On Compression of Generalized Slant Toeplitz Operators to $H^2(\mathbb{T}^n)$

Sur la compression des opérateurs de Stant Toeplitz généralisés à $H^2(\mathbb{T}^n)$

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ABSTRACT. In the paper, we introduce the notion of compression of generalized slant Toeplitz operators to the Hardy space of n-dimensional torus $\mathbb{T}^n$. It deals with characterizations of introduced operator with specific as well as general symbols. Certain algebraic and structural properties of considered operators are also investigated. Finally, we discuss few results related to essentially k^{th}-order λ-slant Toeplitz operator.
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1. Introduction

Slant Toeplitz operators on the Lebesgue space of unit circle $\mathbb{T}$ are given as a solution of operator equation $M_zX = XM_{z^2}$. Its collection forms a significant class of operators, which has close links with the collections of multiplication operators and composition operators. In 1996, Ho [11] had introduced this operator and done a systematic and fruitful study in the direction of slant Toeplitz operators. There he discussed not only simple operator theoretic aspects, but also analyzed advanced view points (namely, C*-algebraic view points) related to these operators. More precisely, the study covers spectral properties, norms, isometric behaviour and other operator theoretic properties for these operators. The structural properties of C*-algebra generated by slant Toeplitz operators have also been discussed by Ho [11].

In the view of structure of slant Toeplitz operators, it is evident that the matrix representation of a slant Toeplitz operator with respect to $\{e_n\}_{n=-\infty}^{\infty}$ (an orthonormal basis for $L^2(\mathbb{T})$) can be obtained by eliminating alternate rows of the corresponding doubly infinite Laurent matrix (which gives a multiplication operator on $L^2(\mathbb{T})$).

It is important to point out that the operator equation prove to be an easy mode to illustrate many generalizations of the class of slant Topelitz operators. We would like to refer here the investigations of Barría and Halmos [3] and Avendaño [2], wherein they gave generalized version of operator equations available for the classes of Toeplitz and Hankel operators respectively on $H^2(\mathbb{T})$. In [3], solutions of the operator relation $U^*XU - X \in \mathcal{K}$ have been investigated, where $\mathcal{K}$ is the set of all compact operators on $H^2(\mathbb{T})$. These solutions are referred as essentially Toeplitz operators. It can also be viewed as the essential commutant of the unilateral forward shift operator, whereas in [2], essentially Hankel operator has been introduced and characterized by the operator relation $U^*X - XU \in \mathcal{K}$, where $\mathcal{K}$ is again the same set as given above and U represents the unilateral forward shift operator on the Hardy space $H^2(\mathbb{T})$. Further, Barría and Halmos [3] raised a question related to the solution of $U^*XU = \lambda X$, which leads in the direction of generalization of notion of Toeplitz operator for a fixed complex number λ. Sun [14] solved this equation completely and termed the solutions as λ-Toeplitz operators. By another point of view, the non-zero solution of $U^*XU = \lambda X$, can be seen as an eigenvector of the operator $T : H^2 \longrightarrow H^2$ given by $T(X) = U^*XU$ corresponding to λ. Other generalizations of Toeplitz operators (namely, dual Toeplitz and commuting Toeplitz operators) have also been taken into consideration on the bi-disk by several researchers (see [7, 8, 9]). In last few years, these convictions were adjoined to the slant Toeplitz operators and the following classes have been investigated on $L^2(\mathbb{T})$ or $H^2(\mathbb{T})$.

1. Essentially slant Toeplitz operators: $M_zX - XM_{z^2} = K_1$
2. Essentially k^{th}-order slant Toeplitz operators: $M_zX - XM_{z^k} = K_2$
3. Generalized λ-slant Toeplitz operators: $\lambda M_zX = XM_{z^k}$
4. Compression of k- order slant Toeplitz operators: $X = T_z^*XT_{z^k}$ on $H^2(\mathbb{T})$
5. Essentially generalized λ-slant Toeplitz operators: $\lambda M_zX - XM_{z^k} = K_3$,

where K_i, $1 \leq i \leq 3$, are compact operators on $L^2(\mathbb{T})$. Here, as well as from now onwards, k and n denote integers greater or equal to 2 and 1, respectively. The symbol λ is a fixed complex number. Papers [1, 4, 14] and the references therein are referred for more details about these classes of operators.

Because of enough features and applications (see [15]), the study of slant Toeplitz operators and its compression is further enhanced and lifted to the Lebesgue space $L^2(\mathbb{T}^n)$ of n-torus in [5, 6]. It is generalized to investigate and explain k^{th}-order λ-slant Toeplitz operators on $L^2(\mathbb{T}^n)$, where $\mathbb{T}^n$ denotes n-torus and is a subset of $\mathbb{C}^n$.

Motivated by the work initiated by several mathematicians listed above, in this paper, we introduce and analyze the class of compression of k^{th}-order λ-slant Toeplitz operators on the Hardy space $H^2(\mathbb{T}^n)$ and essentially k^{th}-order λ-slant Toeplitz operators on $L^2(\mathbb{T}^n)$. Before moving ahead, the essential terminologies and prerequisites are provided, which will be needed for further study.

Let the set of all non-negative integers and the set of all integers are denoted by $\mathbb{Z}_+$ and $\mathbb{Z}$ respectively. The Lebesgue space $L^2(\mathbb{T}^n)$ (the collection of all Lebesgue measurable complex valued functions defined on $\mathbb{T}^n$ such that $\int_{\mathbb{T}^n} |f|^2 \, d\sigma < \infty$ holds) can be expressed as

$$L^2(\mathbb{T}^n) = \left\{ f : f(z_1, z_2, \cdots, z_n) = \sum_{(m_1,m_2,\cdots,m_n)\in\mathbb{Z}^n} f_{m_1,m_2,\cdots,m_n} z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}, \right.$$
$$\left. \sum_{(m_1,m_2,\cdots,m_n)\in\mathbb{Z}^n} |f_{m_1,m_2,\cdots,m_n}|^2 < \infty \right\},$$

by employing multiple Fourier series [13] on $\mathbb{T}^n$. Here, $d\sigma$ denotes the normalized Lebesgue measure. The Lebesgue space $L^2(\mathbb{T}^n)$ is a Hilbert space with the inner product

$$\langle f, g \rangle = \frac{1}{(2\pi)^n} \underbrace{\int_0^{2\pi}\int_0^{2\pi} \cdots \int_0^{2\pi}}_{n-times} f(e^{i\theta_1}, e^{i\theta_2}, \cdots, e^{i\theta_n}) \overline{g(e^{i\theta_1}, e^{i\theta_2}, \cdots, e^{i\theta_n})} \, d\theta_1 d\theta_2 \ldots d\theta_n,$$

In a similar way, it can be seen that the space $H^2(\mathbb{T}^n)$ (a closed subspace of $L^2(\mathbb{T}^n)$) has the same kind of structure and inner product as those of $L^2(\mathbb{T}^n)$. An orthonormal basis for the space $H^2(\mathbb{T}^n)$ is given by $\{e_{m_1,m_2,\cdots,m_n} : (m_1, m_2, \ldots, m_n) \in \mathbb{Z}_+^n\}$, where $e_{m_1,\cdots,m_n}(z_1, z_2, \cdots, z_n) = z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n}$. Note that we oftenly write $z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n}$ as a basis element of $L^2(\mathbb{T}^n)$ or $H^2(\mathbb{T}^n)$ instead of $e_{m_1,m_2,\cdots,m_n}$, whenever there is no notational confusion. For $n \geq 1$, the Hardy space $H^2(\mathbb{D}^n)$ [10, 12] over the open unit polydisc $\mathbb{D}^n$ is the set of all analytic functions on $\mathbb{D}^n$ such that

$$\|f\| := \left(\sup_{0 \leq r < 1} \int_{\mathbb{T}^n} |f(re^{i\theta_1}, re^{i\theta_2}, \ldots, re^{i\theta_n})|^2 d\theta_1 d\theta_2 \ldots d\theta_n \right)^{\frac{1}{2}},$$

is finite, where $d\theta_1 d\theta_2 \ldots d\theta_n$ represents the normalized Lebesgue measure on $\mathbb{T}^n$.

One can identify the Hardy space $H^2(\mathbb{D}^n)$ and the space $H^2(\mathbb{T}^n)$ via the radial limits of functions of $H^2(\mathbb{D}^n)$. Henceforth, a function f in $L^2(\mathbb{T}^n)$ is termed as analytic if its Fourier coefficients $f_{m_1,m_2,\ldots,m_n} = 0$, whenever $m_j < 0$ for at least one j, $1 \leq j \leq n$. A function g of $L^2(\mathbb{T}^n)$ is called co-analytic if its complex conjugate $\bar{g}$ is analytic. The Banach space of all essentially bounded measurable functions on $\mathbb{T}^n$ with essential supremum norm $\|\cdot\|_\infty$ is denoted by $L^\infty(\mathbb{T}^n)$. We fix the notations $\mathfrak{B}(L^2(\mathbb{T}^n))$ and $\mathcal{K}(L^2(\mathbb{T}^n))$ to represent the algebra of all bounded linear operators and the ideal of all compact operators on $L^2(\mathbb{T}^n)$. The symbol B_n is used to express the standard basis of $\mathbb{R}^n$.

