Advances in Pure and Applied Mathematics
2024, vol. 15, n° 1, 44-69 pages, DOI: 10.21494/ISTE.OP.2023.1049 ISTE OpenScience

On the solutions of Fermat type quadratic trinomial equations in $\mathbb{C}^2$ generated by first order linear c-shift and partial differential operators

Sur les solutions des equations de type Fermat quadratiques dans $\mathbb{C}^2$ engendrées par des opérateurs différentiels C-SHIFT d'ordre un

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ABSTRACT. This article is devoted to explore various forms of transcendental entire solution of different quadratic trinomials generated by first order linear c-shift operator. We also investigate the forms of solutions of certain quadratic trinomials under linear and mixed partial differential operators. Our paper improves the results of Li-Xu [Axioms, **126**(10)(2021), 1-19] in two directions. In addition, in a corollary, deducted from one of our main result, we extend a result of Zhang et al. [Aims Math., **7**(2022), 11597-11613]. A series of examples have been exhibited to justify the existence and forms of transcendental entire solution of such equations. In the last section of the paper we have put a relevant question for future research.
2010 Mathematics Subject Classification. 30D35, 39A10, 32W50, 35M30.
KEY WORDS AND PHRASES. Nevanlinna theory, entire solution, Fermat-type equation, partial differential equation, trinomial equation.

1 Background

We assume that the readers are familiar with the basic notations of Nevanlinna theory [6] such as $T(r,f), m(r,f), N(r,f), N(r,\frac{1}{f})$. By $S(r,f)$ we will mean any quantity satisfying $S(r,f) = o(T(r,f))$, $r \to \infty$, outside possibly an exceptional set of finite logarithmic measure. The order of a meromorphic function $f(z)$ in the complex plane $\mathbb{C}$ is defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log\ r}.$$

Let us take $\alpha^2\ (\neq 0,1)$. For convenience, throughout the paper we use

$$A_1 = \frac{1}{2\sqrt{1+\alpha}} - \frac{i}{2\sqrt{1-\alpha}},\quad A_2 = \frac{1}{2\sqrt{1+\alpha}} + \frac{i}{2\sqrt{1-\alpha}}. \tag{1.1}$$

Unless otherwise stated, by z, c we mean $z = (z_1, z_2)$, $c = (c_1, c_2)$. We also use $z+c = (z_1+c_1, z_2+c_2)$.

The study of Fermat-type functional equation

$$f(z)^2 + g(z)^2 = 1, \tag{1.2}$$

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was started almost sixty years ago. In 1966, Gross [5] discussed the existence of solutions of equation (1.2) and showed that the entire solutions of (1.2) are $f(z) = \cos a(z)$, $g(z) = \sin a(z)$, where $a(z)$ is an entire function. In recent years, there have been a gradual development of difference analougue lemmas of meromorphic function and some of their applications in Nevanlinna theory (see [2, 4, 7, 8] and the references therein). On the basis of this progress, various types of functional equations derived from Fermat-type functional equation have been studied by many authors.

In 2005, Li [10] studied Fermat-type partial differential equation of the form

$$(u_{z_1})^m + (u_{z_2})^n = e^g,$$

where g is a polynomial or an entire function.

In 2013, Saleeby [16] first investigated about the entire and meromorphic solutions of the Fermat-type quadratic trinomial equations. This result was the main inspiration among the researchers to study different variants of Fermat-type quadratic trinomial equations more exhaustively and in course of time a series of papers appeared in the literature to enrich the field ([1], [11], [13], [14], [18]).

Recently, Luo *et al.* [14] dealt with the solution of a particular shift differential equation in $\mathbb{C}$ to obtain the following result:

Theorem A. ([14], Theorem 2.3). *Let $\alpha^2 \neq 0, 1$, $c \neq 0$, $\alpha \in \mathbb{C}$ and $g(z)$ be a non-constant polynomial in $\mathbb{C}$. If the shift differential equation*

$$f(z+c)^2 + 2\,\alpha\, f(z+c)\, f'(z) + f'(z)^2 = e^{g(z)},$$

admits a finite order transcendental entire solution $f(z)$, then $g(z)$ must be of the form $g(z) = az + b$, where $a(\neq 0)$, $b \in \mathbb{C}$ are constants. Further $f(z)$ satisfying one of the following conditions:

(i) $f(z) = \dfrac{\sqrt{2}}{a}\left(A_1\eta^{-1} + A_2\eta\right) e^{\frac{az+b}{2}},$

where $\eta(\neq 0) \in \mathbb{C}$ and a, c, A_1, A_2, η are constants satisfying

$$e^{\frac{ac}{2}} = \frac{a(A_1\eta + A_2\eta^{-1})}{2(A_2\eta + A_1\eta^{-1})};$$

(ii) $f(z) = \dfrac{1}{\sqrt{2}}\left(\dfrac{A_2}{a_1}e^{a_1 z + b_1} + \dfrac{A_1}{a_2}e^{a_2 z + b_2}\right),$

where $a_j(\neq 0)$, $b_j \in \mathbb{C}$ are constants, $(j = 1, 2)$ and satisfying $a_1 \neq a_2$, $g(z) = (a_1 + a_2)z + b_1 + b_2 = az + b$, and $e^{a_1 c} = \frac{A_2}{A_1}a_1$, $e^{a_2 c} = \frac{A_2}{A_1}a_2$, $e^{ac} = a_1 a_2$.

Inspired by Theorem A, Zhang *et al.* [18] considered the following equation

$$\Delta_c f(z)^2 + 2\,\alpha\,\Delta_c f(z)\, f(z) + f(z)^2 = e^{g(z)}, \tag{1.3}$$

where $\Delta_c f(z) = f(z+c) - f(z)$, to investigate its solution in $\mathbb{C}$ as follows:

Theorem B. ([18], Theorem 1). *Let $g(z)$ be a non-constant polynomial and $f(z)$ be a finite order transcendental entire solution of the difference equation* (1.3) *then $g(z)$ must be of the form $g(z) = dz+b$ and $f(z) = Ae^{\frac{dz}{2}}$, where $d(\neq 0)$, b, $A(\neq 0)$ are constants satisfying $A^2[e^{dc}+2(\alpha-1)e^{\frac{dc}{2}}-2(\alpha-1)] = e^b$.*

In 2021, Luo *et al.* [14] explored the existence and forms of entire solutions of several quadratic trinomial shift differential equations with more general forms in $\mathbb{C}$. In the same year, Li-Xu [11] further carried forward the investigations of [14] for trinomial shift differential equations in $\mathbb{C}^2$ and established the forms of entire solutions in the following manner.

Theorem C. ([11], Theorem 5). *Let $g(z)$ be a polynomial in $\mathbb{C}^2$ and $\alpha^2 \neq 0, 1$, $c(\neq 0) \in \mathbb{C}^2$. If the c-shift equation*

$$f(z+c)^2 + 2\,\alpha\, f(z+c)f(z) + f(z)^2 = e^{g(z)},$$

admits a transcendental entire solution of finite order, then $g(z_1, z_2)$ must be of the form $g(z_1, z_2) = L(z_1,\ z_2) + H(c_2z_1 - c_1z_2)$, where $L(z_1, z_2)$ is a linear form of $a_1z_1 + a_2z_2$, $H(s)$ is a polynomial in $s = c_2z_1 - c_1z_2$ and a_1, $a_2 \in \mathbb{C}$ are constants. Further $f(z_1, z_2)$ must satisfy one of the following cases:

(i) $$f(z_1, z_2) = \frac{1}{\sqrt{2}}\left(A_1\xi + A_2\xi^{-1}\right) e^{\frac{1}{2}[L(z_1,\ z_2)+H(c_2z_1-c_1z_2)]},$$

where $\xi(\neq 0) \in \mathbb{C}$ is a constant and a_1, a_2, c_1, c_2, ξ, A_1, A_2 satisfying

$$e^{\frac{1}{2}(a_1c_1+a_2c_2)} = \frac{A_2\xi + A_1\xi^{-1}}{A_1\xi + A_2\xi^{-1}};$$

(ii) $$f(z_1, z_2) = \frac{1}{\sqrt{2}}\left(A_1e^{[L_1(z_1,\ z_2)+H_1(c_2z_1-c_1z_2)]} + A_2e^{[L_2(z_1,\ z_2)+H_2(c_2z_1-c_1z_2)]}\right),$$

where $L_1(z_1,\ z_2) = a_{11}z_1 + a_{12}z_2$, $L_2(z_1,\ z_2) = a_{21}z_1 + a_{22}z_2$, $H_j(s)$ are polynomial in $s = c_2z_1 - c_1z_2$, $a_{ij} \in \mathbb{C}$ are constants, $(i = 1, 2; j = 1, 2)$ satisfying $L_1(z_1, z_2) \neq L_2(z_1, z_2)$,

$$\begin{aligned} g(z_1, z_2) &= L_1(z_1, z_2) + L_2(z_1, z_2) + H_1(c_2z_1 - c_1z_2) + H_2(c_2z_1 - c_1z_2) \\ &= L(z_1, z_2) + H(c_2z_1 - c_1z_2), \end{aligned}$$

and

$$e^{a_{11}c_1+a_{12}c_2} = \frac{A_2}{A_1},\ e^{a_{21}c_1+a_{22}c_2} = \frac{A_1}{A_2},\ e^{a_1c_1+a_2c_2} = 1.$$

Theorem D. ([11], Theorem 8). *Let $\alpha^2 \neq 0, 1$, $c_2 \neq 0$ and $g(z)$ be a non-constant polynomial in $\mathbb{C}^2$, not in the form of $\phi(z_2)$, ϕ is a polynomial in z_2. If the c-shift equation*

$$f(z+c)^2 + 2\,\alpha\, f(z+c)\,\frac{\partial f}{\partial z_1} + \left(\frac{\partial f}{\partial z_1}\right)^2 = e^{g(z)},$$

admits a transcendental entire solution $f(z)$ of finite order, then $g(z)$ must be of the form $g(z) = a_1z_1 + a_2z_2 + b$, where $a_1(\neq 0)$, a_2, $b \in \mathbb{C}$ are constants. Further $f(z)$ must satisfy one of the following cases:

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(i) $$f(z)=\frac{\sqrt{2}}{a_1}\left(A_1\xi^{-1}+A_2\xi\right)e^{\frac{a_1z_1+a_2z_2+b}{2}},$$

where $\xi(\neq 0)\in\mathbb{C}$ *is a constant and* a_1, a_2, b, c_1, c_2, ξ, A_1, A_2 *satisfying*

$$e^{\frac{a_1c_1+a_2c_2}{2}}=\frac{a_1\left(A_1\xi+A_2\xi^{-1}\right)}{2\left(A_2\xi+A_1\xi^{-1}\right)};$$

(ii) $$f(z)=\frac{1}{\sqrt{2}}\left(\frac{A_2}{a_{11}}e^{a_{11}z_1+a_{12}z_2+b_1}+\frac{A_1}{a_{21}}e^{a_{21}z_1+a_{22}z_2+b_2}\right),$$

where $a_j(\neq 0)$, b_j, $(j=1,2)$, $\in\mathbb{C}$, *are constants satisfying* $a_{11}z_1+a_{12}z_2\neq a_{21}z_1+a_{22}z_2$, $g(z)=(a_{11}+a_{21})z_1+(a_{12}+a_{22})z_2+b_1+b_2$ *and* $e^{a_{11}c_1+a_{12}c_2}=\frac{A_2}{A_1}a_{11}$, $e^{a_{21}c_1+a_{22}c_2}=\frac{A_1}{A_2}a_{21}$, $e^{a_1c_1+a_2c_2}=a_{11}a_{21}$.

