ISTE OpenScience
Advances in Pure and Applied Mathematics
2023, vol. 14, n° 1, 50-69 pages, DOI: 10.21494/ISTE.OP.2023.0906

An inverse problem for the Schrödinger equation with Neumann boundary condition

Un problème inverse pour l'opérateur de Schrödinger avec condition au bord de type Neumann

Atef Saci and Salah-Eddine Rebiai*

LTM, University of Batna 2, Batna, Algeria
s.rebiai@univ-batna2.dz

ABSTRACT. This article concerns the inverse problem of the recovery of unknown potential coefficient for the Schrödinger equation, in a bounded domain of $\mathbb{R}^n$ with non-homogeneous Neumann boundary condition from a time-dependent Dirichlet boundary measurement. We prove uniqueness and Lipschitz stability for this inverse problem under certain convexity hypothesis on the geometry of the spatial domain and under weak regularity requirements on the data. The proof is based on a Carleman estimate in [12] for Schrödinger equations and its resulting implication, a continuous observability inequality.
Mathematics Subject Classification. 35R30, 35Q40, 49K20.
KEYWORDS. Inverse problems, Uniqueness, Stability, Schrödinger equations, Carleman estimates.

1 Introduction

Inverse problems for partial differential equations (PDE in short) have attracted much attention over the past decades (sees for example [5] and the references therein). Both results of unique and stable recovery of coefficients or sources have been considered for various PDEs and with various measurements. Because of the subject of the present paper, we focus here our attention to references that dealt with inverse problems for Schrödinger equations defined in a bounded domain of $\mathbb{R}^n$ or of a Riemannian manifold.

The stability issue in the inverse problem of determining the time-independent electric potential in the Schrödinger equation with Dirichlet boundary condition from a Neumann boundary measurement on a part of the boundary was treated in [2] and [14]. Baudouin and Puel [2] invoked the method of Carleman estimate of [6] while Mercado et al [14] established a Carleman estimate with a relaxed pseudo-convexity condition. This then yields a reduction of the portion of the Neumann boundary observation over [2] in specific geometries. In these two papers, the Neumann datum is observed on a part of the boundary satisfying the geometric control condition of Bardos, Lebeau and Rauch [1]. This condition was relaxed in [3] at the price of requiring knowledge of the potential near the boundary. In [7], Cristofol and Soccorsi proved Lipschitz stability in the recovery of the time-dependent magnetic potential appearing in the Schrödinger equation, from a finite number of Neumann observations measured on a subboundary for different choices of the initial state by a method based essentially on an appropriate Carleman estimate. This result was extended in [10] to the recovery of the complex-valued electric potential and the direction of the magnetic field appearing in the dynamic Schrödinger equation with static coefficients by finitely many partial boundary measurements of the solution. Bellasoued and Choulli [4] considered the Schrödinger equation with magnetic potential and Dirichlet boundary datum. By employing a Dirichlet-

*Corresponding author.

to- Neumann approach, they obtained a stability estimate of the so-called magnetic field. Yuan and Yamamoto [19] proved Carleman estimates for the Schrödinger equation in Sobolev spaces of negative orders and used these estimates to derive the uniqueness in the inverse problem of determining an L^p-potential locally from some suitable local observation data. Deng [8] used some tools from Riemannian geometry to obtain a Carleman estimate with a pseudo-convex weight function for a Schrödinger equation with variable coefficients in both the elliptic part and in the first order terms and then get uniqueness and stability for the inverse problem consisting in recovering the potential from a Neumann measurement on a portion of the boundary. Triggiani and Zhang [18] established uniqueness and stability results for the inverse problem of determining the electric potential coefficient for a Schrödinger equation with magnetic potential defined on a bounded connected set of a Riemannian manifold, subject to a non-homogenous Dirichlet boundary condition by means of a Neumann boundary measurement on a portion of the boundary. Liu and Triggiani [13] considered the inverse problem of determining simultaneously two unknown electric potential coefficients for a system of two strongly coupled Schrödinger equations, with magnetic potential terms, and with Neumann boundary conditions, from single Dirichlet measurements on a portion of the boundary. Under suitable geometrical assumptions on the complementary unobserved portion of the boundary, they showed uniqueness in determining the two unknown potential coefficients from respective Dirichlet boundary measurements over an arbitrarily short time interval. Their proof is based on a Carleman estimate in [12] for single Schrödinger equations. To our knowledge, there is no result available in the mathematical literature on the recovery of unknown potential coefficient for a Schrödinger equation with Neumann boundary condition from a time-dependent Dirichlet boundary measurement. In this paper, we use a global Carleman estimate for Schrödinger equations due to Lasiecka et al [12] and the related continuous observability inequality to prove uniqueness and Lipschitz stability for this inverse problem under certain convexity hypothesis on the geometry of the spatial domain and under weak regularity requirements on the data.