Now, we recall from a reference of [6] that a k^{th}-order slant Toeplitz operator is given by $A_{\phi,k,n} = E_{k,n} M_\phi$, where the linear operator $E_{k,n}$ on $L^2(\mathbb{T}^n)$ is defined as

$$E_{k,n}(z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n}) = \begin{cases} z_1^{\frac{m_1}{k}} z_2^{\frac{m_2}{k}} \ldots z_n^{\frac{m_n}{k}} & \text{if each } m_i \text{ is a multiple of } k, 1 \leq i \leq n \\ 0 & \text{otherwise,} \end{cases}$$

and M_ϕ denotes the multiplication operator on the Lebesgue space $L^2(\mathbb{T}^n)$ with symbol $\phi \in L^\infty(\mathbb{T}^n)$. It is shown that the system of operator equations $M_{z_1^{j_1} z_2^{j_2} \ldots z_n^{j_n}} X = X M_{z_1^{kj_1} z_2^{kj_2} \ldots z_n^{kj_n}}$, for $(j_1, j_2, \ldots, j_n) \in B_n$, characterizes k^{th}-order slant Toeplitz operator. The work done in [1, 2, 4] and [14] motivate us to carry out this study and derive some notions from the slant Toeplitz operator on $L^2(\mathbb{T}^n)$ in the present paper.

2. Compression of k^{th}-order λ-slant Toeplitz operator

The section begins with few definitions and elementary results related to slant Toeplitz operator on $L^2(\mathbb{T}^n)$ and its compression on $H^2(\mathbb{T}^n)$. Initially, define a linear operator on $L^2(\mathbb{T}^n)$ for a fixed λ in the unit circle, given by

$$D_{\bar{\lambda}}(f)(z_1, z_2, \ldots, z_n) = f(\lambda z_1, \lambda z_2, \ldots, \lambda z_n). \tag{1}$$

One dimensional version of the operator $D_{\bar{\lambda}}$ is utilized in the description of the solution of equation $U^*TU = \lambda T$, where U is unilateral shift operator of $L^2(\mathbb{T})$, analyzed by Sun [14]. The use of the operator

$D_{\bar{\lambda}}$ gives rise a similar kind of operators, discussed by Ho [11] and Sun [14], in higher dimensional setup. From now onwards, λ is chosen as a complex number with unit modulus unless otherwise stated.

Definition 2.1. *[6] A k^{th}-order λ-slant Toeplitz operator $A_{\phi,k,\lambda}$, with symbol $\phi \in L^\infty(\mathbb{T}^n)$, on $L^2(\mathbb{T}^n)$ can be defined as*

$$A_{\phi,k,\lambda}(f) = D_{\bar{\lambda}} A_{\phi,k,n}(f), \text{ for all } f \in L^2(\mathbb{T}^n),$$

where $D_{\bar{\lambda}}$ is same as defined in (1) and $A_{\phi,k,n}$ is a k^{th}-order slant Toeplitz operator with symbol ϕ.

Recall that a k^{th}-order λ-slant Toeplitz operator arises as a solution of system of operator equations in the following form:

$$\lambda^{(j_1+j_2+\ldots+j_n)} M_{z_1^{j_1} z_2^{j_2} \ldots z_n^{j_n}} X = X M_{z_1^{kj_1} z_2^{kj_2} \ldots z_n^{kj_n}}, \tag{2}$$

for $(j_1, j_2, \ldots, j_n) \in \mathbb{Z}^n$. We know that, for $|\lambda| = 1$, the above system (2) has non-trivial solutions. Further, in [6], it is refined and shown that the condition given by (2) for $(j_1, j_2, \ldots, j_n) \in B_n$ becomes a necessary and sufficient condition for an operator X on $L^2(\mathbb{T}^n)$ to be a k^{th}-order λ-slant Toeplitz operator. This characterization should be compared with its counterpart in one variable case for λ-Toeplitz operator and λ-slant Toeplitz operator derived by Sun [14] and Datt, Aggarwal [4] respectively. It is worth mentioning that the non-zero solution of the system of operator equations $\lambda X = M^*_{z_1^{j_1} z_2^{j_2} \ldots z_n^{j_n}} X M_{z_1^{kj_1} z_2^{kj_2} \ldots z_n^{kj_n}}$, for $(j_1, j_2, \ldots, j_n) \in B_n$, can be viewed as common eigenvector of operators $T_{j_1,j_2,\ldots,j_n} : L^2(\mathbb{T}^n) \longrightarrow L^2(\mathbb{T}^n)$ given by

$$T_{j_1,j_2,\ldots,j_n}(X) = M^*_{z_1^{j_1} z_2^{j_2} \ldots z_n^{j_n}} X M_{z_1^{kj_1} z_2^{kj_2} \ldots z_n^{kj_n}},$$

corresponding to λ.

Again, recall that an operator T on $H^2(\mathbb{T}^n)$ is said to be a compression of k^{th}-order slant Toeplitz operator if it can be written as

$$T(f) = P A_{\phi,k,n}(f), \qquad \text{for all } f \in H^2(\mathbb{T}^n),$$

where $\phi \in L^\infty(\mathbb{T}^n)$ and P is an orthogonal projection from $L^2(\mathbb{T}^n)$ onto the space $H^2(\mathbb{T}^n)$. Equivalently, $V_{\phi,k,n} = P A_{\phi,k,n}|_{H^2(\mathbb{T}^n)}$. Throughout the paper, a k^{th}-order slant Toeplitz operator and its compression with inducing function ϕ is denoted by $A_{\phi,k,n}$ and $V_{\phi,k,n}$, respectively. Next, we deal with the compression of k^{th}-order λ-slant Toeplitz operator to the Hardy space $H^2(\mathbb{T}^n)$ and also look for its characterizations and some of its properties.

Definition 2.2. *For $\phi \in L^\infty(\mathbb{T}^n)$, an operator $V_{\phi,k,\lambda}$ on $H^2(\mathbb{T}^n)$ such that $V_{\phi,k,\lambda} = P A_{\phi,k,\lambda}|_{H^2(\mathbb{T}^n)}$ is known as the compression of k^{th}-order λ-slant Toeplitz operator $A_{\phi,k,\lambda}$, where P is an orthogonal projection from $L^2(\mathbb{T}^n)$ onto the space $H^2(\mathbb{T}^n)$.*

The linearity and boundedness of the operator $V_{\phi,k,\lambda}$ follow from the fact that P as well as $A_{\phi,k,\lambda}$ are bounded linear operators on $L^2(\mathbb{T}^n)$ for $\phi \in L^\infty(\mathbb{T}^n)$. The following assertion highlights the linkage between compressions $V_{\phi,k,\lambda}$ and $V_{\phi,k,n}$ to the Hardy space of the operators $A_{\phi,k,\lambda}$ and $A_{\phi,k,n}$ respectively.

Lemma 2.3. *Let $\lambda \in \mathbb{T}$, then $V_{\phi,k,\lambda} = D_{\bar{\lambda}} V_{\phi,k,n}$ for $\phi \in L^\infty(\mathbb{T}^n)$.*

Proof. To furnish a quick proof, for $(i_1, i_2, \ldots, i_n) \in \mathbb{Z}^n$, we observe that

$$\begin{aligned} PD_{\bar{\lambda}}(z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n}) &= \begin{cases} \lambda^{(i_1+i_2+\ldots+i_n)} z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n}, & \text{if each } i_j \geq 0 \text{ for } 1 \leq j \leq n \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} D_{\bar{\lambda}}(z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n}), & \text{if each } i_j \geq 0 \text{ for } 1 \leq j \leq n \\ D_{\bar{\lambda}}(0), & \text{otherwise} \end{cases} \\ &= D_{\bar{\lambda}} P(z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n}). \end{aligned}$$

This observation immediately yields that $PD_{\bar{\lambda}} = D_{\bar{\lambda}}P$. Therefore, $V_{\phi,k,\lambda}$ can be rewritten as $V_{\phi,k,\lambda} = PA_{\phi,k,\lambda}|_{H^2(\mathbb{T}^n)} = D_{\bar{\lambda}} PA_{\phi,k,n}|_{H^2(\mathbb{T}^n)} = D_{\bar{\lambda}} V_{\phi,k,n}$. This completes the proof. □

The observation made in the Lemma (2.3) instantly provides that the composition operator $D_{\bar{\lambda}}$ commutes with the projection P. Therefore, $H^2(\mathbb{T}^n)$ $(= Range(P))$ is a reducing subspace of $D_{\bar{\lambda}}$. Similarly, we can show that $H^2(\mathbb{T}^n)$ is also a reducing subspace of $E_{k,n}$.

