho_v$ *(3) Black hole* $\rho_{BH} = \infty$ *[37]*

The temperature limits $(T_{BH} \rightarrow 0 , T_{WH} \rightarrow \infty)$ in Figure 9 correspond to formation of black hole BH (white hole WH) at the circumference (center) of Poincaré disk [77. 117]. Also, the model of physical space shown in Figure 9 is in harmony with previous study [34] concerning some implications of the modified form of first and second laws of thermodynamics to the important problem of black hole mechanics [151-158] emphasized by Wald [157]

> *"It appears that these laws of "black hole mechanics" and the laws of thermodynamics are two major pieces of a puzzle that fit together so perfectly that there can be little doubt that this "fit" is of deep significance"*

In particular, rather than being related to the surface area of its horizon according to Bekenstein [153], entropy of black hole relates to its total thermal energy namely its enthalpy $H_{BH} = T_{BH}S_{BH} = 4kN_{BH}T_{BH}$ given in Equation (79) as $S_{BH} = 4kN_{BH}$ in exact agreement with prediction of Major and Setter [159].

6. Nernst, Planck, Poincaré, and the Third Law of Thermodynamics

The third law of thermodynamics concerns the absolute minimum basis for both energy and entropy of the system. Nernst-Planck statement of the third law of thermodynamics is that both entropy and absolute temperature can reach absolute zero only through infinite number of finite increments [34, 108, 125]. Because by Equations (20) and (91) absolute temperature and entropy are respectively related to Wien wavelength and number of Heisenberg-Kramers [51] virtual oscillators, Nernst-Planck statement involving step-wise approach to zero is in harmony with quantized nature of both T_β and S_β.

As discussed in Section 3, absolute temperature or Wien wavelength $T_\beta = \lambda_{w\beta}$ provides an internal "*measure*" of spatial extensions. It is most interestingly that in his description of *hyperbolic geometry*, Poincaré [117] anticipated the role of absolute thermodynamic temperature as *measure* or *metric* of physical space,

> *"Suppose, for example, a world enclosed in a large sphere and subject to the following laws: --The temperature is not uniform; it is greatest at the center, and gradually decreases as we move towards the circumference of the sphere, where it is absolute zero. The law of this temperature is as follows: -- If R be the radius of the sphere, and r the distance of the point considered from the center, the absolute temperature will be proportional to $R^2 - r^2$. Further, I shall suppose that in this world all bodies have the same coefficient of dilatation, so that the linear dilatation of any body is proportional to its absolute temperature. Finally, I shall assume that a body transported form one point to another of different temperature is instantaneously in thermal equilibrium with its new environment. There is nothing in these hypotheses either contradictory or unimaginable. A moving object will become smaller and smaller at it approaches the circumference of the sphere. Let us observe, in the first place, that although from the point of view of our*

ordinary geometry this world is finite, to the inhabitants it will appear infinite. As they approach the surface of the sphere, they become colder, and at the same time smaller and smaller. The steps they take are therefore smaller and smaller, so that they can never reach the boundary of the sphere. If to us geometry is only the study of the laws according to which invariable solids move, to these imaginary beings it will be the study of the laws of motion of solids deformed by the differences of temperature alluded to."

Therefore, $T_\beta = \lambda_{w\beta}$ is called Poincaré *thermal measure* of extension.

By Equation (36), one introduces the hierarchy of quantized spatial coordinate $x_\beta = x'_\beta / \lambda_{w\beta} = N_{x\beta}$ at scale β that is normalized with respect to the *measure* given by its temprature $T_\beta = \lambda_{w\beta}$ thus arriving at the hierarchy [36]

$$
\begin{array}{l}
0_\beta \underline{\hspace{2cm}} 1_\beta \underline{\hspace{3cm}} \infty_\beta \quad T_\beta \\
\quad 0_{\beta\text{-}1} \underline{\hspace{1.5cm}} 1_{\beta\text{-}1} \underline{\hspace{2.5cm}} \infty_{\beta-1} \quad T_{\beta\text{-}1} \\
\quad\quad 0_{\beta\text{-}2} \underline{\hspace{0.5cm}} 1_{\beta\text{-}2} \underline{\hspace{2cm}} \infty_{\beta-2} \quad T_{\beta\text{-}2}
\end{array}
\tag{95}
$$

For example, for statistical fields at (ECD, EMD, EAD) scales shown in Figure 5, the cascade in Equation (95) corresponds to distributions for different atomic masses at the same temperature $T = 300$ [m]. However, if one considers a constant atomic mass but varying temperatures, the hierarchy of statistical fields with different measures in Equation (95) will correspond to the cascade

$$
\begin{aligned}
T_\beta = 0 &\Leftrightarrow T_{\beta-1} = 1 \Leftrightarrow T_{\beta-2} = \infty \\
T_{\beta-1} = 0 &\Leftrightarrow T_{\beta-2} = 1 \Leftrightarrow T_{\beta-3} = \infty \\
T_{\beta-2} = 0 &\Leftrightarrow T_{\beta-3} = 1 \Leftrightarrow T_{\beta-4} = \infty
\end{aligned}
\tag{96}
$$

Hence, to reach absolute zero temperature requires infinite number of steps, each step corresponding to a generation of statistical field with its own Nernst-Planck limits $(T_\beta \to 0_\beta, S_\beta \to 0_\beta)$ expressed as

$$
T_\beta = <\lambda_{w\beta}^2>^{1/2} \to 0_\beta \qquad , \qquad S_\beta = 4N_\beta k \to 0_\beta
\tag{97}
$$