1.1 Statement of the problem and main assumptions

Let $T > 0$ and let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be an open bounded domain with smooth boundary Γ of class C^2 consisting of the closure of two disjoint parts Γ_0 and Γ_1 both relatively open in Γ. Namely $\Gamma = \overline{\Gamma_0 \cup \Gamma_{1,}}, \Gamma_0 \cap \Gamma_1 = \emptyset$.

Let ν be the unit outward normal vector on Γ and let $\frac{\partial .}{\partial \nu} = \nabla . \nu$ denote the normal derivative.

As in [11], [12], [17],[16], and [13], we make the following assumption.

Assumption (**H**).
There exists a strictly convex (real-valued) non-negative function $d : \overline{\Omega} \longrightarrow \mathbb{R}_+$ of class $C^3\left(\overline{\Omega}\right)$, such that if we set $h(x) = \nabla d(x)$:

(i)

$$\frac{\partial d}{\partial \nu} = h.\nu = 0 \text{ on } \Gamma_0, \tag{1.1}$$

(ii) the (symmetric) Hessian matrix H_d of d is strictly positive definite on $\overline{\Omega}$, $i.e.$ there exists a constant $\rho > 0$ such that for all $x \in \overline{\Omega}$:

$$H_d\,(x) \geq \rho I\ .$$

(iii) d has no critical point in $\overline{\Omega}$

$$\inf_{x\in\Omega} |h(x)| = s > 0.$$

Remark 1.1. *Assumption (**H**) holds true for large classes of triples* $\{\Omega, \Gamma_0, \Gamma_1\}$*, see the appendices in [11].*

In Ω, we consider the Schrödinger equation :

$$iz_t(x,t) + \Delta z(x,t) = q(x) z(x,t), \qquad \text{in } Q = \Omega \times [0,T], \tag{1.2a}$$

$$z(., \frac{T}{2}) = z_0(x), \qquad \text{in } \Omega, \tag{1.2b}$$

$$\frac{\partial z(x,t)}{\partial \nu} = g(x,t), \qquad \text{in } \Sigma = \Gamma \times [0,T]. \tag{1.2c}$$

Here z_0 is the given initial condition and g is the given Neumann boundary condition. Instead, the potential q is the time-independent unknown coefficient.

Remark 1.2. *In* $(1.2a)$-$(1.2c)$ *and* $(1.4a)$-$(1.4c)$ *we consider* $t = \frac{T}{2}$ *to be the initial time. Because this is appropriate to apply the Carleman estimate specified in [12] for Schrödinger equation, which uses a pseudo-convex function* φ *in* (3.1) *centered around* $\frac{T}{2}$*.*

Let $z = z(q)$ be a solution to problem $(1.2a)$-$(1.2c)$. This paper treats the inverse problem which consists in determining the potential q from the observations of the Dirichlet boundary datum z on $\Sigma_1 = \Gamma_1 \times [0,T]$. We will prove uniqueness and stability of the nonlinear inverse problem characterized by the nonlinear map

$$q \to z|_{\Sigma_1}$$

We will more precisely answer the following questions

Uniqueness in the nonlinear inverse problem

Under the geometrical Assumption (**H**) imposed on Γ_0, does the equality

$$z(q) = z(p) \text{ on } \Sigma_1$$

imply

$$q = p \text{ in } \Omega$$

Stability in the nonlinear inverse problem.

Let $\{z(q), z(p)\}$ be solutions to $(1.2a)$-$(1.2c)$. Under Assumption (**H**), is it possible to estimate $(q-p)|_\Omega$ by $(z(q) - z(p))|_{\Sigma_1}$ in suitable norms ?