Now, we look up the actions of compression of k^{th}-order λ-slant Toeplitz operator and its adjoint on the basis elements to have a systematic study as well as structure of these operators. Note that for $\phi(z_1, z_2, \ldots, z_n) = \sum_{(m_1,m_2,\ldots,m_n)\in\mathbb{Z}^n} \phi_{m_1,m_2,\ldots,m_n} z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n} \in L^\infty(\mathbb{T}^n)$, and $(i_1, i_2, \ldots, i_n) \in \mathbb{Z}^n$, we have

$$A_{\phi,k,\lambda}\left(z_1^{i_1} \ldots z_n^{i_n}\right) = \sum_{(m_1,\ldots,m_n)\in\mathbb{Z}^n} \phi_{km_1-i_1,\ldots,km_n-i_n} \lambda^{(m_1+\ldots+m_n)} z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n}. \quad (3)$$

Utilizing the relation (3), the action of compression $V_{\phi,k,\lambda}$ of k^{th}-order λ-slant Toeplitz operator on basis elements of $H^2(\mathbb{T}^n)$ can be written as

$$\begin{aligned} V_{\phi,k,\lambda}\left(z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n}\right) &= P\left(\sum_{(m_1,\ldots,m_n)\in\mathbb{Z}^n} \phi_{km_1-i_1,\ldots,km_n-i_n} \lambda^{(m_1+\ldots+m_n)} z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n}\right) \\ &= \sum_{(m_1,\ldots,m_n)\in\mathbb{Z}_+^n} \phi_{km_1-i_1,\ldots,km_n-i_n} \lambda^{(m_1+\ldots+m_n)} z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n}. \end{aligned}$$

To find out the adjoint $V_{\phi,k,\lambda}^*$ of $V_{\phi,k,\lambda}$ on $H^2(\mathbb{T}^n)$, we compute the following, for $(i_1, \ldots, i_n), (j_1, \ldots, j_n) \in \mathbb{Z}_+^n$

$$\begin{aligned} &\left\langle V_{\phi,k,\lambda}^*(z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n}),\, z_1^{j_1} z_2^{j_2} \ldots z_n^{j_n}\right\rangle \\ &\quad = \left\langle z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n},\, \sum_{(m_1,\ldots,m_n)\in\mathbb{Z}_+^n} \phi_{km_1-j_1,\ldots,km_n-j_n} \lambda^{(m_1+\ldots+m_n)} z_1^{m_1} \ldots z_n^{m_n}\right\rangle \\ &\quad = \bar{\lambda}^{(i_1+i_2+\ldots+i_n)} \bar{\phi}_{ki_1-j_1,ki_2-j_2,\ldots,ki_n-j_n} \\ &\quad = \left\langle \sum_{(m_1,\ldots,m_n)\in\mathbb{Z}_+^n} \bar{\phi}_{ki_1-m_1,\ldots,ki_n-m_n} \bar{\lambda}^{(i_1+\ldots+i_n)} z_1^{m_1} \ldots z_n^{m_n},\, z_1^{j_1} z_2^{j_2} \ldots z_n^{j_n}\right\rangle. \end{aligned} \quad (4)$$

Equation (4) readily provides that

$$V^*_{\phi,k,\lambda}(z_1^{i_1} z_2^{i_2} \dots z_n^{i_n}) = \sum_{(m_1,m_2,\dots,m_n)\in\mathbb{Z}_+^n} \bar{\phi}_{ki_1-m_1,\dots,ki_n-m_n} \bar{\lambda}^{(i_1+\dots+i_n)} z_1^{m_1} z_2^{m_2} \dots z_n^{m_n},$$

for each $(i_1, i_2, \dots, i_n) \in \mathbb{Z}_+^n$. Equivalently, one can show that $V^*_{\phi,k,\lambda} = PA^*_{\phi,k,\lambda}|_{H^2(\mathbb{T}^n)}$. Also, using the fact $D_{\bar{\lambda}} M_{D_\lambda(\phi)} = M_\phi D_{\bar{\lambda}}$, it can be seen that

$$\begin{aligned} T_{\phi,n} D_{\bar{\lambda}} E_{k,n}|_{H^2(\mathbb{T}^n)} &= P D_{\bar{\lambda}} M_{D_\lambda(\phi)} E_{k,n}|_{H^2(\mathbb{T}^n)} \\ &= P D_{\bar{\lambda}} E_{k,n} M_\zeta|_{H^2(\mathbb{T}^n)} \\ &= V_{\zeta,k,\lambda}, \end{aligned} \tag{5}$$

where $\zeta = E^*_{k,n} D_\lambda(\phi) \in L^\infty(\mathbb{T}^n)$. The relation (5) yields the connection between Toeplitz operator and compression of k^{th}-order λ-slant Toeplitz operator. Before discussing the characterization, we examine an elementary but important fact about most frequently used operators (namely, $E_{k,n}$ and $D_{\bar{\lambda}}$) throughout the paper. Let λ be a complex number of unit modulus and $k \geq 2$. Consider two cases:

1. When there is at least one j_0 with $1 \leq j_0 \leq n$ such that m_{j_0} is not a multiple of k. Then, we have $E_{k,n} D_{\bar{\lambda}}(z_1^{m_1} z_2^{m_2} \dots z_n^{m_n}) = D_{\bar{\lambda}} E_{k,n}(z_1^{m_1} z_2^{m_2} \dots z_n^{m_n}) = 0$.
2. When all m_j's are in the multiple of k i.e. $m_j = ki_j$. Then,

$$\begin{aligned} D_{\bar{\lambda}} E_{k,n}(z_1^{ki_1} z_2^{ki_2} \dots z_n^{ki_n}) &= D_{\bar{\lambda}}(z_1^{i_1} z_2^{i_2} \dots z_n^{i_n}) \\ &= \lambda^{(i_1+i_2+\dots+i_n)} z_1^{i_1} z_2^{i_2} \dots z_n^{i_n} \\ &= E_{k,n}\left(\lambda^{(i_1+i_2+\dots+i_n)} z_1^{ki_1} z_2^{ki_2} \dots z_n^{ki_n}\right) \\ &= \bar{\lambda}^{(k-1)(i_1+i_2+\dots+i_n)} E_{k,n} D_{\bar{\lambda}} \left(z_1^{ki_1} z_2^{ki_2} \dots z_n^{ki_n}\right). \end{aligned}$$

It reveals that operators $E_{k,n}$ and $D_{\bar{\lambda}}$ do not commute because of $k \geq 2$.

Now, consider a mapping $\Psi : L^\infty(\mathbb{T}^n) \longrightarrow \mathfrak{B}(H^2(\mathbb{T}^n))$, given by $\Psi(\phi) = V_{\phi,k,\lambda}$. By its definition, it is clear that the mapping Ψ is linear. Since, $D_{\bar{\lambda}}$ is a unitary operator and the map $\phi \mapsto V_{\phi,k,n}$ is one-one. Therefore, we can conclude that the function Ψ is linear as well as injective.

An instant consequence can be retrieved from the preceding discussion, which can be written as follows: “The operator $V_{\phi,k,\lambda}$ is the zero operator if and only if the inducing function is the zero function".

Before furnishing the characterization for compression of k^{th}-order λ-slant Toeplitz operator, we investigate the necessary and sufficient conditions for its particular case (namley, $\lambda = 1$), which will be helpful in characterizing most generalized version that is, compression of k^{th}-order λ-slant Toeplitz operator. Note that the necessary part of the following theorem is already proved in [5]. Here, we not only provide a different proof for the necessary part but also prove sufficient part of the following theorem.

Theorem 2.4. *A bounded operator $V \in \mathfrak{B}\left(H^2(\mathbb{T}^n)\right)$ is a compression of k^{th}-order slant Toeplitz operator to the Hardy space $H^2(\mathbb{T}^n)$ if and only if the following conditions*

$$V = T^*_{z_1^{p_1} z_2^{p_2} \dots z_n^{p_n},n} V T_{z_1^{kp_1} z_2^{kp_2} \dots z_n^{kp_n},n} \text{ for each } (p_1, p_2, \dots, p_n) \in B_n, \tag{6}$$

hold.

Proof. Assume that the operator $V \in \mathfrak{B}\left(H^2(\mathbb{T}^n)\right)$ satisfies $V = T^*_{z_1^{p_1}\ldots z_n^{p_n},n} V T_{z_1^{kp_1} z_2^{kp_2}\ldots z_n^{kp_n},n}$ for $(p_1, p_2, \ldots, p_n) \in B_n$. Hence we obtain that $V = \left(T^m_{z_1 z_2 \ldots z_n,n}\right)^* V T^m_{z_1^k z_2^k \ldots z_n^k,n}$ for each non-negative integer m. Again, for non-negative integer m, define an operator on $L^2(\mathbb{T}^n)$ to be given as

$$V_m = S^{*m} V P S^{km},$$

where S is the multiplication operator $M_{z_1 z_2 \ldots z_n}$ with symbol $\phi \equiv z_1 z_2 \ldots z_n$ and P is an orthogonal projection from the space $L^2(\mathbb{T}^n)$ onto $H^2(\mathbb{T}^n)$. If n-tuples $(i_1, i_2, \ldots, i_n)$, $(j_1, j_2, \ldots, j_n)$ belong to $\mathbb{Z}^n_+$, then we have

$$\begin{aligned}\left\langle V_m(z_1^{j_1} z_2^{j_2} \ldots z_n^{j_n}),\, z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n}\right\rangle &= \left\langle VP(z_1^{km+j_1} \ldots z_n^{km+j_n}),\, z_1^{i_1+m} z_2^{i_2+m} \ldots z_n^{i_n+m}\right\rangle \\ &= \left\langle V(z_1^{km+j_1} \ldots z_n^{km+j_n}),\, z_1^{i_1+m} z_2^{i_2+m} \ldots z_n^{i_n+m}\right\rangle \\ &= \left\langle V(z_1^{j_1} z_2^{j_2} \ldots z_n^{j_n}),\, z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n}\right\rangle. \end{aligned} \tag{7}$$