However, by Equation (95), zero of scale β $(\lambda_{w\beta} = 0_\beta, N_\beta = 0_\beta)$ is unity $(\lambda_{w\beta-1} = 1_{\beta-1}, N_{\beta-1} = 1_{\beta-1})$ of scale $\beta-1$, leading to an infinite regression thus making absolute zero unattainable by finite number of steps. The stepwise (-quantized) approach of temperature, entropy, and Gibbs free energy $G_\beta = H_\beta - T_\beta S_\beta$ towards absolute zero is schematically shown in Figure 10 [34].
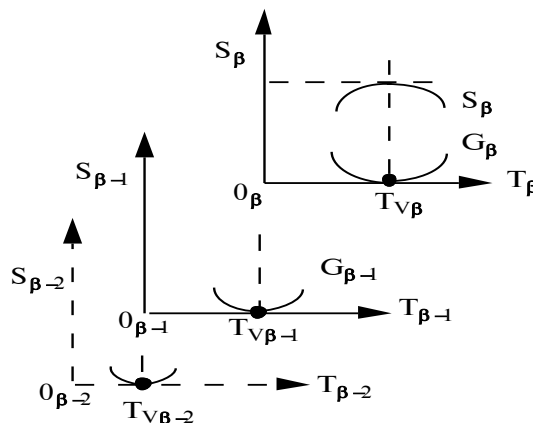

Figure 10. *Hierarchy of absolute "zero" $T_\beta \to 0_\beta$ and "vacuum" $T_{v\beta}$ temperatures and associated entropy S_β and Gibbs free energy G_β [34]*

As emphasized by the quotation from Anaxagoras in Section 3, absolute zero extension hence temperature and duration hence time constitute singularities of the field and as idealized Aristotle limits in Equation (97) are never actualized. Acccording to Figure 10, each scale has a Casimir vacuum temperature that is finite and a vaccuum-vacuum temperature corresponding to the zero-measure of that scale. Also, as discussed in [36] and shown in the hierarchy of scales in Figure 10

$$(\text{vacuum} - \text{vacuum})_\beta = (\text{vacuum})_{\beta-1} \tag{98}$$

The quantized approach towards the limit $(T_\beta \rightarrow 0_\beta, S_\beta \rightarrow 0_\beta)$ viewed from a different perspective also represents quantized approach towards Casimir [75] zero-point energy $(T_\beta S_\beta \rightarrow 0_\beta)$ because of the definition of enthalpy

$$\Delta T_\beta \Delta S_\beta = T_{\text{H}\beta} S_{\text{H}\beta} - T_{\text{L}\beta} S_{\text{L}\beta} = H_{\text{H}\beta} - H_{\text{L}\beta} = \Delta H_\beta \tag{99}$$

as schematically shown on (S-T) diagram in Figure 11
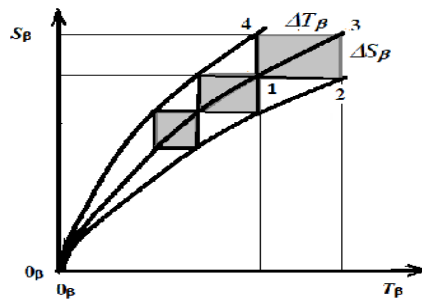

Figure 11. *Stepwise asymptotic approach of entropy S_β and temperature T_β, hence enthalpy H_β, towards absolute zero [34].*

In reference to Carnot cycle shown in Figure 4, the net energy removed from the system at each step in Figure 11 corresponds to *isentropic cooling* by expansion in turbine $(3 \rightarrow 4)$ and *isothermal cooling* in condenser $(4 \rightarrow 1)$ hence the total area $\Delta H = A_{1234}$ shown on S-T diagram in Figure 11.

7. Concluding Remarks

A universal scale-invariant model of Boltzmann statistical mechanics and the associated four laws of generalized thermodynamics were described. Connecting Planck and Boltzmann universal constants to spatio-temporal aspects of Casimir vacuum fluctuations, Kelvin absolute temperature T (degrees kelvin) was identified as Wien wavelength $T_\beta = \lambda_{\text{w}\beta}$ of thermal oscillations. It was shown that given temperature and atomic mass of the field $\mathbb{F}_\beta(T_\beta, m_\beta)$ provides the background space $\mathbb{F}_\beta = \mathbb{S}_{\beta+1}$ required for definitions of *external* space and time coordinates and atomic-mass-unit of $\mathbb{F}_{\beta+1}(x_{\beta+!}, t_{\beta+!}, m_{\beta+1})$. The thermodynamic spacetime and Boltzmann factor are in harmony with modern concepts of deterministic dissipative quantum gravity [73]. De Pretto number 8338 and its connections to classical Joule-Mayer mechanical equivalent of heat $J_c = 4.169 \, \text{kJ/kcal}$ and universal gas constant $R^o = 8338 \, \text{J/kcal.m}$ were identified leading to introduction of modified mechanical equivalent of heat $J = 2J_c = 8338 \, \text{J/kcal}$.

Helmholtz decomposition of total thermal energy into *free heat U* and *latent heat pV* and identification of work as Helmholtz free energy $W = F = U - TS$ resulted in modified form of the first law of thermodynamics $Q = H = U- W = U + pV$. Introducing a modified expressions for the number of photons in a given volume of Casimir vacuum, total energy under Planck energy distribution was shown to be enthalpy *H* rather than internal energy *U*. In harmony with a modified van der Waals law of corresponding states, a model of physical space as a compressible tachyon fluid from infinite rarefaction (*white hole* WH) to infinite compression (*black hole* BH) was described. It is found that, rather than being related to the surface area of its horizon according to Bekenstein [153], entropy of black hole is related to its total thermal energy namely its enthalpy and given by $S_{BH} = 4kN_{BH}$ in exact agreement with prediction of Major and Setter [159].

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