The corresponding linear inverse problem

We notice that the nonlinear inverse problem can be reduced to a linear inverse problem. In fact, if we set :

$$\begin{aligned} f(x) &= q(x) - p(x), \\ R(x,t) &= z(p)(x,t), \\ u(x,t) &= z(q)(x,t) - z(p)(x,t), \end{aligned} \tag{1.3}$$

Then after subtracting problem $(1.2a)$-$(1.2c)$ with coefficient q from problem $(1.2a)$-$(1.2c)$ with coefficient p, we obtain :

$$iu_t(x,t) + \Delta u(x,t) = q(x) u(x,t) + f(x) R(x,t), \quad \text{in } Q, \tag{1.4a}$$

$$u(.,\frac{T}{2}) = 0, \qquad \text{in } \Omega, \tag{1.4b}$$

$$\frac{\partial u(x,t)}{\partial \nu} = 0, \qquad \text{on } \Sigma. \tag{1.4c}$$

where, the term f is the unknown time-independent coefficient. The u-problem has the advantage over the original z-problem in $(1.2a)$-$(1.2c)$ that the *map* $f \to u(f)$, where $u = u(f)$ is a solution to problem $(1.4a)$-$(1.4c)$, is linear. Here we also face two problems.

Uniqueness in the linear inverse problem: Under Assumption (**H**), is the following implication true.

$$u(f) = 0 \text{ on } \Sigma_1 \ \Rightarrow f = 0 \text{ in } \Omega$$

Stability in the linear inverse problem: Under the geometric Assumption (**H**), is it possible to estimate $f|_{\Omega}$ from the knowledge of the Dirichlet traces $u(f)|_{\Sigma_1}$?

2 Main Results

The main results are the following theorems.

Theorem 2.1. *Suppose that Assumption (**H**) holds true. Let*

$$T > 0. \tag{2.1}$$

For the u-problem $(1.4a)$-$(1.4c)$, *assume*

$$\begin{aligned} q \in W^{1.\infty}(\Omega), \ f \in W^{1.\infty}(\Omega), \ R, R_t, R_{tt} \in C\left([0.T], H^1(\Omega)\right), \\ R_{x_j}\left(., \tfrac{T}{2}\right), R_{x_j x_k}\left(., \tfrac{T}{2}\right), R_{x_j x_k x_l}\left(., \tfrac{T}{2}\right) \in L^\infty(\Omega), \qquad 1 \leq j,k,l \leq n, \end{aligned} \tag{2.2}$$

$$fR, \ fR_t, \ fR_{tt} \in C\left([0.T], H^1(\Omega)\right), \tag{2.3}$$

$$\left|R(.,\frac{T}{2})\right| \geq r > 0, \quad x \in \overline{\Omega}, \textit{ for some constant } r. \tag{2.4}$$

Assume also that the solution $u = u(f)$ *to problem* $(1.4a)$-$(1.4c)$ *has the following regularity*

$$u, u_t, u_{tt} \in C\left([0.T], H^1(\Omega)\right), \tag{2.5}$$

If

$$u(f)(x,t) = 0, \quad (x,t) \in \Sigma_1, \tag{2.6}$$

then

$$f(x) = 0, \qquad x \in \Omega.$$

Next, we state a stability result for the linear inverse problem.

Theorem 2.2. *Consider problem* $(1.4a)$-$(1.4c)$*, subject to Assumption (**H**) and to* (2.1)*,* (2.2) *,* (2.4) *.Then there exists a constant* $C = C(\Omega, T, \Gamma_1, \varphi, q, R) > 0$*, such that*

$$\|f\|_{H^1(\Omega)} \leq C\{\|u_t\|_{L^2[\Sigma_1]} + \|u_{tt}\|_{L^2[\Sigma_1]}\}, \tag{2.7}$$

Now, we give a uniqueness result for the nonlinear inverse problem.