Now, consider the case when $(i_1, i_2, \ldots, i_n)$, $(j_1, j_2, \ldots, j_n) \in \mathbb{Z}^n$ with at least one i_r or j_t negative, $1 \le r \le n$ and $1 \le t \le n$. Let I be the collection of all these negative components of n-tuples. Again, suppose that $m_0 = max\{-s : s \in I\}$. Then, this gives that $km_0 + j_p$ and $m_0 + i_p$ are non-negative integers for all $1 \le p \le n$ and

$$\begin{aligned}&\left\langle V_{m_0}(z_1^{j_1} z_2^{j_2} \ldots z_n^{j_n}),\, z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n}\right\rangle \\ &\qquad = \left\langle VP(z_1^{km_0+j_1} z_2^{km_0+j_2} \ldots z_n^{km_0+j_n}),\, z_1^{i_1+m_0} z_2^{i_2+m_0} \ldots z_n^{i_n+m_0}\right\rangle \\ &\qquad = \left\langle V(z_1^{km_0+j_1} z_2^{km_0+j_2} \ldots z_n^{km_0+j_n}),\, z_1^{i_1+m_0} z_2^{i_2+m_0} \ldots z_n^{i_n+m_0}\right\rangle.\end{aligned}$$

Similarly, it is easy to calculate

$$\begin{aligned}&\left\langle V_{m_0+1}(z_1^{j_1} z_2^{j_2} \ldots z_n^{j_n}),\, z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n}\right\rangle \\ &\quad = \left\langle V(z_1^{km_0+j_1+k} z_2^{km_0+j_2+k} \ldots z_n^{km_0+j_n+k}),\, z_1^{i_1+m_0+1} z_2^{i_2+m_0+1} \ldots z_n^{i_n+m_0+1}\right\rangle \\ &\quad = \left\langle V(z_1^{km_0+j_1} z_2^{km_0+j_2} \ldots z_n^{km_0+j_n}),\, z_1^{i_1+m_0} z_2^{i_2+m_0} \ldots z_n^{i_n+m_0}\right\rangle \\ &\quad = \left\langle V_{m_0}(z_1^{j_1} z_2^{j_2} \ldots z_n^{j_n}),\, z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n}\right\rangle.\end{aligned}$$

Analogously, one can verify that

$$\left\langle V_m(z_1^{j_1} z_2^{j_2} \ldots z_n^{j_n}),\, z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n}\right\rangle = \left\langle V_{m_0}(z_1^{j_1} z_2^{j_2} \ldots z_n^{j_n}),\, z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n}\right\rangle, \text{ for all } m \ge m_0.$$

This provides that the sequence $\left(\left\langle V_m(z_1^{j_1} z_2^{j_2} \ldots z_n^{j_n}),\, z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n}\right\rangle\right)_{m\in\mathbb{Z}_+}$ is constant for sufficiently large m and hence convergent for each n-tuple $(i_1, i_2, \ldots, i_n)$, $(j_1, j_2, \ldots, j_n) \in \mathbb{Z}^n$. By linearity, it also follows that the $(\langle V_m p, q\rangle)_{m\in\mathbb{Z}_+}$ is also constant for large value of m and hence convergent for all trigonometric polynomials p, q. We define a sesquilinear form Φ on D, a subset of $L^2(\mathbb{T}^n)$ consisting of trigonometric polynomials, given by

$$\Phi(p, q) = \lim_{m\to\infty} \langle V_m p, q\rangle.$$

Clearly, Φ is a bounded sesquilinear form on D, because of $\Phi(p,q) \leq \|V\|\|p\|\|q\|$. Also, it can be extended to a bounded sesquilinear form Ψ on $L^2(\mathbb{T}^n)$. Therefore, there exists a bounded linear operator V_∞ on $L^2(\mathbb{T}^n)$, which is unique and satisfies $\Psi(f,g) = \langle V_\infty f, g\rangle$ on $L^2(\mathbb{T}^n)$. Again, for $f, g \in L^2(\mathbb{T}^n)$, it implies that

$$\lim_{m\to\infty} \langle V_m f, g\rangle = \langle V_\infty f, g\rangle.$$

Now, our aim is to show that V_∞ is a k-th order slant Toeplitz operator whose compression is the given operator V. For $(i_1, i_2, \ldots, i_n)$, $(j_1, j_2, \ldots, j_n) \in \mathbb{Z}^n$ and $q \in \{1, 2, \ldots, n\}$, we have the following for sufficiently large m (depending on i_p, j_p, q such that $1 \leq p \leq n$)

$$\begin{aligned}
&\left\langle V_m(z_1^{j_1} \ldots z_q^{j_q+k} \ldots z_n^{j_n}),\, z_1^{i_1} \ldots z_q^{i_q+1} \ldots z_n^{i_n} \right\rangle \\
&\qquad = \left\langle VP(z_1^{j_1+km} \ldots z_q^{j_q+k+km} \ldots z_n^{j_n+km}),\, z_1^{i_1+m} \ldots z_q^{i_q+1+m} \ldots z_n^{i_n+m} \right\rangle \\
&\qquad = \left\langle T_{z_q}^* V T_{z_q^k}(z_1^{j_1+km} \ldots z_q^{j_q+km} \ldots z_n^{j_n+km}),\, z_1^{i_1+m} \ldots z_q^{i_q+m} \ldots z_n^{i_n+m} \right\rangle \\
&\qquad = \left\langle VP(z_1^{j_1+km} \ldots z_q^{j_q+km} \ldots z_n^{j_n+km}),\, z_1^{i_1+m} \ldots z_q^{i_q+m} \ldots z_n^{i_n+m} \right\rangle \\
&\qquad = \left\langle V_m(z_1^{j_1} \ldots z_q^{j_q} \ldots z_n^{j_n}),\, z_1^{i_1} \ldots z_q^{i_q} \ldots z_n^{i_n} \right\rangle. \qquad (8)
\end{aligned}$$

The expression (8) resulted in the following form for $q \in \{1, 2, \ldots, n\}$.

$$\begin{aligned}
\left\langle M_{z_q}^* V_\infty M_{z_q^k}(z_1^{j_1} \ldots z_n^{j_n}),\, z_1^{i_1} \ldots z_n^{i_n} \right\rangle &= \left\langle V_\infty(z_1^{j_1} \ldots z_q^{j_q+k} \ldots z_n^{j_n}),\, z_1^{i_1} \ldots z_q^{i_q+1} \ldots z_n^{i_n} \right\rangle \\
&= \lim_{m\to\infty} \left\langle V_m(z_1^{j_1} \ldots z_q^{j_q+k} \ldots z_n^{j_n}),\, z_1^{i_1} \ldots z_q^{i_q+1} \ldots z_n^{i_n} \right\rangle \\
&= \lim_{m\to\infty} \left\langle V_m(z_1^{j_1} z_2^{j_2} \ldots z_n^{j_n}),\, z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n} \right\rangle \\
&= \left\langle V_\infty(z_1^{j_1} z_2^{j_2} \ldots z_n^{j_n}),\, z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n} \right\rangle.
\end{aligned}$$

This yields that $V_\infty = M_{z_q}^* V_\infty M_{z_q^k}$ for each integer q such that $1 \leq q \leq n$. Using characterization of k^{th}-order slant Toeplitz operator, this confirms that V_∞ is a k-th order slant Toeplitz operator. Finally, let f and g be two functions in $H^2(\mathbb{T}^n)$, then in the view of (7), we get that

$$\langle PV_\infty|_{H^2(\mathbb{T}^n)} f, g\rangle = \langle V_\infty f, g\rangle = \lim_{m\to\infty} \langle V_m f, g\rangle = \lim_{m\to\infty} \langle Vf, g\rangle = \langle Vf, g\rangle.$$

Hence, $V = PV_\infty|_{H^2(\mathbb{T}^n)}$ that is, V is a compression of k^{th}-order slant Toeplitz operator.

Conversely, let $V = PA_{\phi,k,n}\big|_{H^2(\mathbb{T}^n)}$ for some $\phi \in L^\infty(\mathbb{T}^n)$, where $A_{\phi,k,n}$ is a k^{th}-order slant Toeplitz operator induced by ϕ. Using the fact that $PM_{\bar{z}_j}P = PM_{\bar{z}_j}$, $1 \leq j \leq n$, we find that

$$\begin{aligned}
T_{z_j}^* V T_{z_j^k}(f) &= T_{z_j}^* PA_{\phi,k,n} T_{z_j^k}(f) \\
&= PM_{z_j}^* PA_{\phi,k,n} PM_{z_j^k}(f) \\
&= PM_{z_j}^* A_{\phi,k,n} M_{z_j^k}(f) = PA_{\phi,k,n}(f) = V(f),
\end{aligned}$$

for each integer j, $1 \leq j \leq n$ and $f \in H^2(\mathbb{T}^n)$. It proves that $T_{z_j}^* V T_{z_j^k} = V$ for $1 \leq j \leq n$. This completes the proof. □

Now, we are in a position to provide a result, which initially furnish a characterization for the compression of k^{th}-order λ-slant Toeplitz operator on $H^2(\mathbb{T}^n)$ for general inducing function. After that, we also deal with an assertion, which not only equipped with necessary condition but also sufficient condition for a bounded operator on $H^2(\mathbb{T}^n)$ to be compression of k^{th}-order λ-slant Toeplitz operator with some specific symbol.