Very recently, Zhang *et al.* [18] discussed about the existence and forms of transcendental entire solutions of several quadratic trinomial differential-difference equations involving $f(z)$, $f'(z)$, $\Delta_cf(z)$ in $\mathbb{C}$. Next we give the proper motivation to consider the equations to find the exact form of solutions.

2 Motivation

This section is built up on the basis of the following definition:

Definition 2.1. ***First order linear c-shift operator.***

Let c *be a non zero constant in* $\mathbb{C}^2$. *We define first order linear* c *shift operator of* f *as*

$$L(z,f)=a_0f(z)+a_1f(z+c),$$

where a_0, $a_1(\not\equiv 0)$ *are constants in* $\mathbb{C}$.

We also define first order partial differential operator of f *as*

$$P_L(z,f)=b_1\frac{\partial f}{\partial z_1}+b_2\frac{\partial f}{\partial z_2},$$

where b_1, b_2, *with* $\mid b_1\mid+\mid b_2\mid\neq 0$ *are constants in* $\mathbb{C}$ *not all zero.*

The goal of this paper is to explore the existence and possible forms of finite order transcendental entire solution of the quadratic trinomial equation generated by first order linear c-shift operator

$$L(z,f)^2+2\,\alpha\,L(z,f)\,f(z)+f(z)^2=e^{g(z)},\tag{2.1}$$

quadratic trinomial partial differential equation generated by first order linear c-shift operator

$$L(z,f)^2+2\,\alpha\,L(z,f)\,P_L(z,f)+P_L(z,f)^2=e^{g(z)},\tag{2.2}$$

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and the quadratic trinomial mixed partial differential equation

$$f(z+c)^2 + 2\,\alpha\, f(z+c)\left(\sum_{l+m=1}^{2} b_{lm}\frac{\partial^{(l+m)} f(z)}{\partial z_\beta^l \partial z_\zeta^m}\right) + \left(\sum_{l+m=1}^{2} b_{lm}\frac{\partial^{(l+m)} f(z)}{\partial z_\beta^l \partial z_\zeta^m}\right)^2 = e^{g(z)}, \tag{2.3}$$

where $g(z)$ is a polynomial in $\mathbb{C}^2$, $\beta = 1, 2$; $\zeta = 1, 2$.

The aim of the paper is to generalize and improve the previous results to get larger as well as higher dimensional solution space. To this end, it is high time to thoroughly investigate the solutions of (2.1)-(2.3) so that all the previous results mentioned so far can be brought under an umbrella.

In this section, we will present the following theorems which are the main results of this paper.

3 Main results and relevant examples

Theorem 3.1. *Let* $\alpha^2 \neq 0, 1$, $c(\neq 0) \in \mathbb{C}$ *be constants and* $g(z)$ *be a polynomial in* $\mathbb{C}^2$. *If the first order linear* c*-shift equation* (2.1) *admits a finite order transcendental entire solution* $f(z)$, *then* $g(z)$ *and* $f(z)$ *can take one of the following forms*:

(i) $g(z) = L(z) + H(s)$,

$$f(z) = \frac{1}{\sqrt{2}}\left(A_2\lambda + A_1\lambda^{-1}\right) e^{\frac{1}{2}[L(z)+H(s)]},$$

where $L(z) = d_1z_1 + d_2z_2$, $H(s)$ *is a polynomial in* $s = e_2z_1 + e_1z_2$ *with* $e_2c_1 + e_1c_2 = 0$; $\lambda(\neq 0) \in \mathbb{C}$ *is a constant and* d_1, d_2, c_1, c_2, a_0, a_1, λ, A_1, A_2 *are constants satisfying*

$$e^{\frac{1}{2}(d_1c_1+d_2c_2)} = -\frac{a_0}{a_1} + \frac{A_1\lambda + A_2\lambda^{-1}}{(A_2\lambda + A_1\lambda^{-1})a_1};$$

(ii) $g(z) = L_1(z) + L_2(z) + H_1(s) + H_2(\tilde{s})$;

$$f(z) = \frac{1}{\sqrt{2}}\left(A_2 e^{[L_1(z)+H_1(s)]} + A_1 e^{[L_2(z)+H_2(\tilde{s})]}\right),$$

where $L_1(z) = d_{11}z_1 + d_{12}z_2$, $L_2(z) = d_{21}z_1 + d_{22}z_2$, $H_1(s)$ *polynomial in* $s = e_{12}z_1 + e_{11}z_2$ *with* $e_{12}c_1 + e_{11}c_2 = 0$, $H_2(\tilde{s})$ *is a polynomial in* $\tilde{s} = e_{22}z_1 + e_{21}z_2$ *with* $e_{22}c_1 + e_{21}c_2 = 0$, *where* d_{ij}, e_{ij} $(i, j = 1, 2)$ *are constants in* $\mathbb{C}$ *such that* $L_1(z) \neq L_2(z)$ *and*

$$e^{d_{11}c_1+d_{12}c_2} = \left(\frac{A_1 - a_0A_2}{a_1A_2}\right), \quad e^{d_{21}c_1+d_{22}c_2} = \left(\frac{A_2 - a_0A_1}{a_1A_1}\right).$$

Corollary 3.1. *We see that if in Theorem 3.1 we put* $a_0 = 0$, $a_1 = 1$, *then we obtain* Theorem C. *So Theorem 3.1 generalizes* Theorem C.

The following examples show that both the forms of solution of *Theorem 3.1* hold.

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Example 3.1. *Let $l = 0$, $\lambda = 1$, $a_0 = a_1 = \frac{1}{2}$, $d_1 = 3$, $d_2 = -1$, $c = (\pi i, 3\pi i)$, $H(s) = k$, $\alpha = \frac{1}{2}$. Then $A_1 = \frac{\sqrt{2}}{\sqrt{3}}\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)$, $A_2 = \frac{\sqrt{2}}{\sqrt{3}}\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$. Then $f(z) = \frac{1}{\sqrt{3}} e^{\frac{3z_1 - z_2 + k}{2}}$ is a solution of (2.1) under $i)$ of Theorem 3.1, where $g(z) = 3z_1 - z_2 + k$.*

Example 3.2. *Let $l_1 = 0$, $l_2 = 0$, $a_0 = -1$, $a_1 = 1$, $H_1(s) = k_1$, $H_2(\tilde{s}) = k_2$, $c = (\pi i, \frac{\pi i}{3})$, $d_{11} = 2$, $d_{12} = -1$, $d_{21} = 3$, $d_{22} = -2$, $\alpha = \frac{1}{2}$. Then $A_1 = \frac{\sqrt{2}}{\sqrt{3}} e^{-i\frac{\pi}{3}}$, $A_2 = \frac{\sqrt{2}}{\sqrt{3}} e^{i\frac{\pi}{3}}$ and*

$$f(z) = \frac{1}{\sqrt{3}}\left(e^{2z_1 - z_2 + \frac{i\pi}{3} + k_1} + e^{3z_1 - 2z_2 - \frac{i\pi}{3} + k_2}\right),$$

is a solution of (2.1) of Theorem 3.1 under $ii)$ where, $g(z) = 5z_1 - 3z_2 + k_1 + k_2$.

As Theorem 3.1 is motivated from Theorem B, so we think the following observation worth to be mentioned.

Observation 3.1. *The next example shows that in* Theorem B, *another form of $f(z)$ exists and hence* Theorem B *is incomplete in some sense.*

Example 3.3. *Let $\alpha = \frac{1}{2}$. Then $A_1 = \frac{\sqrt{2}}{\sqrt{3}} e^{-\frac{\pi i}{3}}$ and $A_2 = \frac{\sqrt{2}}{\sqrt{3}} e^{\frac{\pi i}{3}}$. Let us take $a_1 = 1, a_0 = -1$, d_1, d_2, c be constants in $\mathbb{C}$ satisfying $e^{d_1 c} = 1 + e^{\frac{-2\pi i}{3}}$, $e^{d_2 c} = 1 + e^{\frac{2\pi i}{3}}$. Then $f(z) = \frac{1}{\sqrt{3}}\left(e^{d_1 z + \frac{\pi i}{3}} + e^{d_2 z - \frac{\pi i}{3}}\right)$ is a solution of the equation*

$$\Delta_c f(z)^2 + 2\,\alpha\, f(z)\,\Delta_c f(z) + f(z)^2 = e^{g(z)},$$

with $g(z) = (d_1 + d_2)z$.

Following the same procedure as done in the proof of *Theorem 3.1* the analogous result of *Theorem 3.1* on $\mathbb{C}$ can be obtained as follows:

Corollary 3.2. Statement of Theorem 3.1 in $\mathbb{C}$.

Let $\alpha^2 \neq 0, 1$, $c(\neq 0)$ be constants in $\mathbb{C}$ and $g(z)$ be a polynomial in $\mathbb{C}$. If the quadratic trinomial equation generated by first order linear c-shift operator

$$L(z, f)^2 + 2\,\alpha\, L(z, f) f(z) + f(z)^2 = e^{g(z)},$$

admits a finite order transcendental entire solution $f(z)$, then $g(z)$ and $f(z)$ can take one of the following forms:

(i) *$g(z) = dz + b$;*

$$f(z) = \frac{1}{\sqrt{2}}\left(A_2\lambda + A_1\lambda^{-1}\right) e^{\frac{dz+b}{2}},$$

where d, b, $\lambda(\neq 0) \in \mathbb{C}$ are constants and d, b, a_0, a_1, c, A_1, A_2 satisfying

$$e^{\frac{dc}{2}} = -\frac{a_0}{a_1} + \frac{A_1\lambda + A_2\lambda^{-1}}{a_1\left(A_2\lambda + A_1\lambda^{-1}\right)};$$

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(ii) $g(z)=(d_1+d_2)z+b_1+b_2$;

$$f(z)=\frac{1}{\sqrt{2}}\left(A_2e^{d_1z+b_1}+A_1e^{d_2z+b_2}\right),$$

where d_j, $b_j\in\mathbb{C}$, $(j=1,2)$ *are constants,* $d_1\neq d_2$ *and* d_1, d_2, a_0, a_1, c, A_1, A_2 *satisfying*

$$e^{d_1c}=\frac{A_1-a_0A_2}{a_1A_2},\ e^{d_2c}=\frac{A_2-a_0A_1}{a_1A_1}.$$

Note 3.1. *Putting* $a_0=-1$ *and* $a_1=1$ *in* Corollary 1.2 *we get the actual corrected form of* Theorem B.