Theorem 2.3. *Suppose that Assumptions (**H**) holds true. Consider problem* $(1.2a)$-$(1.2c)$ *on* $[0.T]$*, with* $T > 0$ *as in* (2.1)*, one time with potential coefficient* q *and one time with potential coefficient* p*, assume further*

$$q,\ p \in W^{1.\infty}(\Omega)\,. \tag{2.8}$$

Let $z = z(x, t)$ *be the solution of the corresponding problem* $(1.2a)$-$(1.2c)$*, such that*

$$z, z_t, z_{tt} \ \in\ C\left([0.T], H^1(\Omega)\right), \tag{2.9}$$

$$z_{0x_i},\ z_{0x_ix_j},\ z_{0x_ix_jx_k} \ \in\ L^\infty(\Omega)\,, \quad 1 \leq\ i, j, k \leq n. \tag{2.10}$$

Moreover, let

$$|z_0(x)| \geq r\ > 0, \quad x \in \overline{\Omega.} \tag{2.11}$$

If

$$z(q)(x,t) = z(p)(x,t)\,, \quad (x,t) \in \Sigma_1. \tag{2.12}$$

then

$$q(x) = p(x)\,, \qquad x \in \Omega.$$

Finally, we state a stability result for the nonlinear inverse problem.

Theorem 2.4. *Suppose that Assumption (**H**) is satisfied. Consider problem* $(1.2a)$-$(1.2c)$ *on* $[0.T]$*, with* $T > 0$ *as in* (2.1)*, onece with potential coefficient* $q \in W^{1.\infty}(\Omega)$ *and once with potential coefficient* $p \in W^{1.\infty}(\Omega)$*, and let* $z(q), z(p) \in C\left([0.T], H^1(\Omega)\right)$ *denote the corresponding solutions. Moreover, assume that the initial datum* z_0 *is subject to* (2.10) *and* (2.11)*. Then there exists a constant* $C = C(\Omega, T, \Gamma_1, \varphi, \|q\|_{L^\infty(\Omega)}, z_0, g) > 0$*, such that*

$$\|q - p\|_{H^1(\Omega)} \leq C\{\|z_t(q)(x,t) - z_t(p)(x,t)\|_{L^2[\Sigma_1]} + \|z_{tt}(q) - z_{tt}(p)\|_{L^2[\Sigma_1]}\}, \tag{2.13}$$

The proofs of these theorems are based on a Carleman estimate for Schrödinger equations and a consequent continuous observability inequality.

3 Carleman estimates for Schrödinger equations

In this section, we recall a Carleman estimate for the Schrödinger equation at the H^1 -level established in [12].

3.1 Pseudo-convex function (see [12], p. 46).

Suppose that the function d introduced in Assumption (**H**) satisfies $d(x) \geq d_0 > 0$, and we define a pseudo-convex function $\varphi(.,.)$ by

$$\varphi(x,t) = d(x) - c\left(t - \frac{T}{2}\right)^2, \quad x \in \Omega, \ \ t \in [0.T], \tag{3.1}$$

where $T > 0$ is arbitrary and $c = c_T$ is chosen large enough so that

$$cT^2 > 4\max_{x\in\overline{\Omega}} d(x) \quad \text{so that} \quad cT^2 > 4\max_{x\in\overline{\Omega}} d(x) + 4\delta, \tag{3.2}$$

for a suitably small $\delta > 0$, henceforth kept fixed. The function φ has the following properties.

(a) For the constant $\delta > 0$ fixed in (3.2) and for any $t > 0$,

$$\begin{aligned} \varphi(x,t) &\leq \varphi\left(x, \frac{T}{2}\right) \quad \text{for any } t > 0 \text{ and any } x \in \Omega, \\ \varphi(x,0) &= \varphi(x,T) = d(x) - c\frac{T^2}{4} \leq -\delta \ \text{ uniformly in } x \in \Omega. \end{aligned} \tag{3.3}$$

(b) There are t_0 and t_1, with $0 < t_0 < \frac{T}{2} < t_1 < T$, such that for all $x \in \Omega$

$$\min_{x\in\overline{\Omega},\, t\in[t_0,t_1]} \varphi(x,t) \geq \sigma > 0. \tag{3.4}$$

since $\varphi\left(x, \frac{T}{2}\right) = d(x) \geq d_0 > 0$. for all $x \in \Omega$ (in fact, only the weaker property: $\min_{x\in\overline{\Omega},\, t\in[t_0,t_1]} \varphi(x,t) \geq \sigma > -\delta$ is actually needed).