Theorem 2.5. *The necessary condition for an operator $V \in \mathfrak{B}\left(H^2(\mathbb{T}^n)\right)$ to be a compression of k^{th}-order λ-slant Toeplitz operator on the space $H^2(\mathbb{T}^n)$ is that*

$$\lambda^{(p_1+p_2+\ldots+p_n)}V = T^*_{z_1^{p_1} z_2^{p_2}\ldots z_n^{p_n},n} V T_{z_1^{kp_1} z_2^{kp_2}\ldots z_n^{kp_n},n} \text{ for each } (p_1, p_2, \ldots, p_n) \in B_n. \tag{9}$$

Moreover, the condition given by (9) also acts as sufficient condition for compression of k^{th}-order λ-slant Toeplitz operator.

Proof. Suppose that $V = V_{\phi,k,\lambda}$, for some $\phi \in L^\infty(\mathbb{T}^n)$ of the form

$$\phi(z_1, z_2, \ldots, z_n) = \sum_{(m_1,m_2,\ldots,m_n)\in\mathbb{Z}^n} \phi_{m_1,m_2,\ldots,m_n} z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n}.$$

Now, consider the expression $T^*_{z_1^{p_1} z_2^{p_2}\ldots z_n^{p_n},n} V T_{z_1^{kp_1} z_2^{kp_2}\ldots z_n^{kp_n},n}(z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n})$ for $(i_1, i_2, \ldots, i_n)$, $(p_1, p_2, \ldots, p_n) \in \mathbb{Z}^n_+$. By a simple computation, it can be written as

$$\begin{aligned}
&T^*_{z_1^{p_1} z_2^{p_2}\ldots z_n^{p_n},n} V T_{z_1^{kp_1} z_2^{kp_2}\ldots z_n^{kp_n},n}(z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n}) \\
&\quad = T^*_{z_1^{p_1}\ldots z_n^{p_n},n}\left[\sum_{(m_1,\ldots,m_n)\in\mathbb{Z}^n_+} \phi_{km_1-i_1-kp_1,\ldots,km_n-i_n-kp_n} \lambda^{m_1+\ldots+m_n} z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n}\right] \\
&\quad = P\left[\sum_{(m_1,\ldots,m_n)\in\mathbb{Z}^n_+} \phi_{km_1-i_1-kp_1,\ldots,km_n-i_n-kp_n} \lambda^{(m_1+\ldots+m_n)} z_1^{m_1-p_1} \ldots z_n^{m_n-p_n}\right]. \\
&\quad = P\left[\sum_{m_j=-p_j, 1\le j\le n}^{\infty} \phi_{km_1-i_1,\ldots,km_n-i_n} \lambda^{((m_1+p_1)\ldots+(m_n+p_n))} z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n}\right] \\
&\quad = \lambda^{(p_1+\ldots+p_n)} \sum_{(m_1,\ldots,m_n)\in\mathbb{Z}^n_+} \phi_{km_1-i_1,\ldots,km_n-i_n} \lambda^{(m_1+\ldots+m_n)} z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n} \\
&\quad = \lambda^{(p_1+\ldots+p_n)} V_{\phi,k,\lambda}(z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n}) = \lambda^{(p_1+\ldots+p_n)} V(z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n}),
\end{aligned}$$

which gives the desired necessary condition.

Conversely, assume that $V \in \mathfrak{B}\left(H^2(\mathbb{T}^n)\right)$ satisfies (9). Pre-multiplying the given equation (9) by D_λ and using the fact discussed in (14), we get that

$$(D_\lambda V) = T^*_{z_1^{p_1} z_2^{p_2}\ldots z_n^{p_n},n} (D_\lambda V) T_{z_1^{p_1 k} z_2^{p_2 k}\ldots z_n^{p_n k},n},$$

for each $(p_1, p_2, \ldots, p_n) \in \mathbb{Z}^n_+$. By using the Theorem 2.4, it gives that $D_\lambda V$ is a compression of k^{th}-order slant Toeplitz operator $A_{\phi,k,n}$ induced by some symbol $\phi \in L^\infty(\mathbb{T}^n)$. Thus, we have $V = D_{\overline{\lambda}} V_{\phi,k,n}$. Hence, the proof is completed. ∎

Now, we look for a characterization of compression of $A_{\phi,k,\lambda}$ with specific symbol, which can be written in the following form.

Theorem 2.6. *An operator V on $H^2(\mathbb{T}^n)$ is a compression of k^{th}-order λ-slant Toeplitz operator with inducing function ϕ in the following form*

$$\phi(z_1, z_2, \dots, z_n) = \sum_{m_j=-(k-1), 1\le j\le n}^{\infty} \phi_{m_1,m_2\dots,m_n} z_1^{m_1} z_2^{m_2} \dots z_n^{m_n} \in L^\infty(\mathbb{T}^n), \tag{10}$$

if and only if it holds the conditions

$$\lambda T_{z_1^{p_1} z_2^{p_2} \dots z_n^{p_n}, n} V = V T_{z_1^{kp_1} z_2^{kp_2} \dots z_n^{kp_n}, n},$$

for each $(p_1, p_2, \dots, p_n) \in B_n$.

Proof. Let a bounded operator V be a compression of k^{th}-order λ-slant Toeplitz operator with symbol $\phi \in L^\infty(\mathbb{T}^n)$, given by (10). The form of ϕ, given in (10), can be expressed as

$$\phi(z_1, z_2, \dots, z_n) = \sum_{(m_1,m_2,\dots,m_n)\in\mathbb{Z}^n} \phi_{m_1,m_2\dots,m_n} z_1^{m_1} z_2^{m_2} \dots z_n^{m_n},$$

with $\phi_{m_1,m_2,\dots,m_n} = 0$, whenever at least one m_j is less or equal to $-k$. Now, for $(i_1, i_2, \dots, i_n) \in \mathbb{Z}_+^n$ and $(0, 0, \dots, 0) \ne (p_1, p_2, \dots, p_n) \in \mathbb{Z}_+^n$, we have

$$\begin{aligned}
VT_{z_1^{kp_1} z_2^{kp_2} \dots z_n^{kp_n}, n}(z_1^{i_1} z_2^{i_2} \dots z_n^{i_n}) \;&=\; V_{\phi,k,\lambda}(z_1^{i_1+kp_1} z_2^{i_2+kp_2} \dots z_n^{i_n+kp_n}) \\
&= \sum_{(m_1,\dots,m_n)\in\mathbb{Z}_+^n} \phi_{km_1-i_1-kp_1,\dots,km_n-i_n-kp_n} \lambda^{(m_1+\dots+m_n)} z_1^{m_1} z_2^{m_2} \dots z_n^{m_n} \\
&= \sum_{m_j=p_j, 1\le j\le n}^{\infty} \phi_{km_1-i_1-kp_1,\dots,km_n-i_n-kp_n} \lambda^{(m_1+\dots+m_n)} z_1^{m_1} z_2^{m_2} \dots z_n^{m_n} \\
&+ \sum_{\substack{(m_1,\dots,m_n)\in\mathbb{Z}_+^n,\ \text{at least one } m_{j_0}\le p_{j_0}-1, \\ 1\le j_0\le n \text{ for which } p_{j_0}\ne 0}} \phi_{km_1-i_1-kp_1,\dots,km_n-i_n-kp_n} \lambda^{(m_1+\dots+m_n)} z_1^{m_1} \dots z_n^{m_n} \\
&= \sum_{m_j=p_j, 1\le j\le n}^{\infty} \phi_{km_1-i_1-kp_1,\dots,km_n-i_n-kp_n} \lambda^{(m_1+\dots+m_n)} z_1^{m_1} z_2^{m_2} \dots z_n^{m_n},
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
T_{z_1^{p_1} \dots z_n^{p_n}, n} V(z_1^{i_1} z_2^{i_2} \dots z_n^{i_n}) \;&=\; T_{z_1^{p_1} \dots z_n^{p_n}, n} V_{\phi,k,\lambda}(z_1^{i_1} z_2^{i_2} \dots z_n^{i_n}) \\
&= T_{z_1^{p_1} \dots z_n^{p_n}, n} \left[\sum_{(m_1,\dots,m_n)\in\mathbb{Z}_+^n} \phi_{km_1-i_1,\dots,km_n-i_n} \lambda^{(m_1+\dots+m_n)} z_1^{m_1} \dots z_n^{m_n} \right] \\
&= \sum_{(m_1,\dots,m_n)\in\mathbb{Z}_+^n} \phi_{km_1-i_1,\dots,km_n-i_n} \lambda^{(m_1+\dots+m_n)} z_1^{m_1+p_1} z_2^{m_2+p_2} \dots z_n^{m_n+p_n} \\
&= \bar{\lambda}^{(p_1+\dots+p_n)} \sum_{m_j=p_j, 1\le j\le n}^{\infty} \phi_{km_1-i_1-kp_1,\dots,km_n-i_n-kp_n} \lambda^{(m_1+\dots+m_n)} z_1^{m_1} \dots z_n^{m_n}.
\end{aligned} \tag{12}$$

Equations (11) and (12) immediately give that

$$\lambda^{(p_1+p_2...+p_n)}T_{z_1^{p_1}z_2^{p_2}...z_n^{p_n},n}V = VT_{z_1^{kp_1}z_2^{kp_2}...z_n^{kp_n},n}, \tag{13}$$

for all $(p_1, \ldots, p_n) \in \mathbb{Z}_+^n$. Thus, we get the desired condition, particularly for $(p_1, \ldots, p_n) \in B_n$.