Note 3.2. *Corollary 3.2 shows that order of* $f(z)$ *is one. But Theorem 3.1 shows that order of* $f(z)$ *can be any finite number. This means* (2.1) *possesses solution of first order in* $\mathbb{C}$ *but in* $\mathbb{C}^2$ *order can be greater than or equal to one.*

Theorem 3.2. *Let* $\alpha^2\neq 0,1$, $c\ (\neq 0)\in\mathbb{C}$ *be a constant such that* $c_1b_2-c_2b_1\neq 0$ *and* $g(z)$ *be a non-constant polynomial in* $\mathbb{C}^2$ *not in the form of* $\phi(b_1z_2-b_2z_1)$. *If the equation* (2.2) *admits a finite order transcendental entire solution* $f(z)$, *then* $g(z)$ *and* $f(z)$ *can take one of the following forms*:

(i) $g(z)=L(z)+H(s)$;

$$f(z)=\frac{\sqrt{2}\left(A_2\lambda+A_1\lambda^{-1}\right)}{b_1(d_1+le_2)+b_2(d_2+le_1)}e^{\frac{L(z)+H(s)}{2}},$$

where $L(z)=d_1z_1+d_2z_2$, $H(s)=l(e_2z_1+e_1z_2)+d$, *where* d_1, d_2, e_1, e_2, d, $\lambda(\neq 0)\in\mathbb{C}$, l *are constants in* $\mathbb{C}$ *with* $e_2c_1+e_1c_2=0$ *and* d_1, d_2, c_1, c_2, e_1, e_2, λ, A_1, A_2, l *satisfying*

$$b_1(d_1+le_2)+b_2(d_2+le_1)\neq 0,$$
$$e^{\frac{d_1c_1+d_2c_2}{2}}=\frac{1}{2}\left[\frac{\left(A_1\lambda+A_2\lambda^{-1}\right)}{a_1\left(A_2\lambda+A_1\lambda^{-1}\right)}\{b_1(d_1+le_2)+b_2(d_2+le_1)\}\right]-\frac{a_0}{a_1};$$

(ii) $g(z)=L_1(z)+L_2(z)+H_1(s)+H_2(\tilde{s})$;

$$\begin{aligned}f(z)= \quad & \frac{1}{\sqrt{2}}\left(\frac{A_2}{b_1(d_{11}+l_1e_{12})+b_2(d_{12}+l_1e_{11})}e^{L_1(z)+H_1(s)}\right.\\ & \left.+\frac{A_1}{b_1(d_{21}+l_2e_{22})+b_2(d_{22}+l_2e_{21})}e^{L_2(z)+H_2(\tilde{s})}\right),\end{aligned}$$

where $L_1(z)=d_{11}z_1+d_{12}z_2$, $L_2(z)=d_{21}z_1+d_{22}z_2$ *such that* $L_1(z)\neq L_2(z)$; $H_1(s)=l_1(e_{12}z_1+e_{11}z_2)+d_1$ *with* $e_{12}c_1+e_{11}c_2=0$, $H_2(\tilde{s})=l_2(e_{22}z_1+e_{21}z_2)+d_2$ *with* $e_{22}c_1+e_{21}c_2=0$, e_{ij} *are constants* $\in\mathbb{C}$, $(i,j=1,2)$ *and* d_{ij}, e_{ij}, l_j, $(i,j=1,2)$, b_1, b_2 *satisfying*

$$b_1(d_{11}+l_1e_{12})+b_2(d_{12}+l_1e_{11})\neq 0,$$
$$b_1(d_{21}+l_2e_{22})+b_2(d_{22}+l_2e_{21})\neq 0,$$

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$$e^{d_{11}c_1+d_{12}c_2} = \left[\frac{A_1}{a_1A_2}\{b_1(d_{11}+l_1e_{12})+b_2(d_{12}+l_1e_{11})\}-\frac{a_0}{a_1}\right],$$
$$e^{d_{21}c_1+d_{22}c_2} = \left[\frac{A_2}{a_1A_1}\{b_1(d_{21}+l_2e_{22})+b_2(d_{22}+l_2e_{21})\}-\frac{a_0}{a_1}\right].$$

The above theorem articulates the following corollary.

Corollary 3.3. *Let $\alpha^2 \neq 0, 1$, be a constant in $\mathbb{C}$. If $g(z)$ is a polynomial of degree greater than one and $c_1b_2 - c_2b_1 \neq 0$ then equation (2.2) can not admit transcendental entire solution with finite order.*

Corollary 3.4. *Setting $a_0 = 0$, $a_1 = 1$, $b_1 = 1$, $b_2 = 0$; $H(s)$, $H_1(s)$, $H_2(\tilde{s})$ as constant polynomials we get* Theorem D. *This implies that our result is a generalized version of* Theorem D.

Corollary 3.5. *Considering $a_0 = 0$, $a_1 = 1$, $b_1 = b_2 = 1$, $H(s)$, $H_1(s)$, $H_2(\tilde{s})$ as constant polynomials we get the similar results as* Theorem 9 *of [11]. This shows that our result is an extension of* Theorem 9 *of [11].*

The following examples show that both the forms of the solutions of *Theorem 3.2* hold.

Example 3.4. *Let $l = 1$, $a_0 = 2$, $a_1 = 1$, $b_1 = b_2 = 1$, $(c_1, c_2) = (\pi i, -\pi i)$, $d_1 = 2$, $d_2 = 2$, $e_1 = 1$, $e_2 = 1$, $d = k$, $\lambda = 1$, $\alpha = \frac{1}{2}$. Then*

$$f(z) = \frac{1}{3\sqrt{3}}e^{\frac{3z_1+3z_2+k}{2}}$$

is a solution of (2.2) *under form $i)$ of Theorem 3.2 with $g(z) = 3z_1 + 3z_2 + d$.*

Example 3.5. *Let $l_1 = 0$, $l_2 = 0$, $a_0 = -1$, $a_1 = 1$, $b_1 = 1$, $b_2 = 1$, $d_{11} = 2$, $d_{12} = -1$, $d_{21} = -3$, $d_{22} = 2$, $H_1(s) = d_1$, $H_2(\tilde{s}) = d_2$ $c = (\pi i, \frac{\pi i}{3})$, $\alpha = \frac{1}{2}$. Then*

$$f(z) = \frac{1}{\sqrt{3}}\left(e^{2z_1-z_2+\frac{\pi i}{3}+d_1} - e^{-3z_1+2z_2-\frac{\pi i}{3}+d_2}\right),$$

is a solution of (2.2) *under form $ii)$ of Theorem 3.2 with $g(z) = -z_1 + z_2 + d_1 + d_2$.*

A close look into the proof of *Theorem 3.2* afterwards, will reveal the fact that the condition $c_1b_2 - c_2b_1 \neq 0$ is used only to ensure that, the degree of $H_1(s) \leq 1$ and degree $H_2(\tilde{s}) \leq 1$. On the other hand, under the situation $c_1b_2 - c_2b_1 = 0$, the following example justifies that the solution of equation (2.2) exists but that is different form those given in *Theorem 3.2*.

Example 3.6. *Let $c_1 = c_2 = b_1 = b_2 = 1$, $a_0 = a_1 = 1$. Also let d_{11}, d_{12}, d_{21}, d_{22} are constants satisfying*

$$e^{d_{11}+d_{12}} = e^{-\frac{2\pi i}{3}}(d_{11}+d_{12}) - 1,$$
$$e^{d_{21}+d_{22}} = e^{\frac{2\pi i}{3}}(d_{21}+d_{22}) - 1.$$

We see that

$$f(z) = \frac{1}{\sqrt{3}}\left(\frac{1}{(d_{11}+d_{12})}e^{d_{11}z_1+d_{12}z_2+(z_1-z_2)^2+\frac{\pi i}{3}} + \frac{1}{(d_{21}+d_{22})}e^{d_{21}z_1+d_{22}z_2+(z_1-z_2)^3-\frac{\pi i}{3}}\right),$$

is a solution of equation (2.2) *with $g(z) = (d_{11}+d_{21})z_1 + (d_{21}+d_{22})z_2 + (z_1-z_2)^2 + (z_1-z_2)^3$, but degree of $g(z)$ is properly greater than one.*

Using similar arguments as done in the proof of *Theorem* 3.2, we get the analogous result of *Theorem* 3.2 in $\mathbb{C}$ as follows.

Corollary 3.6. Statement of Theorem 3.2 in $\mathbb{C}$.

Let $\alpha^2 \neq 0, 1$, $c \neq 0$ *be constants in* $\mathbb{C}$ *and* $g(z)$ *be a non-constant polynomial in* $\mathbb{C}$. *If the quadratic trinomial equation generated by first order linear* c*-shift*

$$L(z,f)^2 + 2\ \alpha\ L(z,f)\ f'(z) + f'(z)^2 = e^{g(z)},$$

admits a finite order transcendental entire solution $f(z)$, *then* $g(z)$ *and* $f(z)$ *can take one of the following forms:*

(i) $g(z) = dz + b$;

$$f(z) = \frac{\sqrt{2}}{d}\left(A_2\lambda + A_1\lambda^{-1}\right) e^{\frac{dz+b}{2}},$$

$d(\neq 0), b$, $\lambda(\neq 0)$, $\in \mathbb{C}$ *are constants and* A_1, A_2, d, c a_0, a_1, λ *satisfying*

$$e^{\frac{dc}{2}} = -\frac{a_0}{a_1} + \frac{d\left(A_1\lambda + A_2\lambda^{-1}\right)}{2a_1\left(A_2\lambda + A_1\lambda^{-1}\ \right)}.$$

(ii) $g(z) = (d_1 + d_2)z + b_3 + b_4$;

$$f(z) = \frac{1}{2}\left(\frac{A_2}{d_1}e^{d_1 z + b_3} + \frac{A_1}{d_2}e^{d_2 z + b_4}\right),$$

where $d_j(\neq 0)$, $(j = 1, 2)$, b_k, $(k = 3, 4)$, $\in \mathbb{C}$ *are constants such that* $d_1 \neq d_2$ *and* d_1, d_2, a_0, a_1, c, A_1, A_2 *satisfying* $e^{d_1 c} = \frac{A_1 d_1 - a_0 A_2}{a_1 A_2}$, $e^{d_2 c} = \frac{A_2 d_2 - a_0 A_1}{a_1 A_1}$.

In the following theorem we denote by

$$M = \tfrac{1}{2}b_{10}\tfrac{\partial}{\partial z_1} + \tfrac{1}{2}b_{01}\tfrac{\partial}{\partial z_2} + b_{11}\left(\tfrac{1}{2}\tfrac{\partial^2}{\partial z_1 \partial z_2} + \tfrac{1}{4}\tfrac{\partial}{\partial z_1}\tfrac{\partial}{\partial z_2}\right) + b_{20}\left(\tfrac{1}{2}\tfrac{\partial^2}{\partial z_1^2}\ + \tfrac{1}{4}\left(\tfrac{\partial}{\partial z_1}\right)^2\right) + b_{02}\left(\tfrac{1}{2}\tfrac{\partial^2}{\partial z_2^2} + \tfrac{1}{4}\left(\tfrac{\partial}{\partial z_2}\right)^2\right),$$

$$A = -b_{10}c_2 + b_{01}c_1 - \frac{1}{2}b_{11}(d_2c_2 - d_1c_1) - b_{20}d_1c_2 + b_{02}d_2c_1$$

$$A' = -b_{10}c_2 + b_{01}c_1 - b_{11}(d_{12}c_2 - d_{11}c_1) - 2b_{20}d_{11}c_2 + 2b_{02}d_{12}c_1,$$

$$A'' = -b_{10}c_2 + b_{01}c_1 - b_{11}(d_{22}c_2 - d_{21}c_1) - 2b_{20}d_{21}c_2 + 2b_{02}d_{22}c_1,$$

$$B = b_{20}c_2^2 + b_{02}c_1^2 - b_{11}c_1c_2,$$

where d_1, d_2, d_{11}, d_{12}, d_{21}, d_{22} are constants in $\mathbb{C}$.