3.2 Carleman estimate at the H^1- level

Consider the Schrödinger equation

$$iu_t(x,t) + \Delta u(x,t) = q(x)\, u(x,t) + F(x,t) \ \text{ in } \ Q, \tag{3.5}$$

where $q \in L^\infty(\Omega)$, $F \in L^2(Q) \equiv L^2\left(0,T,L^2(\Omega)\right)$.

We consider solutions of (3.5) at first without imposing boundary conditions. They will be taken initially in the class

$$u \in H^{2.2}(Q) \equiv L^2\left(0,T,H^2(\Omega)\right) \cap H^2\left(0,T,L^2(\Omega)\right) \tag{3.6}$$

so that

$$\frac{\partial u}{\partial \nu} \in L^2\left(0,T,H^{\frac{1}{2}}(\Gamma)\right), u_t \in L^2\left(0,T,H^1(\Omega)\right), \ \ u_t|_\Gamma \in L^2\left(0,T,H^{\frac{1}{2}}(\Gamma)\right).$$

The following Carleman estimate holds true for the solutions of (3.5) (see ([12], Theorem 5.1, p. 74).

Theorem 3.1. . *Suppose that $F \in L^2(Q)$ and that Assumption (**H**) is satisfied. Let u be a solution of (3.5) in the class (3.6). Then the following one parameter family of estimates holds true, for all $\tau > 0$ sufficiently large*

$$B|_{\Sigma}(u) + 4\int\limits_{Q} e^{2\tau\varphi}|F|^2\,dQ \geq \widetilde{C}_{1,\tau}\int\limits_{Q} e^{2\tau\varphi}\left[|\nabla u|^2\right]dQ + \widetilde{C}_{2,\tau}\int\limits_{Q} e^{2\tau\varphi}|u|^2\,dQ - C_{d;T}\tau e^{-2\tau\delta}\left[E_u(0) + E_u(T)\right], \tag{3.7}$$

where

$$\widetilde{C}_{1,\tau} = \delta_0\left(2\tau\rho - \frac{1}{2}\right) - 4C_T,\ \widetilde{C}_{2,\tau} = 4\tau^3\rho s^2(1-\delta_0) + \grave{O}\left(\tau^2\right) - 4C_T.$$

Here $0 < \delta_0 < 1$ is some fixed number while $C_{d,T}$ is a positive constant depending on $d(x)$ and T. In particular, C_T depends on the $L^\infty(\Omega)$-norm of the coefficient q. Moreover the boundary terms $B|_{\Sigma}(u)$, are given explicitly as follows,

$$\begin{aligned}
B|_{\Sigma}(u) = {} & 2\tau\int_0^T\int_{\Gamma_1} e^{2\tau\varphi}\left[2\tau^2|h|^2 + \Phi\right]|u|^2\,h.\nu d\Gamma dt \\
& -4c\tau\int_0^T\int_{\Gamma} e^{2\tau\varphi}\left(t - \frac{T}{2}\right)\left[\eta\frac{\partial\xi}{\partial\nu} - \xi\frac{\partial\eta}{\partial\nu}\right]d\Gamma dt \\
& -2\tau\int_0^T\int_{\Gamma_1} e^{2\tau\varphi}\left[\xi_t\eta - \xi\eta_t\right]h.\nu d\Gamma dt \\
& +\int_0^T\int_{\Gamma} e^{2\tau\varphi}\left[2\tau^2|h|^2 - \Phi\right]\left[\overline{u}\frac{\partial u}{\partial v} + u\frac{\partial\overline{u}}{\partial\nu}\right]d\Gamma dt \\
& +2\tau\int_0^T\int_{\Gamma} e^{2\tau\varphi}h\left[\nabla\overline{u}\frac{\partial u}{\partial\nu} + \nabla u\frac{\partial\overline{u}}{\partial\nu}\right]d\Gamma dt \\
& -2\tau\int_0^T\int_{\Gamma_1} e^{2\tau\varphi}|\nabla u|^2\,h.\nu d\Gamma dt.
\end{aligned} \tag{3.8}$$

where $\xi = \Re(u)$, $\eta = \Im(u)$, c is the constant in (3.2), while the function Φ may be taken to satisfy either $\Phi \equiv 0$, or else $\Phi = \tau\Delta d(x)$. Moreover, E_u is the energy function defined by

$$E_u(t) = \int\limits_{\Omega}\left[|\nabla u(x,t)|^2 + |u(x,t)|^2\right]d\Omega.$$