Conversely, let $V \in \mathfrak{B}(H^2(\mathbb{T}^n))$ satisfies the system of operator equations, given by (13) for $(p_1, p_2, \ldots, p_n) \in B_n$. Equivalently, one can show that the operator V satisfies

$$\lambda^{(p_1+p_2...+p_n)}T_{z_1^{p_1}z_2^{p_2}...z_n^{p_n},n}V = VT_{z_1^{kp_1}z_2^{kp_2}...z_n^{kp_n},n},$$

for all $(p_1, p_2, \ldots, p_n) \in \mathbb{Z}_+^n$. Now, we observe that for each $(p_1, \ldots, p_n) \in \mathbb{Z}_+^n$ and $f \in H^2(\mathbb{T}^n)$

$$\begin{aligned} T_{z_1^{p_1}z_2^{p_2}...z_n^{p_n},n}D_{\overline{\lambda}}f(z_1, \cdots, z_n) &= z_1^{p_1}z_2^{p_2}\ldots z_n^{p_n}f(\lambda z_1, \lambda z_2, \cdots, \lambda z_n) \\ &= \overline{\lambda}^{p_1+\ldots+p_n}(\lambda z_1)^{p_1}(\lambda z_2)^{p_2}\ldots(\lambda z_n)^{p_n}f(\lambda z_1, \lambda z_2, \cdots, \lambda z_n) \\ &= \overline{\lambda}^{p_1+\ldots+p_n}D_{\overline{\lambda}}T_{z_1^{p_1}z_2^{p_2}...z_n^{p_n},n}f(z_1, z_2, \cdots, z_n). \end{aligned} \tag{14}$$

Equation (14) gives that $D_\lambda T_{z_1^{p_1}z_2^{p_2}...z_n^{p_n},n} = \overline{\lambda}^{p_1+\ldots+p_n}T_{z_1^{p_1}z_2^{p_2}...z_n^{p_n},n}D_\lambda$ for each $(p_1, p_2, \ldots, p_n) \in \mathbb{Z}_+^n$. Pre-multiplying the given equation (13) by D_λ and employing the above observed fact, we get that

$$T_{z_1^{p_1}z_2^{p_2}...z_n^{p_n},n}(D_\lambda V) = (D_\lambda V)T_{z_1^{p_1k}z_2^{p_2k}...z_n^{p_nk},n},$$

for each $(p_1, p_2, \ldots, p_n) \in \mathbb{Z}_+^n$. It implies that $D_\lambda V$ is a compression of k^{th}-order slant Toeplitz operator in the form $V_{\phi,k,n}$ induced by some special symbol $\phi \in L^\infty(\mathbb{T}^n)$ written into the form (10) (see [5]). Thus, we have $V = D_{\overline{\lambda}}V_{\phi,k,n}$, which is a compression of k^{th}-order λ-slant Toeplitz operator with desired symbol. This completes the proof. □

Note that the necessary part of Theorem 2.5 is proved with straight forward computation. However, we can also derive it using the technique suggested in Theorem 2.4. One should compare the conditions obtained in all three preceding Theorems with their counterparts in one variable case discussed by several researchers.

Remark 2.7. *If an operator $V \in \mathfrak{B}(H^2(\mathbb{T}^n))$ satisfies the conditions given by (13) for $(p_1, p_2, \ldots, p_n) \in \mathbb{Z}_+^n$. Then, it automatically satisfies those given in (9). But converse need not be true in general. It means that conditions given in (9) are not equivalent to those given by (13) for $(p_1, p_2, \ldots, p_n) \in \mathbb{Z}_+^n$.*

Counter example in the support of above assertion can be constructed by taking an operator, which is a compression of k^{th}-order λ-slant Toeplitz operator induced by symbol $\phi \in L^\infty(\mathbb{T}^n)$, not in the form as given in (10). In particular, one can choose $\phi = z_1^{-t}$ for $t \geq k$. Then, this operator certainly agrees with the stipulations in (9). But it does not agree with the stipulations given by (13) for all $(p_1, p_2, \ldots, p_n) \in \mathbb{Z}_+^n$.

Before moving to the properties of compression of k^{th}-order λ-slant Toeplitz operators, we observe the fact that $T_{\phi,n}T_{\psi,n} = T_{\phi\psi,n}$ provided that either ψ is analytic or ϕ is co-analytic. If ψ is analytic, then conclusion is obvious. Under the assumption that ϕ is co-analytic, consider the expression $(T_{\phi,n}T_{\psi,n})^*$. It can be rewritten as

$$(T_{\phi,n}T_{\psi,n})^* = T_{\psi,n}^*T_{\phi,n}^* = PM_{\bar{\psi}}PM_{\bar{\phi}}|_{H^2(\mathbb{T}^n)}. \tag{15}$$

In the view of relation (15), we obtain that $(T_{\phi,n}T_{\psi,n})^* = T^*_{\psi\phi,n}$, which reduces to desired conclusion. This observation is frequently used in the subsequent results.

Theorem 2.8. *Let λ be an element of the unit circle $\mathbb{T}$ in the complex plane and $\phi, \psi \in L^\infty(\mathbb{T}^n)$. Then, the following conclusion can be drawn:*

(a) The relation $E_{k,n}V^*_{\phi,k,\lambda} = D_\lambda T_{\Psi,n}$ holds, where $T_{\Psi,n}$ is the Toeplitz operator induced by symbol $\Psi = D_{\bar{\lambda}}E_{k,n}(\bar{\phi})$.

(b) If $\bar{\phi}$ (or ψ) is analytic then $V_{\phi,k,\lambda}T_{\psi,n} = V_{\phi\psi,k,\lambda}$ and $T_{\psi,n}V_{\phi,k,\lambda} = V_{\Phi,k,\lambda}$, where $\Phi = \phi \cdot E^*_{k,n}D_\lambda(\psi) \in L^\infty(\mathbb{T}^n)$.

(c) The operator $V_{\phi,k,\lambda}V^*_{\psi,k,\lambda}$ is a Toeplitz operator provided that $\bar{\phi}$ (or $\bar{\psi}$) is analytic function.

Proof. Suppose that ϕ, ψ belong to $L^\infty(\mathbb{T}^n)$ and $|\lambda| = 1$. Then,

(a) Consider the expression $E_{k,n}V^*_{\phi,k,\lambda}$, which gives that

$$\begin{aligned} E_{k,n}V^*_{\phi,k,\lambda}(f) &= E_{k,n}V^*_{\phi,k,n}D_\lambda(f) \\ &= E_{k,n}PM_{\bar{\phi}}E^*_{k,n}PD_\lambda(f) \\ &= PM_{E_{k,n}(\bar{\phi})}D_\lambda(f) \\ &= PD_\lambda M_{D_{\bar{\lambda}}E_{k,n}(\bar{\phi})}(f) = D_\lambda T_{\Psi,n}(f), \end{aligned} \tag{16}$$

where $\Psi = D_{\bar{\lambda}}E_{k,n}(\bar{\phi}) \in L^\infty(\mathbb{T}^n)$ and $f \in H^2(\mathbb{T}^n)$. It provides the desired expression.

(b) Let $\bar{\phi}$ (or ψ) be analytic. In the view of observation, made just above the theorem, and under the assumption, the operator $V_{\phi,k,\lambda}T_{\psi,n}$ can be expressed as

$$\begin{aligned} V_{\phi,k,\lambda}T_{\psi,n} &= D_{\bar{\lambda}}E_{k,n}T_{\phi,n}T_{\psi,n} \\ &= D_{\bar{\lambda}}E_{k,n}T_{\phi\psi,n} = V_{\phi\psi,k,\lambda}. \end{aligned}$$

This implies that the operator $V_{\phi,k,\lambda}T_{\psi,n}$ resulted into a compression of k^{th}-order λ-slant Toeplitz operator $V_{\phi\psi,k,\lambda}$ with symbol $\phi\psi \in L^\infty(\mathbb{T}^n)$. For another part, consider the operator $T_{\psi,n}V_{\phi,k,\lambda}$. Similarly, it can be seen that

$$\begin{aligned} T_{\psi,n}V_{\phi,k,\lambda} &= T_{\psi,n}D_{\bar{\lambda}}V_{\phi,k,n} \\ &= D_{\bar{\lambda}}T_{D_\lambda(\psi),n}E_{k,n}T_{\phi,n} \\ &= D_{\bar{\lambda}}E_{k,n}T_{\phi\cdot E^*_{k,n}D_\lambda(\psi),n} = V_{\Phi,k,\lambda}, \end{aligned}$$

which is again a compression of k^{th}-order λ-slant Toeplitz operator with inducing function $\Phi = \phi \cdot E^*_{k,n}D_\lambda(\psi)$.

(c) Assume that $\bar{\phi}$ (or $\bar{\psi}$) is analytic. Again, with the assumption as well as discussion done just before the theorem, we see that

$$\begin{aligned} V_{\phi,k,\lambda}V^*_{\psi,k,\lambda}(f) &= D_{\bar{\lambda}}E_{k,n}T_{\phi,n}T^*_{\psi,n}E^*_{k,n}D_\lambda(f) \\ &= D_{\bar{\lambda}}E_{k,n}T_{\phi\bar{\psi},n}E^*_{k,n}D_\lambda(f) \\ &= D_{\bar{\lambda}}T_{E_{k,n}(\phi\bar{\psi}),n}D_\lambda(f) \\ &= D_{\bar{\lambda}}D_\lambda T_{D_{\bar{\lambda}}E_{k,n}(\phi\bar{\psi}),n}(f) = T_{\Psi,n}(f), \end{aligned}$$

for $f \in H^2(\mathbb{T}^n)$. This clearly shows that the operator $V_{\phi,k,\lambda}V^*_{\psi,k,\lambda}$ represents Toeplitz operator induced by $\Psi = D_{\bar{\lambda}}E_{k,n}(\phi\bar{\psi}) \in L^\infty(\mathbb{T}^n)$.