Theorem 3.3. *Let $\alpha^2 \neq 0, 1$, $\alpha \in \mathbb{C}$, $g(z)$ be a non-constant polynomial in $\mathbb{C}^2$, does not satisfy the partial differential equation $Mg = 0$. Let $c(\neq 0)$ be such that*
a) $B \neq 0$ *or*
b) $B = 0$, $A \neq 0$ *or*
c) $B = 0$, $A' \neq 0$, $A'' \neq 0$.

If equation (2.3) *admits a finite order transcendental entire solution $f(z)$, then $f(z)$ and $g(z)$ can take one of the following forms:*

(i) $g(z) = L(z) + H(s)$;

$$f(z) = \frac{p}{q} e^{\frac{L(z)+H(s)}{2}},$$

where $p = \sqrt{2}\left(A_2\lambda + A_1\lambda^{-1}\right)$, $q = b_{10}(d_1 + le_2) + b_{01}(d_2 + le_1) + \frac{1}{2}b_{11}(d_1 + le_2)(d_2 + le_1) + \frac{1}{2}b_{20}(d_1 + le_2)^2 + \frac{1}{2}b_{02}(d_2 + le_1)^2$, $L(z) = d_1z_1 + d_2z_2$, $H(s) = l(e_2z_1 + e_1z_2) + d$, *where* d_1, d_2, e_1, e_2, d, l, $\lambda(\neq 0)$ $\in \mathbb{C}$ *are constants with* $e_2c_1 + e_1c_2 = 0$ *and* d_1, d_2, e_1, e_2, c_1, c_2, b_{10}, b_{01}, A_1, A_2, λ, l *satisfying*

$$e^{\frac{L(c)}{2}} = \frac{A_1\lambda + A_2\lambda^{-1}}{2\left(A_2\lambda + A_1\lambda^{-1}\right)} q.$$

(ii) $g(z) = L_1(z) + L_2(z) + H_1(s) + H_2(\tilde{s})$;

$$f(z) = \frac{1}{\sqrt{2}}\left(\frac{A_2}{P} e^{L_1(z)+H_1(s)} + \frac{A_1}{Q} e^{L_2(z)+H_2(\tilde{s})}\right),$$

where $P = \{b_{10}(d_{11} + l_1e_{12}) + b_{01}(d_{12} + l_1e_{11}) + b_{11}(d_{11} + l_1e_{12})(d_{12} + l_1e_{11}) + b_{20}(d_{11} + l_1e_{12})^2 + b_{02}(d_{12} + l_1e_{11})^2\}$ *and* $Q = b_{10}(d_{21} + l_2e_{22}) + b_{01}(d_{22} + l_2e_{21}) + b_{11}(d_{21} + l_2e_{22})(d_{22} + l_2e_{21}) + b_{20}(d_{21} + l_2e_{22})^2 + b_{02}(d_{22} + l_2e_{21})^2$, $L_1(z) = d_{11}z_1 + d_{12}z_2$, $L_2(z) = d_{21}z_1 + d_{22}z_2$ *with* $L_1(z) \neq L_2(z)$; $H_1(s) = l_1(e_{12}z_1 + e_{11}z_2) + d_1$ *with* $e_{12}c_1 + e_{11}c_2 = 0$, $H_2(\tilde{s}) = l_2(e_{22}z_1 + e_{21}z_2) + d_2$ *with* $e_{21}c_1 + e_{22}c_2 = 0$ *and* d_{ij}, e_{ij}, l_j, b_{ij}, $\in \mathbb{C}$, $(i, j = 1, 2)$ *are constants satisfying*

$$e^{L_1(c)} = \frac{A_1}{A_2} P,$$

$$e^{L_2(c)} = \frac{A_2}{A_1} Q.$$

Corollary 3.7. *Setting* $b_{11} = b_{20} = b_{02} = 0$; $b_{10} = b_{01} = 1$ *in Theorem 3.3 we can get* Theorem 9 *of [11], while choosing* $b_{11} = b_{20} = b_{02} = b_{01} = 0$ *and* $b_{10} = 1$ *we get* Theorem D.

The following examples show that both the forms of solutions of equation (2.3) under *Theorem 3.3* hold.

Example 3.7. *Let* $l = 0$, $b_{10} = -2$, $b_{01} = 1$, $b_{11} = -2$, $b_{20} = 2$ $b_{20} = 2$, $\lambda = 1$, $d_1 = 1$, $d_2 = -1$, $d = -2$, $c = (\pi i, \pi i)$, $\alpha = \frac{1}{2}$. *Then* $f(z) = \frac{1}{\sqrt{3}} e^{\frac{z_1 - z_2 - 2}{2}}$ *is a solution of* (2.3) *under* $a) - (i)$ *of* Theorem 3.3 *where* $g(z) = z_1 - z_2 - 2$.

Example 3.8. *Let* $l=0$, $b_{10}=3$, $b_{01}=1$, $b_{11}=2$, $b_{20}=1$ $b_{20}=1$, $\lambda=1$, $d_1=1$, $d_2=-1$, $d=-2$, $c=(1,1)$, $\alpha=\frac{1}{2}$. *Then* $f(z)=\frac{1}{\sqrt{3}}e^{\frac{z_1-z_2-2}{2}}$ *is a solution of* (2.3) *under* $b)-(i)$ *of* Theorem 3.3 *where* $g(z)=z_1-z_2-2$.

Example 3.9. *Let* $l=0$, $b_{10}=-1$ $b_{01}=1$, $b_{11}=1$, $b_{20}=1$, $b_{02}=-1$, $d_{11}=-1$, $d_{12}=-1$, $d_{21}=3$, $d_{22}=-1$, $\alpha=\frac{1}{2}$, $c=(\frac{\pi i}{3},\frac{\pi i}{3})$. *Then*

$$f(z)=\frac{1}{\sqrt{3}}\left(e^{-z_1-z_2+\frac{\pi i}{3}}+e^{3z_1-z_2-\frac{\pi i}{3}}\right)$$

is a solution of (2.3) *under* $a)-(ii)$ *of Theorem 3.3 where* $g(z)=2z_1-2z_2$.

We see that the under the conditon $B=0$, $A=0$, the equation (2.3) has different form of solution.

Example 3.10. *Let* $l=0$, $\alpha=\frac{1}{2}$, $b_{10}=1$, $b_{01}=1$, $b_{11}=2$, $b_{20}=1$, $b_{02}=1$, $c_1=1$, $c_2=1$, d_1, d_2 *be constants satisfying*

$$e^{\frac{d_1+d_2}{2}}=\frac{1}{2}\left(d_1+d_2+d_1d_2+\frac{1}{2}d_1^2+\frac{1}{2}d_2^2\right),$$

$$f(z)=\frac{2}{\sqrt{3}\left(d_1+d_2+d_1d_2+\frac{1}{2}d_1^2+\frac{1}{2}d_2^2\right)}e^{\frac{d_1z_1+d_2z_2+G(-z_1+z_2)}{2}},$$

where $G(-z_1+z_2)$ *is a polynomial of degree greater than one, is a solution of* (2.3) *with* $g(z)=d_1z_1+d_2z_2+G(-z_1+z_2)$.

From the proof of *Theorem 3.3*, we see that if $A'=0$, $A''=0$ and $B=0$ then the solution $f(z)$ of (2.3) is different from those given in *Theorem 3.3* and the degree of $g(z)$ is ≥ 2.

Example 3.11. *Let* $l=0$, $b_{10}=b_{01}=1$, $b_{11}=2$, $b_{20}=b_{02}=1$ $c_1=c_2=1$, $\alpha=-\frac{1}{2}$, $d_{11},d_{12},d_{21},d_{22}$ *be constants satisfying*

$$e^{d_{11}+d_{12}}=e^{-\frac{2\pi i}{6}}\left(d_{11}+d_{12}+2d_{11}d_{12}+d_{11}^2+d_{12}^2\right),$$
$$e^{d_{21}+d_{22}}=e^{\frac{2\pi i}{6}}\left(d_{21}+d_{22}+2d_{21}d_{22}+d_{21}^2+d_{22}^2\right),$$

We see that

$$\begin{aligned}f(z)&=\frac{1}{\sqrt{3}}\left(\frac{1}{d_{11}+d_{12}+2d_{11}d_{12}+d_{11}^2+d_{12}^2}e^{d_{11}z_1+d_{12}z_2+F_1(-z_1+z_2)+\frac{\pi i}{6}}\right.\\&\left.+\frac{1}{d_{21}+d_{22}+2d_{21}d_{22}+d_{21}^2+d_{22}^2}e^{d_{21}z_1+d_{22}z_2+F_1(-z_1+z_2)-\frac{\pi i}{6}}\right),\end{aligned}$$

where $F_1(z)$, $F_2(z)$ *are polynomials of degree properly greater than one is a solution of Theorem 3.3 with* $g(z)=(d_{11}+d_{21})z_1+(d_{12}+d_{22})z_2+2F_1(-z_1+z_2)$ *and hence the degree of* $g(z)$ *is not one.*

Corollary 3.8. *Let* $c(\neq 0)\in\mathbb{C}$, $\alpha^2\neq 0,1$ *and conditions* a) *or* b) *or* c) *of Theorem 3.3 is satisfied. If degree of* $g(z)$ *in Theorem 3.3 is greater than one, then the equation* (2.3) *can not have any transcendental entire solution of finite order.*

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4 Lemmas and Proofs of the theorems

4.1 Lemmas:

In proving our results we need the following lemmas.

Lemma 4.1. ([9], Lemma 2) *Let $f_j(\not\equiv 0)$, $j = 1, 2, 3$ be meromorphic function in $\mathbb{C}^m$ such that f_1 is not constant, $f_1 + f_2 + f_3 = 1$ and*

$$\sum_{j=1}^{3}\left\{N_2\left(r, \frac{1}{f_j}\right) + 2\overline{N}(r, f_j)\right\} < \zeta T(r, f_1) + o(log^+T(r, f_1)),$$

for all r outside possibly a set with finite logarithmic measure, where $\zeta < 1$ is a positive number, then either $f_2 \equiv 1$ or $f_3 \equiv 1$, where $N_2\left(r, \frac{1}{f}\right)$ is counting function of zeros of f in $|z| \le r$ and simple zero is counted once and multiple zero is counted twice.

Lemma 4.2. ([15]). *For an entire function F on $\mathbb{C}^n$, $F(0) \neq 0$ and put $\rho(n_F) = \rho < \infty$. Then there exists a canonical function $g_F \in \mathbb{C}^n$ such that $F(z) = f_F(z)e^{g_F(z)}$. For special case $n = 1$, $f_F(z)$ is the canonical product of Weierstrass.*

Remark 4.1. *Here $\rho(n_F)$ is the order of counting function of zeros of F.*

Lemma 4.3. ([3], Lemma 3.2). *Let f be a non-constant meromorphic function in $\mathbb{C}^n$. Then for any $I \in (Z^+)^n$, $T(r, \partial^I f) = o(T(r, f))$ for all r except possibly a set of finite Lebesgue measure, where $I = (i_1, \ldots, i_n) \in (Z^+)^n$ denotes a multiple index with $|I| = i_1 + i_2 + \cdots + i_n$, $Z^+ = \{0, 1, 2, ...\}$ and $\partial^I f = \frac{\partial^I f}{\partial i^n \xi_n \ldots \partial^{i_1}\xi_1}$.*

Lemma 4.4. ([9], Lemma 3.1). *Suppose that $a_0(z), a_1(z), \ldots, a_n(z)$, $n \ge 1$, are meromorphic on $\mathbb{C}^m$ and $g_0(z)$, $g_2(z)$, $\ldots, g_n(z)$ are entire on $\mathbb{C}^m$. $g_j(z) - g_k(z)$ non constant for $0 \le j < k \le n$. If*

$$\sum_{j=0}^{n} a_j(z)e^{g_j(z)} = 0$$

and $T(r, a_j) = o\left(T(r)\right) \quad j = 0, 1, 2 \ldots, n,$

$$T(r) = \min_{0 \le j < k \le n} \quad T(r, e^{g_k - g_j}),$$

then $a_j \equiv 0$.