We also recall the following extension of the Carleman estimate (3.7) to solutions of (3.5) in the class;

$$u \in C\left([0.T], H^1(\Omega)\right), \frac{\partial u}{\partial\nu}, u_t \in L^2\left(0, T, L^2(\Gamma)\right). \tag{3.9}$$

Theorem 3.2. *([12], Theorem 7.3.1, p. 92). With reference to* (3.5)*, let* $F \in L^2\left(0, T, H^1(\Omega)\right)$ *, and let the function* d *be subject to the Assumption (**H**). Then, the Carleman estimate* (3.7) *holds true also for a solution of* (3.5)*, in the class* (3.9)*.*

4 Proof of Theorem 2.1

Step 1. We return to $(1.4a)$ for $u = u(f)$, i.e., we consider the equation

$$iu_t(x,t) + \Delta u(x,t) = q(x)\, u(x,t) + f(x)\, R(x,t)\,. \tag{4.1}$$

By matching (4.1) with (3.5), we find

$$F = fR \in C\left([0.T], H^1(\Omega)\right). \tag{4.2}$$

Thus $F \in L^2\left(0, T, H^1(\Omega)\right)$. So, we apply Theorem 3.2, i.e. Carleman's estimate (3.7) to $(4.1), (4.2)$ and get

$$\begin{aligned} & B|_\Sigma(u) + 4\int_Q e^{2\tau\varphi} |f(x)\, R(x,t)|^2\, dQ \geq \\ & \widetilde{C}_{1,\tau} \int_Q e^{2\tau\varphi} \left[|\nabla u|^2\right] dQ + \widetilde{C}_{2,\tau} \int_Q e^{2\tau\varphi} |u|^2\, dQ - C_{d;T}\tau e^{-2\tau\delta}\left[E_u(0) + E_u(T)\right], \end{aligned} \tag{4.3}$$

From $(1.4c)$ and (2.6), we have

$$B|_\Sigma(u) = 0. \tag{4.4}$$

Substituting (4.4) on the left hand side of (4.3), we find the following estimate, for all τ sufficiently large

$$\begin{aligned} & \widetilde{C}_{1,\tau} \int_Q e^{2\tau\varphi} \left[|\nabla u|^2\right] dQ + \widetilde{C}_{2,\tau} \int_Q e^{2\tau\varphi} |u|^2\, dQ \\ \leq\ & 4\int_Q e^{2\tau\varphi} |f(x)\, R(x,t)|^2\, dQ + C_{d;T}\tau e^{-2\tau\delta}\left[E_u(0) + E_u(T)\right]. \end{aligned} \tag{4.5}$$

Step 2. We differentiate problem $(1.4a)$-$(1.4c)$ with respect to time to obtain

$$i(u_t)_t(x,t) + \Delta(u_t)(x,t) = q(x)\, u_t(x,t) + f(x)\, R_t(x,t)\,, \quad \text{in } Q, \tag{4.6a}$$

$$u_t\left(., \frac{T}{2}\right) = -i\, f(.)\, R\left(., \frac{T}{2}\right), \qquad \text{in } \Omega, \tag{4.6b}$$

$$\frac{\partial u_t(x,t)}{\partial \nu} = 0, \qquad \text{on } \Sigma. \tag{4.6c}$$

and proceed as in the previous step to find for all τ sufficiently large

$$\widetilde{C}_{1,\tau}\int\limits_{Q} e^{2\tau\varphi}\left[\left|\nabla u_t\right|^2\right]dQ+\widetilde{C}_{2,\tau}\int\limits_{Q} e^{2\tau\varphi}\left|u_t\right|^2 dQ$$
$$\leq 4\int\limits_{Q} e^{2\tau\varphi}\left|f\left(x\right)R_t\left(x,t\right)\right|^2 dQ+C_{d;T}\tau e^{-2\tau\delta}\left[E_{u_t}\left(0\right)+E_{u_t}\left(T\right)\right]. \tag{4.7}$$