Thus, the proof is completed. ☐

A particular choice to the inducing function (i.e. $\phi = \psi$) in part (c) of the above theorem quickly yields the following result.

Corollary 2.9. *Let ϕ be a co-analytic function in $L^\infty(\mathbb{T}^n)$. The following assertion can be made:*

(i) The operator $V_{\phi,k,\lambda}$ is co-isometry if and only if $D_{\bar{\lambda}}E_{k,n}\left(|\phi|^2\right) = 1$.

(ii) If ϕ is an inner function then $V_{\phi,k,\lambda}$ is co-isometry.

Now, we discuss a result, which is similar to that for various class of operators, namely, Toeplitz operator, slant Toeplitz operator, k^{th}-order slant Toeplitz operator etc.

Theorem 2.10. *A compression of k^{th}-order λ-slant Toeplitz operator $V_{\phi,k,\lambda}$ on $H^2(\mathbb{T}^n)$ is compact if and only if inducing function is the zero function.*

Proof. Let ϕ be an element of $L^\infty(\mathbb{T}^n)$, which is given as

$$\phi(z_1, z_2, \ldots, z_n) = \sum_{(m_1,m_2,\ldots,m_n)\in\mathbb{Z}^n} \phi_{m_1,m_2,\ldots,m_n} z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n}.$$

Also, assume that the compression of k^{th}-order λ-slant Toeplitz operator $V_{\phi,k,\lambda}$ is compact. Then, by the part (b) of the Theorem 2.8, the assumption implies that the operators $V_{\phi,k,\lambda}T_{z_1^{p_1}z_2^{p_2}\ldots z_n^{p_n}} = V_{(z_1^{p_1}z_2^{p_2}\ldots z_n^{p_n})\cdot\phi,k,\lambda}$ are also compact for $p_j \in \mathbb{Z}_k, 1 \le j \le n$. Again, by the part (a) of the Theorem 2.8, compactness of $E_{k,n}V^*_{(z_1^{p_1}z_2^{p_2}\ldots z_n^{p_n})\cdot\phi,k,\lambda}$ follows, which yields that $D_\lambda T_{\Psi_{p_1,\ldots,p_n},n}$ is compact operator and hence $T_{\Psi_{p_1,\ldots,p_n},n}$ is compact for each $p_j \in \mathbb{Z}_k, 1 \le j \le n$, where $\Psi_{p_1,\ldots,p_n} = D_{\bar{\lambda}}E_{k,n}\left(z_1^{-p_1}z_2^{-p_2}\ldots z_n^{-p_n}\cdot\bar{\phi}\right)$. By employing the fact that Toeplitz operator is compact if and only if inducing symbol is zero, we obtain that

$$D_{\bar{\lambda}}E_{k,n}\left(z_1^{-p_1}z_2^{-p_2}\ldots z_n^{-p_n}\cdot\bar{\phi}\right)(z_1, z_2, \ldots, z_n) = 0$$

$$E_{k,n}\left\{\sum_{(m_1,\ldots,m_n)\in\mathbb{Z}^n} \bar{\phi}_{-m_1,\ldots,-m_n} z_1^{m_1-p_1} z_2^{m_2-p_2} \ldots z_n^{m_n-p_n}\right\} = 0. \tag{17}$$

Equivalently, the equation (17) reduces to

$$\sum_{(m_1,m_2,\ldots,m_n)\in\mathbb{Z}^n} \bar{\phi}_{-km_1+p_1,-km_2+p_2,\ldots,-km_n+p_n} z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n} = 0.$$

This implies that $\bar{\phi}_{-km_1+p_1,-km_2+p_2,\ldots,-km_n+p_n} = 0$ for all $p_j \in \mathbb{Z}_k, 1 \leq j \leq n$ and $(m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n$. Thus, preceding expression reduces to $\phi_{m_1,m_2,\ldots,m_n} = 0$ for all $(m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n$. Hence, we have $\phi = 0$. Converse part is obvious. □

3. Essentially k^{th}-order λ-slant Toeplitz operator

In this section, we study operators X on $L^2(\mathbb{T}^n)$ satisfying

$$\lambda^{j_1+j_2+\ldots+j_n} M_{z_1^{j_1} z_2^{j_2} \ldots z_n^{j_n}} X - X M_{z_1^{kj_1} z_2^{kj_2} \ldots z_n^{kj_n}} \in \mathcal{K}(L^2(\mathbb{T}^n)), \tag{18}$$

for a given non-zero complex number λ and $(j_1, j_2, \ldots, j_n) \in \mathbb{Z}^n$. Note that we have already discussed the operators satisfying (18) with the restriction that $|\lambda| = 1$ in order to have a characterization for these operators in [6]. These operators are called as essentially k^{th}-order λ-slant Toeplitz operators on $L^2(\mathbb{T}^n)$ and the collection is denoted by (k, λ)-ESTO($L^2(\mathbb{T}^n)$). But it is important to point out that there are operators (namely, compact operators), which satisfy (18) for any non-zero complex number λ. Following illustration depicts significant structure as well as feature of the collection (k, λ)-ESTO($L^2(\mathbb{T}^n)$).

Example 3.1. *The class $\mathcal{K}(L^2(\mathbb{T}^n))$ is a proper subset of (k, λ)-ESTO($L^2(\mathbb{T}^n)$) for $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Moreover, the collection (k, λ)-ESTO($L^2(\mathbb{T}^n)$) is neither an algebra nor self adjoint.*

To ascertain it, take a fixed complex number λ with $|\lambda| = 1$ and consider a linear operator A on $L^2(\mathbb{T}^n)$ defined by

$$A(z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n}) = \begin{cases} z_1 z_2 \ldots z_n, & \text{if each } m_i = 0, 1 \leq i \leq n \\ \lambda^{p_1+p_2+\ldots+p_n} z_1^{p_1} z_2^{p_2} \ldots z_n^{p_n}, & \text{if each } m_i = kp_i - 1, 1 \leq i \leq n \\ 0, & \text{otherwise,} \end{cases}$$

for all $(m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n$. An easy computation justifies that $A = D_{\bar{\lambda}} E_{k,n} M_{z_1 z_2 \ldots z_n} + K$, where

$$K(z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n}) = \begin{cases} z_1 z_2 \ldots z_n, & \text{if each } m_i = 0, 1 \leq i \leq n \\ 0, & \text{otherwise} \end{cases}$$

is a compact operator. In [6], it is shown that A is neither a compact operator nor k^{th}-order λ-slant Toeplitz operator but $A \in (k, \lambda)$-ESTO($L^2(\mathbb{T}^n)$). Here, we claim that neither A^2 nor A^* is in the class $(k, \lambda)-ESTO(L^2(\mathbb{T}^n))$. In order to prove, let if possible that $A^2 \in (k, \lambda)-ESTO(L^2(\mathbb{T}^n))$. Then, each operator $\lambda M_{z_1^{p_1} z_2^{p_2} \ldots z_n^{p_n}} A^2 - A^2 M_{z_1^{kp_1} z_2^{kp_2} \ldots z_n^{kp_n}}$ for $(p_1, p_2, \ldots, p_n) \in B_n$ must be a compact operator. In particular, for $(p_1, p_2, \ldots, p_n) = (1, 0, 0, \ldots, 0)$, the operator $S = \lambda M_{z_1} A^2 - A^2 M_{z_1^k}$ should be compact. Since $e_{m_1,m_2,\ldots,m_n}$(i.e. $z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n}$) converges to 0 weakly as $m_1 \to \infty$ keeping other indices fixed. In particular, for fixed choice of $m_j = k^2 q_j - k - 1, 2 \leq j \leq n$, the sequence $z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n}$ converges to 0 weakly as $m_1 \to \infty$. If S is compact then $S(z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n})$ must converge to 0 strongly as $m_1 \to \infty$. Now, one can verify that

$$\lambda M_{z_1} A^2 \left(z_1^{m_1} z_2^{k^2 q_2 - k - 1} \ldots z_n^{k^2 q_n - k - 1} \right) = \lambda^{(k+1)(q_1+\ldots+q_n)-n+1} \begin{cases} z_1^{q_1+1} z_2^{q_2} \ldots z_n^{q_n}, & if\ m_1 = k^2 q_1 - k - 1 \\ 0, & otherwise \end{cases}$$

and

$$A^2 M_{z_1^k}\left(z_1^{m_1} z_2^{k^2 q_2-k-1}\dots z_n^{k^2 q_n-k-1}\right)$$
$$= \lambda^{(k+1)(\hat{q}_1+\ldots+q_n)-n}\begin{cases} z_1^{\hat{q}_1} z_2^{q_2}\dots z_n^{q_n}, & if\ m_1 = k^2\hat{q}_1 - 2k-1 \\ 0, & otherwise. \end{cases}$$

Since, $k^2\hat{q}_1 - 2k - 1 \neq k^2 q_1 - k - 1$, for any choice of integers $\hat{q}_1, q_1$. Therefore, we get that

$$S\left(z_1^{m_1} z_2^{k^2 q_2-k-1}\dots z_n^{k^2 q_n-k-1}\right)$$
$$= \begin{cases} \lambda^{(k+1)(q_1+\ldots+q_n)-n+1} z_1^{q_1+1} z_2^{q_2}\dots z_n^{q_n}, & if\ m_1 = k^2 q_1 - k - 1 \\ -\lambda^{(k+1)(\hat{q}_1+\ldots+q_n)-n} z_1^{\hat{q}_1} z_2^{q_2}\dots z_n^{q_n}, & if\ m_1 = k^2\hat{q}_1 - 2k - 1 \\ 0, & otherwise. \end{cases}$$

Clearly, it shows that $\left\|S\left(z_1^{m_1} z_2^{k^2 q_2-k-1}\dots z_n^{k^2 q_n-k-1}\right)\right\| \to 1$ as $m_1 \to \infty$. Thus, we can conclude that S is not compact and hence $A^2 \notin (k,\lambda) - ESTO(L^2(\mathbb{T}^n))$. It also infer that the collection $(k,\lambda) - ESTO(L^2(\mathbb{T}^n))$ is not an algebra.