4.2 Proofs of the theorems

<u>Proof of Theorem 3.1</u>. Let f be a finite order transcendental entire solution of (2.1) in $\mathbb{C}^2$. Let us take

$$u = \frac{1}{\sqrt{2}}\left(L(z, f) + f(z)\right), \tag{4.1}$$

$$v = \frac{1}{\sqrt{2}}\left(L(z, f) - f(z)\right). \tag{4.2}$$

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Using (4.1) and (4.2) we get

$$L(z,f)=\frac{1}{\sqrt{2}}\left(u+v\right), \tag{4.3}$$

$$f(z)=\frac{1}{\sqrt{2}}\left(u-v\right). \tag{4.4}$$

Then using (4.3) and (4.4), from (2.1) we get

$$(1+\alpha)u^2+(1-\alpha)v^2=e^{g(z)},$$

which implies

$$\left(\frac{\sqrt{1+\alpha}\;u}{e^{\frac{g(z)}{2}}}\right)^2+\left(\frac{\sqrt{1-\alpha}\;v}{e^{\frac{g(z)}{2}}}\right)^2=1.$$

This leads to

$$\left(\frac{\sqrt{1+\alpha}\;u}{e^{\frac{g(z)}{2}}}+i\frac{\sqrt{1-\alpha}\;v}{e^{\frac{g(z)}{2}}}\right)+\left(\frac{\sqrt{1+\alpha}\;u}{e^{\frac{g(z)}{2}}}-i\frac{\sqrt{1-\alpha}\;v}{e^{\frac{g(z)}{2}}}\right)=1. \tag{4.5}$$

Since f is a finite order transcendental entire function and g is non-constant a polynomial, using *Lemma 4.2* we get a polynomial $p(z)$ such that from (4.5) we get

$$\frac{\sqrt{1+\alpha}\;u}{e^{\frac{g(z)}{2}}}+i\frac{\sqrt{1-\alpha}\;v}{e^{\frac{g(z)}{2}}}=e^{p(z)}, \tag{4.6}$$

$$\frac{\sqrt{1+\alpha}\;u}{e^{\frac{g(z)}{2}}}-i\frac{\sqrt{1-\alpha}\;v}{e^{\frac{g(z)}{2}}}=e^{-p(z)}. \tag{4.7}$$

Let

$$\sigma_1(z)=\frac{g(z)}{2}+p(z),\quad \sigma_2(z)=\frac{g(z)}{2}-p(z). \tag{4.8}$$

From (4.6) and (4.7) we get

$$\sqrt{1+\alpha}\;u=\frac{e^{\sigma_1(z)}+e^{\sigma_2(z)}}{2},$$

$$\sqrt{1+\alpha}\;v=\frac{e^{\sigma_1(z)}-e^{\sigma_2(z)}}{2}.$$

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This gives

$$L(z,f) = \frac{1}{\sqrt{2}}\left[\frac{e^{\sigma_1(z)}+e^{\sigma_2(z)}}{2\sqrt{1+\alpha}}+\frac{e^{\sigma_1(z)}-e^{\sigma_2(z)}}{2i\sqrt{1-\alpha}}\right], \tag{4.9}$$
$$= \frac{1}{\sqrt{2}}\left(A_1e^{\sigma_1(z)}+A_2e^{\sigma_2(z)}\right),$$

$$f(z) = \frac{1}{\sqrt{2}}\left[\frac{e^{\sigma_1(z)}+e^{\sigma_2(z)}}{2\sqrt{1+\alpha}}-\frac{e^{\sigma_1(z)}-e^{\sigma_2(z)}}{2i\sqrt{1-\alpha}}\right], \tag{4.10}$$
$$= \frac{1}{\sqrt{2}}\left(A_2e^{\sigma_1(z)}+A_1e^{\sigma_2(z)}\right),$$

A_1, A_2 are as defined in (1.1). Thus in view of (4.9) and (4.10) we have

$$\left(\frac{A_1-a_0A_2}{a_1A_2}\right)e^{\sigma_1(z)-\sigma_1(z+c)}+\left(\frac{A_2-a_0A_1}{a_1A_2}\right)e^{\sigma_2(z)-\sigma_1(z+c)} \tag{4.11}$$
$$-\frac{A_1}{A_2}e^{\sigma_2(z+c)-\sigma_1(z+c)}=1.$$

We discuss the following cases:

Case 1: Let $\sigma_2(z+c)-\sigma_1(z+c)$ be a constant. From (4.8) we have $-2p(z+c)$ is a constant. Let $\lambda = e^{p(z)}$. Then using (4.9) and (4.10) we get

$$L(z,f)=\frac{1}{\sqrt{2}}\left(A_1\lambda+A_2\lambda^{-1}\right)e^{\frac{g(z)}{2}}, \tag{4.12}$$

$$f(z)=\frac{1}{\sqrt{2}}\left(A_2\lambda+A_1\lambda^{-1}\right)e^{\frac{g(z)}{2}}. \tag{4.13}$$

In view of (4.12) and (4.13) we get

$$a_1\left(A_2\lambda+A_1\lambda^{-1}\right)e^{\frac{g(z+c)}{2}-\frac{g(z)}{2}}=-a_0\left(A_2\lambda+A_1\lambda^{-1}\right)+\left(A_1\lambda+A_2\lambda^{-1}\right). \tag{4.14}$$

Since $e^{g(z+c)-g(z)}$ has no zeros, we see that

$$\left(A_2\lambda+A_1\lambda^{-1}\right)=0$$

and

$$-a_0\left(A_2\lambda+A_1\lambda^{-1}\right)+\left(A_1\lambda+A_2\lambda^{-1}\right)=0$$

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can not hold simulteneously, otherwise it yeilds that $A_1^2 = A_2^2$, which is a contradiction, as $\alpha^2 \neq 0, 1$. Since $a_0, a_1 \neq 0$, it follows that $a_1\left(A_2\lambda + A_1\lambda^{-1}\right) \neq 0$ and

$$-a_0\left(A_2\lambda + A_1\lambda^{-1}\right) + \left(A_1\lambda + A_2\lambda^{-1}\right) \neq 0.$$

Again $g(z)$ being a polynomial (4.14) implies that $g(z+c) - g(z)$ must be a constant, otherwise we get a contradiction from the fact that left of the equation (4.14) is transcendental but right is not. Hence we get $g(z) = L(z) + H(s)$, where $L(z) = d_1z_1 + d_2z_2$, $H(s)$ is a polynomial in s, where $s = e_2z_1 + e_1z_2$, such that $e_2c_1 + e_1c_2 = 0$. Thus it follows from (4.14) $e^{d_1c_1+d_2c_2} = -\frac{a_0}{a_1} + \frac{\left(A_1\lambda+A_2\lambda^{-1}\right)}{a_1(A_2\lambda+A_1\lambda^{-1})}$. From (4.13) $f(z) = \frac{1}{\sqrt{2}}\left(A_2\lambda + A_1\lambda^{-1}\right)e^{\frac{L(z)+H(s)}{2}}$. Hence conclusion (i) of *Theorem 3.1* is proved.

Case 2: Let $\sigma_2(z+c) - \sigma_1(z+c)$ be not-constant. Let $M_1 = \left(\frac{A_1 - a_0A_2}{a_1A_2}\right)$, $M_2 = \left(\frac{A_2 - a_0A_1}{a_1A_2}\right)$. Now we discuss the following cases:

Subcase 2.1: Let $M_1 \equiv 0$ and $M_2 \equiv 0$. Then from (4.11) we get $-\frac{A_1}{A_2}e^{\sigma_2(z+c)-\sigma_1(z+c)} = 1$, which implies that $\sigma_2(z+c) - \sigma_1(z+c)$ is a constant, a contradiction.

Subcase 2.2: Let $M_1 \equiv 0$ and $M_2 \not\equiv 0$, then from (4.11) we get

$$M_2\, e^{\sigma_2(z)-\sigma_1(z+c)} - \frac{A_1}{A_2}\, e^{\sigma_2(z+c)-\sigma_1(z+c)} = 1. \tag{4.15}$$

Since $\sigma_2(z+c) - \sigma_1(z+c)$ is not a constant, it follows that $\sigma_2(z) - \sigma_1(z+c)$ is not a constant. Then from (4.15) we get

$$M_2\, e^{\sigma_2(z)} - \frac{A_1}{A_2}\, e^{\sigma_2(z+c)} - e^{\sigma_1(z+c)} = 0. \tag{4.16}$$

We have $\sigma_2(z+c) - \sigma_2(z)$ is not a constant. Since otherwise, $\sigma_2(z+c) = \sigma_2(z) + k$, k is a constant in $\mathbb{C}$. Hence from (4.16)

$$\left(M_2e^{-k} - \frac{A_1}{A_2}\right)e^{\sigma_2(z+c)-\sigma_1(z+c)} = 1, \tag{4.17}$$

equation (4.17) contradicts that $\sigma_2(z+c) - \sigma_1(z+c)$ is constant, which is a contradiction. Then from (4.15) we have $M_2 \equiv 0$, contradiction.

Subcase 2.3: If $M_1 \not\equiv 0$, $M_2 \equiv 0$, then by the similar argument of **Subcase 2.2** we get a contradiction.

Subcase 2.4: Suppose $M_1 \not\equiv 0$, $M_2 \not\equiv 0$. Then applying *Lemma 4.1* to the equation (4.11) we get

$$M_1e^{\sigma_1(z)-\sigma_1(z+c)} \equiv 1,$$

or

$$M_2e^{\sigma_2(z)-\sigma_1(z+c)} \equiv 1.$$

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Subcase 2.4.1: Let

$$M_1 e^{\sigma_1(z)-\sigma_1(z+c)} \equiv 1. \tag{4.18}$$

From (4.18) we see that $\sigma_1(z) - \sigma_1(z+c)$ is a constant. This leads to $\sigma_1(z) = L_1(z) + H_1(s)$, where $L_1(z) = d_{11}z_1 + d_{12}z_2$, $H_1(s)$ is a polynomial in s, where $s = e_{12}z_1 + e_{11}z_2$, such that $e_{12}c_1 + e_{11}c_2 = 0$ and d_{11}, d_{12} are two constants satisfying that

$$e^{d_{11}c_1+d_{12}c_2} = \left(\frac{A_1 - a_0A_2}{a_1A_2}\right).$$

Since $M_1 e^{\sigma_1(z)-\sigma_1(z+c)} \equiv 1$, then from (4.11) we get

$$M_2 e^{\sigma_2(z)-\sigma_1(z+c)} = \frac{A_1}{A_2} e^{\sigma_2(z+c)-\sigma_1(z+c)}.$$

That is

$$e^{\sigma_2(z)-\sigma_2(z+c)} = \frac{A_1a_1}{A_2 - a_0A_1}. \tag{4.19}$$

Equation (4.19) shows that $\sigma_2(z)-\sigma_2(z+c)$ is a constant, which leads to $\sigma_2(z) = L_2(z)+H_2(\tilde{s})$, where $L_2(z) = d_{21}z_1 + d_{22}z_2$, $H_2(\tilde{s})$ is a polynomial in $\tilde{s}$, where $\tilde{s} = e_{22}z_1 + e_{21}z_2$, such that $e_{22}c_1 + e_{21}c_2 = 0$ and d_{21}, d_{22} are two constants satisfying that

$$e^{d_{21}c_1+d_{22}c_2} = \frac{A_2 - a_0A_1}{a_1A_1}.$$

Since $\sigma_1(z+c) - \sigma_2(z+c)$ is not a constant so $L_1(z) \neq L_2(z)$. Therefore from (4.8) we get $g(z) = L_1(z) + L_2(z) + H_1(s) + H_2(\tilde{s})$ and from (4.10) we get $f(z) = \frac{1}{\sqrt{2}}\left(A_2 e^{[L_1(z)+H_1(s)]} + A_1 e^{[L_2(z)+H_2(\tilde{s})]}\right)$.