Next we show that A^* is not in the set $(k,\lambda) - ESTO(L^2(\mathbb{T}^n))$. Again, let if possible that $A^* \in (k,\lambda) - ESTO(L^2(\mathbb{T}^n))$. Then, operators $\lambda M_{z_1^{p_1} z_2^{p_2}\ldots z_n^{p_n}} A^* - A^* M_{z_1^{kp_1} z_2^{kp_2}\ldots z_n^{kp_n}}$ for all $(p_1, p_2, \ldots, p_n) \in B_n$ should be compact operators. For particular choice $(p_1, p_2, \ldots, p_n) = (1, 0, 0, \ldots, 0)$, the operator $\lambda M_{z_1} A^* - A^* M_{z_1^k}$ must be compact. An easy calculation provides that

$$\begin{aligned} \lambda M_{z_1} A^* - A^* M_{z_1^k}(z_1^{m_1} z_2^{m_2}\dots z_n^{m_n}) &= \bar{\lambda}^{(m_1+\ldots+m_n)-1} z_1^{km_1} z_2^{km_2-1}\dots z_n^{km_n-1} \\ &- \bar{\lambda}^{(m_1+\ldots+m_n+k)-1} z_1^{km_1+k^2-1} z_2^{km_2-1}\dots z_n^{km_n-1}. \end{aligned}$$

It helps in making an inference that $\lambda M_{z_1} A^* - A^* M_{z_1^k}$ is not compact. Hence, the claim follows, which enlighten the fact that the collection of essentially k^{th}-order λ-slant Toeplitz operators is not self adjoint.

Theorem 3.2. *Let λ and μ be distinct complex numbers and k, r be integers ≥ 2 such that $k \neq r$. Then, we have the following:*

1. (k,μ)-$ESTO(L^2(\mathbb{T}^n)) \cap (k,\lambda)$-$ESTO(L^2(\mathbb{T}^n)) = \mathcal{K}(L^2(\mathbb{T}^n))$.
2. (k,μ)-$ESTO(L^2(\mathbb{T}^n)) \cap (r,\lambda)$-$ESTO(L^2(\mathbb{T}^n)) = \mathcal{K}(L^2(\mathbb{T}^n))$, *where* $|\lambda| \neq |\mu|$.

Proof. Let $T \in (k,\mu)$-$ESTO(L^2(\mathbb{T}^n)) \cap (k,\lambda)$-$ESTO(L^2(\mathbb{T}^n))$. Then, $\lambda M_{z_1^{p_1} z_2^{p_2}\ldots z_n^{p_n}} T - T M_{z_1^{kp_1} z_2^{kp_2}\ldots z_n^{kp_n}}$ and $\mu M_{z_1^{p_1} z_2^{p_2}\ldots z_n^{p_n}} T - T M_{z_1^{kp_1} z_2^{kp_2}\ldots z_n^{kp_n}}$ are compact operators for all $(p_1, p_2, \ldots, p_n) \in B_n$. The particular selection $(p_1, p_1, \ldots, p_n) = (1, 0, 0, \ldots, 0)$ yields that the operators $\lambda M_{z_1} T - T M_{z_1^k}$ and $\mu M_{z_1} T - T M_{z_1^k}$ are compact operators. Therefore, we can conclude that $(\lambda - \mu) M_{z_1} T$ is compact and hence T is compact as $\lambda \neq \mu$. The converse inclusion is obvious. Thus, the claim follows.

For second part, suppose that k, r are distinct integers ≥ 2. Also, without lose of generality, assume that $k < r$ and $T \in (k,\mu)$-$ESTO(L^2(\mathbb{T}^n)) \cap (r,\lambda)$-$ESTO(L^2(\mathbb{T}^n))$. Again, as in the part first, we get that the operators given by $\mu M_{z_1^{p_1} z_2^{p_2} \ldots z_n^{p_n}} T - T M_{z_1^{kp_1} z_2^{kp_2} \ldots z_n^{kp_n}}$ and $\lambda M_{z_1^{p_1} z_2^{p_2} \ldots z_n^{p_n}} T - T M_{z_1^{rp_1} z_2^{rp_2} \ldots z_n^{rp_n}}$ are compact for all $(p_1, \ldots, p_n) \in B_n$. In particular, for $(p_1, p_1, \ldots, p_n) = (1, 0, 0, \ldots, 0)$, we get that the operators $\mu M_{z_1} T - T M_{z_1^k}$ and $\lambda M_{z_1} T - T M_{z_1^r}$ are also compact. Further, this helps to infer that $\mu M_{z_1} T (M_{z_1^{r-k}} - \frac{\lambda}{\mu} I)$ is compact operator on $L^2(\mathbb{T}^n)$. Since $|\lambda| \neq |\mu|$ and spectrum of $M_{z_1^m}$ is the unit circle for positive integer m. Therefore, $(M_{z_1^{r-k}} - \frac{\lambda}{\mu} I)$ is invertible and hence T is compact. Converse inclusion vacuously follows. This completes the proof. □

Next we investigate necessary condition for an essentially k^{th}-order λ-slant Toeplitz operator to be a self-adjoint operator. To derive it initially we state a lemma, which can be obtained by some rearrangements and definition.

Lemma 3.3. *Let $T, T^* \in (k, \mu)$-$ESTO(L^2(\mathbb{T}^n))$, then $AT^* - T^* A^* \in \mathcal{K}(L^2(\mathbb{T}^n))$, where $A = \lambda M_z + M_{\bar{z}^k}$.*

Using the above Lemma, we have the following result immediately.

Theorem 3.4. *An essentially k^{th}-order λ-slant Toeplitz operator T is self-adjoint only if the operator $(\lambda M_z + M_{\bar{z}^k}) T$ is essentially self-adjoint.*

Finally, we analyze certain properties of operators on $H^2(\mathbb{T}^n)$ satisfying similar kind of relations as given in (18) on $H^2(\mathbb{T}^n)$. Let the collection of operators X on $H^2(\mathbb{T}^n)$ satisfying

$$\lambda^{j_1 + j_2 + \ldots + j_n} X - T^*_{z_1^{j_1} z_2^{j_2} \ldots z_n^{j_n}} X T_{z_1^{kj_1} z_2^{kj_2} \ldots z_n^{kj_n}} \in \mathcal{K}(H^2(\mathbb{T}^n)), \tag{19}$$

for a given non-zero complex number λ and $(j_1, j_2, \ldots, j_n) \in \mathbb{Z}^n$ be denoted by (k, λ)-ESTO($H^2(\mathbb{T}^n)$). The operator in this collection is termed as essential compression of k^{th}-order λ-slant Toeplitz operator.

Note that if, in the Example 3.1, we replace multiplication operators (namely, $M_{z_1^{p_1} z_2^{p_2} \ldots z_n^{p_n}}$) by Toeplitz operators (namely, $T_{z_1^{p_1} z_2^{p_2} \ldots z_n^{p_n}, n}$) and also define corresponding operators on $H^2(\mathbb{T}^n)$ in the similar fashion, then the same discussion as done in the preceding Example (3.1) helps to point out following conclusions related to essential compression of k^{th}-order λ-slant Toeplitz operators.

(a) The collection $\mathcal{K}(H^2(\mathbb{T}^n))$ of compact operators on $H^2(\mathbb{T}^n)$ is a proper subset of (k, λ)-ESTO($H^2(\mathbb{T}^n)$).

(b) The class (k, λ)-ESTO($H^2(\mathbb{T}^n)$) is not an algebra. Also, it is not a self adjoint class.

(c) The set (k, λ)-ESTO($H^2(\mathbb{T}^n)$) properly contains the collection of all compression of k^{th}-order λ-slant Toeplitz operators.

Remark 3.5. *It is important to observe that for the particular choices of n, k, and λ, the results of the paper gives certain significant outcomes, which are already known. For $n = 1, k = 2, \lambda = 1$, the paper yields certain results of [11] and some conclusions similar to that done in [14]. Also, certain result of [1] can be obtained by substituting $n = 1$ and $\lambda = 1$. Particular choices $n = 1$ and $\lambda = 1$ in the paper give some outcomes of [4] and [5] respectively.*

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