Subcase 2.4.2: Let

$$M_2\, e^{\sigma_2(z)-\sigma_1(z+c)} \equiv 1.$$

then we see that $\sigma_2(z)-\sigma_1(z+c) = \mu_1$, where μ_1 is a constant. Thus it follows from (4.11) $\sigma_1(z)-\sigma_2(z+c) = \mu_2$, where μ_2 is also a constant. Hence we have $\sigma_1(z) - \sigma_2(z) + \sigma_1(z+c) - \sigma_2(z+c) = \mu_2 - \mu_1$, which combining with (4.8) we get $p(z) + p(z+c) = \frac{1}{2}(\mu_2 - \mu_1)$, which contradicts the fact that $\sigma_2(z+c) - \sigma_2(z+c) = 2p(z+c)$ is not a constant. Hence conclusion (ii) of *Theorem3.1* is proved. ☐

<u>**Proof of Theorem 3.2**</u>. Let $f(z)$ be a finite order transcendental entire solution of the equation (2.2). Using the same arguments as done in the proof of *Theorem 3.1* we get (4.9) and

$$P_L(z,f) = \frac{1}{\sqrt{2}}\left(A_2 e^{\sigma_1(z)} + A_1 e^{\sigma_2(z)}\right), \tag{4.20}$$

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where σ_1, σ_2 are defined as in equation (4.8). Using (4.9) and (4.20) we get

$$\begin{aligned}&\frac{\left[A_1\left(b_1\frac{\partial\sigma_1}{\partial z_1}+b_2\frac{\partial\sigma_1}{\partial z_2}\right)-a_0A_2\right]}{a_1A_2}e^{\sigma_1(z)-\sigma_1(z+c)}\\&+\frac{\left[A_2\left(b_1\frac{\partial\sigma_2}{\partial z_1}+b_2\frac{\partial\sigma_2}{\partial z_2}\right)-a_0A_1\right]}{a_1A_2}e^{\sigma_2(z)-\sigma_1(z+c)}-\frac{A_1}{A_2}e^{\sigma_2(z+c)-\sigma_1(z+c)}=1.\end{aligned} \tag{4.21}$$

Now we discuss the following cases:

Case 1: Let $\sigma_2(z+c)-\sigma_1(z+c)$ be a constant, which in view of (4.8) means $p(z)$ is a constant. Let $\lambda = e^{p(z)}$. In view of (4.9) and (4.20) we get

$$L(z,f)=\frac{1}{\sqrt{2}}\left(A_1\lambda+A_2\lambda^{-1}\right)e^{\frac{g(z)}{2}}, \tag{4.22}$$

$$P_L(z,f)=\frac{1}{\sqrt{2}}\left(A_2\lambda+A_1\lambda^{-1}\right)e^{\frac{g(z)}{2}}. \tag{4.23}$$

We deduce from (4.22) and (4.23)

$$\begin{aligned}&\left[\frac{1}{2}\left(A_1\lambda+A_2\lambda^{-1}\right)\left(b_1\frac{\partial g}{\partial z_1}+b_2\frac{\partial g}{\partial z_2}\right)-a_0\left(A_2\lambda+A_1\lambda^{-1}\right)\right]\\&=a_1\left(A_2\lambda+A_1\lambda^{-1}\right)e^{\frac{g(z+c)}{2}-\frac{g(z)}{2}}.\end{aligned} \tag{4.24}$$

If $g(z+c)-g(z)$ is not a constant, then right hand side of (4.24) is transcendental and left hand side is a polynomial, a contradiction. Now two cases can arise

(**i**) $A_2\lambda+A_1\lambda^{-1}=0,\ A_1\lambda+A_2\lambda^{-1}=0$, which implies $A_1^2=A_2^2$, which is a contradiction as $\alpha^2\neq 1$.

(**ii**) $A_2\lambda+A_1\lambda^{-1}=0$ and $b_1\frac{\partial g}{\partial z_1}+b_2\frac{\partial g}{\partial z_2}=0,\ A_1\lambda+A_2\lambda^{-1}\neq 0$. We see that $b_1\frac{\partial g}{\partial z_1}+b_2\frac{\partial g}{\partial z_2}=0$ implies $g=\phi(b_1z_2-b_2z_1)$, which contradicts our assumption. It follows that $g(z+c)-g(z)$ is a constant. Then we have $g(z)=L(z)+H(s)$, where $L(z)=d_1z_1+d_2z_2$, $H(s)$ is a polynomial in s, where $s=e_2z_1+e_1z_2$, such that

$$e_2c_1+e_1c_2=0. \tag{4.25}$$

From (4.24) we get

$$\begin{aligned}&\frac{1}{2}\left(A_1\lambda+A_2\lambda^{-1}\right)\left\{(b_1d_1+b_2d_2)+H'(s)(b_1e_2+b_2e_1)\right\}\\&-a_0\left(A_2\lambda+A_1\lambda^{-1}\right)=a_1\left(A_2\lambda+A_1\lambda^{-1}\right)e^{\frac{d_1c_1+d_2c_2}{2}}.\end{aligned} \tag{4.26}$$

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Using (4.25) and the fact $b_2c_1 - b_1c_2 \neq 0$ we see that deg $H(s)$ can not be greater than one. If so, then one side of (4.26) becomes a non constant polynomial whereas the other side is constant. Hence deg $H(s) \leq 1$. Let $H(s) = l(e_2z_1 + e_1z_2) + d$; l, d be constants in $\mathbb{C}$. Hence from (4.24) we get

$$\frac{1}{2}\left[\frac{\left(A_1\lambda + A_2\lambda^{-1}\right)}{a_1\left(A_2\lambda + A_1\lambda^{-1}\right)}\{b_1(d_1 + le_2) + b_2(d_2 + le_1)\}\right] - \frac{a_0}{a_1} = e^{\frac{d_1c_1 + d_2c_2}{2}}.$$

So the solution becomes

$$f(z) = \frac{\sqrt{2}\left(A_2\lambda + A_1\lambda^{-1}\right)}{b_1(d_1 + ld_2) + b_2(d_2 + ld_1)}e^{\frac{L(z)+H(s)}{2}} + \phi(b_1z_2 - b_2z_1),$$

such that $b_1(d_1 + le_2) + b_2(d_2 + le_1) \neq 0$. Since $\phi(b_1z_2 - b_2z_1)$ is arbitrary, putting $f(z)$ in equation (2.2) we can conclude $\phi(b_1z_2 - b_2z_1) \equiv 0$.

Case 2: Let $\sigma_2(z + c) - \sigma_1(z + c)$ be not-constant.

Take $Q_1 = \frac{\left[A_1\left(b_1\frac{\partial\sigma_1}{\partial z_1} + b_2\frac{\partial\sigma_1}{\partial z_2}\right) - a_0A_2\right]}{a_1A_2}$ and $Q_2 = \frac{\left[A_2\left(b_1\frac{\partial\sigma_2}{\partial z_1} + b_2\frac{\partial\sigma_2}{\partial z_2}\right) - a_0A_1\right]}{a_1A_2}$.

Subcase 2.1: Let $Q_1 \equiv 0$, $Q_2 \equiv 0$. Then from (4.21) we get $-\frac{A_1}{A_2}e^{\sigma_2(z+c)-\sigma_1(z+c)} = 1$, which implies that $\sigma_2(z + c) - \sigma_1(z + c)$ is a constant, which is a contradiction.

Subcase 2.2: Let $Q_1 \equiv 0$ and $Q_2 \not\equiv 0$. Then from (4.21)

$$Q_2e^{\sigma_2(z)-\sigma_1(z+c)} - \frac{A_1}{A_2}e^{\sigma_1(z+c)-\sigma_2(z+c)} = 1. \tag{4.27}$$

Since $\sigma_1(z + c) - \sigma_2(z + c)$ is not a constant then $\sigma_2(z) - \sigma_1(z + c)$ is also non constant. Then from (4.27) we have

$$Q_2e^{\sigma_2(z)} - \frac{A_1}{A_2}e^{\sigma_2(z+c)} - e^{\sigma_1(z+c)} = 0. \tag{4.28}$$

Next we claim that $\sigma_2(z+c) - \sigma_2(z)$ is not a constant. Since otherwise, we get $\sigma_2(z+c) = \sigma_2(z)+k$, where k is a constant in $\mathbb{C}$, and hence from (4.28) we get

$$\left(Q_2e^{-k} - \frac{A_1}{A_2}\right)e^{\sigma_2(z+c)-\sigma_1(z+c)} = 1,$$

which contradicts that $\sigma_2(z + c) - \sigma_1(z + c)$ is non constant. Thus using *Lemma* $4.3,\ 4.4$ on (4.28) we get $Q_2 \equiv 0$, a contradiction.

Subcase 2.3: Let $Q_1 \not\equiv 0$ and $Q_2 \equiv 0$. Then by similar argument as used in **Subcase 2.2** we get a contradiction.

Subcase 2.4: Let $Q_1 \not\equiv 0$, $Q_2 \not\equiv 0$. Then using *Lemma 4.1* in (4.21) we get $Q_1e^{\sigma_1(z)-\sigma_1(z+c)} \equiv 1$, or $Q_2e^{\sigma_2(z)-\sigma_1(z+c)} \equiv 1$.

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Subcase 2.4.1: Let

$$Q_1 e^{\sigma_1(z)-\sigma_1(z+c)} \equiv 1. \tag{4.29}$$

Then we have $\sigma_1(z) - \sigma_1(z+c)$ is a constant. Let $\sigma_1(z) = L_1(z) + H_1(s)$, where $L_1(z) = d_{11}z_1 + d_{12}z_2$; $H_1(s)$ is a polynomial in s, where $s = e_{12}z_1 + e_{11}z_2$; e_{11}, e_{12} are constants such that

$$e_{12}c_1 + e_{11}c_2 = 0. \tag{4.30}$$

Then from (4.29) we get

$$e^{d_{11}c_1+d_{12}c_2} = \left[\frac{A_1}{a_1A_2}\{b_1d_{11} + b_2d_{12} + H_1'(s)(b_1e_{12} + b_2e_{11})\} - \frac{a_0}{a_1}\right]. \tag{4.31}$$

Using $b_2c_1 - b_1c_2 \neq 0$ and (4.30) we get $b_1e_{12} + b_2e_{11} \neq 0$. Hence from (4.31) we see that deg $H_1(s) \leq 1$. Let $H_1(s) = l_1(e_{12}z_1 + e_{11}z_2) + d_1$; d_1, l_1 be two constants in $\mathbb{C}$. Then we get $\sigma_1(z) = L_1(z) + H_1(s)$. From (4.29) we have

$$e^{d_{11}c_1+d_{12}c_2} = \left[\frac{A_1}{a_1A_2}\{b_1(d_{11} + l_1e_{12}) + b_2(d_{12} + l_1e_{11})\} - \frac{a_0}{a_1}\right].$$

Since

$$Q_1 e^{\sigma_1(z)-\sigma_1(z+c)} \equiv 1,$$

then from (4.21) we get

$$Q_2 e^{\sigma_2(z)-\sigma_1(z+c)} = \frac{A_1}{A_2} e^{\sigma_2(z+c)-\sigma_1(z+c)}. \tag{4.32}$$

Now (4.32) yields

$$\left[\frac{A_2}{A_1a_1}\left(b_1\frac{\partial\sigma_2}{\partial z_1} + b_2\frac{\partial\sigma_2}{\partial z_2}\right) - \frac{a_0}{a_1}\right] e^{\sigma_2(z)-\sigma_2(z+c)} = 1. \tag{4.33}$$

So we have $\sigma_2(z) = L_2(z) + H_2(\tilde{s})$, where $L_2(z) = d_{21}z_1 + d_{22}z_2$, $H_2(\tilde{s})$ is a polynomial in $\tilde{s}$, where $\tilde{s} = e_{22}z_1 + e_{21}z_2$; d_{21}, d_{22} are constants such that

$$e_{22}c_1 + e_{21}c_2 = 0. \tag{4.34}$$

From (4.33) we get

$$e^{d_{21}c_1+d_{22}c_2} = \left[\frac{A_2}{a_1A_1}\{b_1d_{21} + b_2d_{22} + H_2'(\tilde{s})(b_1e_{22} + b_2e_{21})\}\right]. \tag{4.35}$$

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Using the fact $b_2c_1 - b_1c_2 \neq 0$ and (4.34) we get $(b_1e_{22} + b_2e_{21}) \neq 0$. Hence from (4.35) we see that $\deg H_2(\tilde{s}) \leq 1$. Let us take $H_2(\tilde{s}) = l_2\,(e_{22}z_1 + e_{21}z_2) + d_2$; l_2, d_2 be constants $\in \mathbb{C}$. Then we get

$$\sigma_2(z) = L_2(z) + H_2(\tilde{s}).$$

So from (4.33) we have

$$e^{d_{21}c_1+d_{22}c_2} = \left[\frac{A_2}{a_1A_1}\left\{b_1(d_{21} + l_2e_{22}) + b_2(d_{22} + l_2e_{21})\right\} - \frac{a_0}{a_1}\right].$$

As $\sigma_2(z+c) - \sigma_1(z+c)$ is not a constant, we have $L_1(z) \neq L_2(z)$. Again from (4.8) we get

$$\begin{aligned} g(z) \quad &= \sigma_1(z) + \sigma_2(z), \\ &= L_1(z) + L_2(z) + H_1(s) + H_2(\tilde{s}). \end{aligned}$$

Hence (4.20) gives

$$\begin{aligned} f(z) \quad &= \frac{1}{\sqrt{2}}\left(\frac{A_2}{b_1(d_{11} + l_1e_{12}) + b_2(d_{12} + l_1e_{11})}e^{L_1(z)+H_1(s)}\right. \\ &\left.+\frac{A_1}{b_1(d_{21} + l_2e_{22}) + b_2(d_{22} + l_2e_{21})}e^{L_2(s)+H_2(\tilde{s})}\right) + \phi_1(b_2z_1 - b_1z_2), \end{aligned}$$

such that $b_1(d_{11}+l_1e_{12})+b_2(d_{12}+l_1e_{11}) \neq 0$, $b_1(d_{21}+l_2e_{22})+b_2(d_{22}+l_2e_{21}) \neq 0$. Since $\phi(b_2z_1-b_1z_2)$ is arbitrary, putting $f(z)$ in equation (2.2) we can conclude that $\phi(b_2z_1 - b_1z_2) \equiv 0$.

Subcase 2.4.2: If $Q_2e^{\sigma_2(z)-\sigma_1(z+c)} \equiv 1$, then $\sigma_2(z)-\sigma_1(z+c)$ is a constant. Let $\sigma_2(z)-\sigma_1(z+c) = \mu_1$, where μ_1 is a constant in $\mathbb{C}$. Then from (4.21) $Q_1e^{\sigma_2(z+c)-\sigma_1(z+c)}$, which implies that $\sigma_1(z)-\sigma_2(z+c) = \mu_2$, where μ_2 is a constant. Hence $\sigma_1(z) - \sigma_2(z+c) - \sigma_2(z) + \sigma_1(z+c) = \mu_2 - \mu_1$, which implies that $p(z) + p(z+c) = \frac{1}{2}(\mu_2 - \mu_1)$, contradicts that $\sigma_1(z+c) - \sigma_2(z+c) = 2p(z+c)$ is not a constant. □

Proof of Theorem 3.3. Let $f(z)$ be a non-constant finite order transcendental entire solution of (2.3). Then by similar arguments as used in *Theorems 3.1* and *3.2* we get

$$f(z+c) = \frac{1}{\sqrt{2}}\left(A_1e^{\sigma_1(z)} + A_2e^{\sigma_2(z)}\right). \tag{4.36}$$

$$\sum_{l+m=1}^{2} b_{lm}\frac{\partial^{(l+m)}f(z)}{\partial z_\beta^l\partial z_\zeta^m} = \frac{1}{\sqrt{2}}\left(A_2e^{\sigma_1(z)} + A_1e^{\sigma_2(z)}\right), \tag{4.37}$$

From (4.36) and (4.37) we get

$$\frac{A_1}{A_2}N_1e^{\sigma_1(z)-\sigma_1(z+c)} + N_2e^{\sigma_2(z)-\sigma_1(z+c)} - \frac{A_1}{A_2}e^{\sigma_2(z+c)-\sigma_1(z+c)} = 1, \tag{4.38}$$

where,
$N_1 = b_{10}\frac{\partial\sigma_1(z)}{\partial z_1} + b_{01}\frac{\partial\sigma_1(z)}{\partial z_2} + b_{11}\left(\frac{\partial^2\sigma_1(z)}{\partial z_1\partial z_2} + \frac{\partial\sigma_1(z)}{\partial z_1}\frac{\partial\sigma_1(z)}{\partial z_2}\right) + b_{20}\left(\frac{\partial^2\sigma_1(z)}{\partial z_1^2} + \left(\frac{\partial\sigma_1(z)}{\partial z_1}\right)^2\right)$
$+ b_{02}\left(\frac{\partial^2\sigma_1(z)}{\partial z_2^2} + \left(\frac{\partial\sigma_1(z)}{\partial z_2}\right)^2\right)$; $N_2 = b_{10}\frac{\partial\sigma_2(z)}{\partial z_1} + b_{01}\frac{\partial\sigma_2(z)}{\partial z_2} + b_{11}\left(\frac{\partial^2\sigma_2(z)}{\partial z_1\partial z_2} + \frac{\partial\sigma_2(z)}{\partial z_1}\frac{\partial\sigma_2(z)}{\partial z_2}\right) +$
$b_{20}\left(\frac{\partial^2\sigma_2(z)}{\partial z_1^2} + \left(\frac{\partial\sigma_2(z)}{\partial z_1}\right)^2\right) + b_{02}\left(\frac{\partial^2\sigma_2(z)}{\partial z_2^2} + \left(\frac{\partial\sigma_2(z)}{\partial z_2}\right)^2\right)$.

Next we consider the following cases:

Case 1: Let $\sigma_2(z+c) - \sigma_1(z+c)$ be a constant. Then using (4.8) we get $-2p(z+c)$ is a constant. Let $\lambda = e^{p(z)}$. Hence from (4.36) and (4.37) we get

$$f(z+c) = \frac{1}{\sqrt{2}}\left(A_1\lambda + A_2\lambda^{-1}\right)e^{\frac{g(z)}{2}}, \tag{4.39}$$

$$\sum_{l+m=1}^{2} b_{lm}\frac{\partial^{(l+m)}f(z)}{\partial z_\beta^l \partial z_\zeta^m} = \frac{1}{\sqrt{2}}\left(A_2\lambda + A_2\lambda^{-1}\right)e^{\frac{g(z)}{2}}. \tag{4.40}$$

From (4.39) and (4.40) we get

$$\left(A_1\lambda + A_2\lambda^{-1}\right)Mg = \left(A_2\lambda + A_1\lambda^{-1}\right)e^{\frac{g(z+c)}{2} - \frac{g(z)}{2}}. \tag{4.41}$$

Let deg $(g(z+c) - g(z)) > 1$. Since $Mg \neq 0$, (4.41) shows that the only possible case is $A_1\lambda + A_2\lambda^{-1} = 0$ and $A_2\lambda + A_1\lambda^{-1} = 0$ which contradics $\alpha^2 \neq 0, 1$. Hence $g(z+c) - g(z)$ must be constant. Let $g(z) = L(z) + H(s)$, where $L(z) = d_1z_1 + d_2z_2$, $H(s)$ is a polynomial in s and $s = e_2z_1 + e_1z_2$ such that

$$e_2c_1 + e_1c_2 = 0, \tag{4.42}$$

d_1, d_2, e_1, e_2 are constants in $\mathbb{C}$. Then from (4.41) we get

$$\begin{aligned}&(A_1\lambda + A_2\lambda^{-1})\left\{\frac{1}{2}b_{10}(d_1 + H'(s)e_2) + \frac{1}{2}b_{01}(d_2 + H'(s)e_1) + b_{11}\left(\frac{1}{2}H''(s)e_1e_2\right.\right.\\ &\left.+\frac{1}{4}(d_1 + H'(s)e_2)(d_2 + H'(s)e_1)\right) + b_{20}\left(\frac{1}{2}H''(s)e_2^2 + \frac{1}{4}\left(d_1 + H'(s)e_2\right)^2\right)\\ &\left.+b_{02}\left(\frac{1}{2}H''(s)e_1^2 + \frac{1}{4}\left(d_2 + H'(s)e_1\right)^2\right)\right\} = (A_2\lambda + A_1\lambda^{-1})e^{\frac{d_1c_1 + d_2c_2}{2}}.\end{aligned} \tag{4.43}$$

For the sake of convenience let us denote B' by the coefficient of $H'(s)$, B'' and C' by that of $H'(s)^2$ and $H''(s)$ and the constant term of (4.43) by C respectively. That is to say $B' = \frac{1}{2}b_{10}e_2 + \frac{1}{2}b_{01}e_1 + \frac{1}{4}b_{11}(d_1e_1 + d_2e_2) + \frac{1}{2}b_{20}d_1e_2 + \frac{1}{2}b_{02}d_2e_1$, $B'' = \frac{1}{4}b_{11}e_1e_2 + \frac{1}{4}b_{20}e_2^2 + \frac{1}{4}b_{02}e_1^2$, $2B'' = C'$, $C = \frac{1}{2}b_{10}d_1 + \frac{1}{2}b_{01}d_2 + \frac{1}{4}b_{11}d_1d_2 + \frac{1}{4}b_{20}d_1^2 + \frac{1}{4}b_{02}d_2^2$.

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Then (4.43) reduces to

$$(A_1\lambda + A_2\lambda^{-1})\left\{C + B'H(s) + B''H(s)^2 + 2B''H''(s)\right\} = (A_2\lambda + A_1\lambda^{-1})e^{\frac{d_1c_1+d_2c_2}{2}}.$$

We discuss the following four possibilities

(i) $B' \neq 0$, $B'' \neq 0$.

(iii) $B' \neq 0$, $B'' = 0$.

(iii) $B' = 0$, $B'' \neq 0$.

(iv) $B' = 0$, $B'' = 0$.

Since one side of (4.43) is constant and one side is polynomial comparing degree of $H(s)$ in both sides of (4.43) under (i) and (ii) it is easy to say that degree of $H(s) \leq 1$. (iii) can also be dealt in the same way and here the degree of $H(s) \leq 1$. Under (iv) $H(s)$ is an arbitrary polynomial.

Now using (4.42) and the fact that $B \neq 0$ and $A \neq 0$, or $A = 0$, we have (i) or (ii). Whereas using (4.42) and the fact that $B = 0$ and $A \neq 0$, we get (iii). So let $H(s) = l(e_2z_1 + e_1z_2) + d$; l, d be constants in $\mathbb{C}$. Therefore $g(z) = L(z) + H(s)$.

From (4.41) we get

$$\begin{aligned} e^{\frac{d_1c_1+d_2c_2}{2}} &= \frac{A_1\lambda + A_2\lambda^{-1}}{2\left(A_2\lambda + A_1\lambda^{-1}\right)}\Big\{b_{10}(d_1 + le_2) + b_{01}(d_2 + le_1) + \frac{1}{2}b_{11}(d_1 + le_2) \\ &\quad (d_2 + le_1) + \frac{1}{2}b_{20}(d_1 + le_2)^2 + \frac{1}{2}b_{02}(d_2 + le_1)^2\Big\}. \end{aligned}$$

i.e.,

$$e^{\frac{d_1c_1+d_2c_2}{2}} = \frac{A_1\lambda + A_2\lambda^{-1}}{2\left(A_2\lambda + A_1\lambda^{-1}\right)}q,$$

where $q = b_{10}(d_1 + le_2) + b_{01}(d_2 + le_1) + \frac{1}{2}b_{11}(d_1 + le_2)(d_2 + le_1) + \frac{1}{2}b_{20}(d_1 + le_2)^2 + \frac{1}{2}b_{02}(d_2 + le_1)^2$. Clearly, $q \neq 0$.

In view of (4.39) we get

$$f(z) = \frac{p}{q}e^{\frac{L(z)+H(s)}{2}},$$

where $p = \sqrt{2}\left(A_2\lambda + A_1\lambda^{-1}\right)$.

Case 2: Let $\sigma_2(z + c) - \sigma_1(z + c)$ be not-constant.

Subcase 2.1: Let $N_1 \equiv 0$ and $N_2 \equiv 0$. Then from (4.38) we see that N_1 and N_2 can not be zero at the same time, otherwise $\sigma_2(z + c) - \sigma_1(z + c)$ is a constant, a contradiction.

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Subcase 2.2: Let $N_1 \equiv 0$ and $N_2 \not\equiv 0$, then from (4.38) it follows

$$N_2 e^{\sigma_2(z)-\sigma_1(z+c)} - \frac{A_1}{A_2} e^{\sigma_2(z+c)-\sigma_1(z+c)} = 1. \tag{4.44}$$

Since $\sigma_2(z+c) - \sigma_1(z+c)$ is not a constant then $\sigma_2(z) - \sigma_1(z+c)$ is non-constant. We rewrite (4.44) as

$$N_2 e^{\sigma_2(z)} - \frac{A_1}{A_2} e^{\sigma_2(z+c)} - e^{\sigma_1(z+c)} = 0.$$

We claim that $\sigma_2(z+c) - \sigma_2(z)$ is not a constant.

Since otherwise, we get $\sigma_2(z+c) = \sigma_2(z) + k$, where k is a constant in $\mathbb{C}$. Hence from (4.44) we get

$$\left(N_2 e^{-k} - \frac{A_1}{A_2} \right) e^{\sigma_2(z+c)-\sigma_1(z+c)} = 1,$$

which contradicts that $\sigma_2(z+c) - \sigma_1(z+c)$ is non constant. Thus using Lemmas $4.3,\ 4.4$ on (4.44) we get $N_2 \equiv 0$, a contradiction.

Subcase 2.3: Let $N_1 \not\equiv 0$ and $N_2 \equiv 0$. By similar arguments as done in **Subcase 2.2**, we get a contradiction.

Subcase 2.4: Let $N_1 \not\equiv 0$ and $N_2 \not\equiv 0$. Then by using *Lemma 4.1* in (4.38) we have

$$\frac{A_1}{A_2} N_1 e^{\sigma_1(z)-\sigma_1(z+c)} \equiv 1,$$

or

$$N_2 e^{\sigma_2(z)-\sigma_1(z+c)} \equiv 1.$$

Subcase 2.4.1: Let

$$\frac{A_1}{A_2} N_1 e^{\sigma_1(z)-\sigma_1(z+c)} \equiv 1, \tag{4.45}$$

we see that $\sigma_1(z)$ must be a one degree polynomial. Thus, $\sigma_1(z) = L_1(z) + H_1(s)$, where $L_1(z) = d_{11}z_1 + d_{12}z_2$, $s = e_{12}z_1 + e_{11}z_2$, $H_1(s)$ is a polynomial in s such that

$$e_{12}c_1 + e_{11}c_2 = 0. \tag{4.46}$$

From (4.45) we get

$$\begin{aligned} &\frac{A_1}{A_2} \{ b_{10}(d_{11} + H_1'(s)e_{12}) + b_{01}(d_{12} + H_1'(s)e_{11}) + b_{11} \left(H_1''(s)e_{11}e_{12} + (d_{11} \right. \\ &+ H_1'(s)e_{12})(d_{12} + H_1'(s)e_{11})) + b_{20} \left(H_1''(s)e_{12}^2 + (d_{11} + H_1'(s)e_{12})^2 \right) + b_{02} \\ &\left(H_1''(s)e_{11}^2 + (d_{12} + H_1'(s)e_{11})^2 \right) \Big\} = e^{d_{11}c_1 + d_{12}c_2}. \end{aligned} \tag{4.47}$$

Now we proceed with the similar arguments as done in **Case 1**. Here using $B \neq 0$ and $A' \neq 0$ or $A' = 0$ together with (4.46) or using $B = 0$ and $A' \neq 0$ with (4.46) we get degree of $H_1(s) \leq 1$. Let

$$H_1(s) = l_1\left(e_{12}z_1 + e_{11}z_2\right) + d_1;$$

d_1, l_1 be constants in $\mathbb{C}$. Then

$$\sigma_1(z) = L_1(z) + H_1(s).$$

From (4.45) we get

$$e^{d_{11}c_1 + d_{12}c_2} = \frac{A_1}{A_2}P,$$

where $P = \{b_{10}(d_{11} + l_1e_{12}) + b_{01}(d_{12} + l_1e_{11}) + b_{11}(d_{11} + l_1e_{12})(d_{12} + l_1e_{11}) + b_{20}(d_{11} + l_1e_{12})^2 + b_{02}(d_{12} + l_1e_{11})^2\}$.

Now in view of (4.38) we get

$$\frac{A_2}{A_1}N_2e^{\sigma_2(z) - \sigma_2(z+c)} \equiv 1. \tag{4.48}$$

In a similar manner we can obtain $\sigma_2(z) = L_2(z) + H_2(\tilde{s})$, where $L_2(z) = d_{21}z_1 + d_{22}z_2$, $H_2(\tilde{s}) = l_2(d_{22}z_1 + d_{21}z_2) + d_2$ is a polynomial in $\tilde{s}$, $\tilde{s} = e_{22}z_1 + e_{21}z_2$ such that

$$e_{22}c_1 + e_{21}c_2 = 0,$$

d_2, l_2 are constants in $\mathbb{C}$.

Following the similar arguments as done in **Case 1** and using $B \neq 0$ or $B = 0$ and $A'' \neq 0$ from (4.48) we get

$$e^{d_{21}c_1 + d_{22}c_2} = \frac{A_2}{A_1}Q,$$

where $Q = \{b_{10}(d_{21} + l_2e_{22}) + b_{01}(d_{22} + l_2e_{21}) + b_{11}(d_{21} + l_2e_{22})(d_{22} + l_2e_{21}) + b_{20}(d_{21} + l_2e_{22})^2 + b_{02}(d_{22} + l_2e_{21})^2\}$.

Next in view of (4.36) we get

$$f(z) = \frac{1}{\sqrt{2}}\left(\frac{A_2}{P}e^{L_1(z)+H_1(s)} + \frac{A_1}{Q}e^{L_2(z)+H_2(\tilde{s})}\right),$$

where $P = \{b_{10}(d_{11} + l_1e_{12}) + b_{01}(d_{12} + l_1e_{11}) + b_{11}(d_{11} + l_1e_{12})(d_{12} + l_1e_{11}) + b_{20}(d_{11} + l_1e_{12})^2 + b_{02}(d_{12} + l_1e_{11})^2\}$ and $Q = \{b_{10}(d_{21} + l_2e_{22}) + b_{01}(d_{22} + l_2e_{21}) + b_{11}(d_{21} + l_2e_{22})(d_{22} + l_2e_{21}) + b_{20}(d_{21} + l_2e_{22})^2 + b_{02}(d_{22} + l_2e_{21})^2\}$.

$$g(z) = L_1(z) + H_1(s) + L_2(z) + H_2(\tilde{s}).$$

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As $\sigma_2(z+c)-\sigma_1(z+c)$ is not a constant, we have $L_1(z)\neq L_2(z)$.

Subcase 2.4.2: Let $N_2e^{\sigma_2(z)-\sigma_1(z+c)}\equiv 1$, then by similar arguments as used in **Subcase 2.4.2** of *Theorem 3.2* we get a contradiction. □

5 An open question

Next observing the structures of the equations discussed throughout the paper, we see that it will be natural to investigate the solutions of the following equation

$$L(z,f)^2+2\,\alpha\,L(z,f)\left(\sum_{l+m=1}^{2} b_{lm}\frac{\partial^{(l+m)}f(z)}{\partial z_\beta^l\partial z_\zeta^m}\right)+\left(\sum_{l+m=1}^{2} b_{lm}\frac{\partial^{(l+m)}f(z)}{\partial z_\beta^l\partial z_\zeta^m}\right)^2=e^{g(z)},$$

or even the equation

$$L_1(z,f)^2+2\,\alpha\,L_1(z,f)\left(\sum_{l+m=1}^{2} b_{lm}\frac{\partial^{(l+m)}f(z)}{\partial z_\beta^l\partial z_\zeta^m}\right)+\left(\sum_{l+m}^{2} b_{lm}\frac{\partial^{(l+m)}f(z)}{\partial z_\beta^l\partial z_\zeta^m}\right)^2=e^{g(z)},$$

where

$$L_1(z,f)=a_0f(z)+a_1f(z+c)+a_2f(z+2c)+\ldots+a_nf(z+nc),$$

where $a_n\neq 0$, so that all the results could be accommodated under a single result. So we place it as the open question for future research.

Acknowledgement: The authors would like to thank the referee for his/her useful suggestions which improved the presentation of the paper. The second author is thankful to Council of Scientific and Industrial Research (India, for financial help under File no.-09/0106(13572)/2022-EMR-I).

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