Step 3. We differentiate the problem $(4.6a)$-$(4.6c)$ with respect to time, we find

$$i\left(u_{tt}\right)_t\left(x,t\right)+\Delta\left(u_{tt}\right)\left(x,t\right)=q\left(x\right)u_{tt}\left(x,t\right)+f\left(x\right)R_{tt}\left(x,t\right),\quad \text{in } Q, \tag{4.8a}$$
$$u_{tt}\left(.,\frac{T}{2}\right)=\Delta\left(f\left(.\right)R\left(.,\frac{T}{2}\right)\right)-q\left(.\right)f\left(.\right)R\left(.,\frac{T}{2}\right)-i\,f\left(.\right)R_t\left(.,\frac{T}{2}\right),\quad \text{in } \Omega, \tag{4.8b}$$
$$\frac{\partial u_{tt}(x,t)}{\partial\nu}=0,\quad \text{on } \Sigma. \tag{4.8c}$$

We note that, from (2.2), $u_{tt}\left(.,\frac{T}{2}\right)$, is in $H^{-1}\left(\Omega\right)$ (see $(4.8b)$ and not in $H^1\left(\Omega\right)$, as needed to invoke Theorem 3.2. As the initial conditions in $(4.8b)$ are very irregular, we will first prove Theorem 2.1 under the following temporary constraints on the data

$$fR\left(.,\frac{T}{2}\right)\in H^3\left(\Omega\right), \tag{4.9}$$

for which the solution

$$u_{tt}\in C\left(\left[0.T\right],H^1\left(\Omega\right)\right). \tag{4.10}$$

(4.9) holds true provided that

$$f\in H^3\left(\Omega\right),\ \text{and } R\left(.,\frac{T}{2}\right)\in\mathcal{M}(H^3\left(\Omega\right)):\text{the set of multipliers in } H^3(\Omega).. \tag{4.11}$$

A characterization for (4.11) is given in ([15], Theorem 1 with $m=l=3$, $p=2$, p. 243). More direct sufficient conditions for (4.11) to hold is that

$$\begin{gathered} f\in H^3\left(\Omega\right), \\ R_{x_j}\left(.,\tfrac{T}{2}\right),\ R_{x_jx_k}\left(.,\tfrac{T}{2}\right),\ R_{x_jx_kx_l}\left(.,\tfrac{T}{2}\right)\in L^\infty\left(\Omega\right),\ 1\leq\ j,k,l\leq n. \end{gathered} \tag{4.12}$$

Then, we will extend the result to all $f\in W^{1.\infty}\left(\Omega\right)$, with R_{x_j}, $R_{x_jx_k}$, $R_{x_jx_kx_l}$ as in (4.12), by using

(i) the continuity of the map

$$\begin{aligned} W^{1,\infty}\left(\Omega\right)&\ \rightarrow\ C\left(\left[0,T\right],H^{\frac{1}{2}}\left(\Gamma\right)\right), \\ \mathcal{T}:\ f&\ \rightarrow\ \mathcal{T}f=u\left|_{\Sigma}\right. \end{aligned}$$

and
(ii) the denseness of $H^3\left(\Omega\right)$ in $W^{1.\infty}\left(\Omega\right)$.

From $\left(2.2\right),\left(2.3\right)$, we have $F=fR_{tt}\in L^2\left(0,T,H^1\left(\Omega\right)\right)$, So, we can apply Theorem 3.2 to the u_{tt}-equation $(4.8a)$, to obtain after recalling $(4.8c)$ and (2.6)

$$\begin{aligned} & \widetilde{C}_{1,\tau} \int\limits_Q e^{2\tau\varphi} \left[|\nabla u_{tt}|^2 \right] dQ + \widetilde{C}_{2,\tau} \int\limits_Q e^{2\tau\varphi} |u_{tt}|^2 dQ \\ \leq\ & 4 \int\limits_Q e^{2\tau\varphi} |f(x) R_{tt}(x,t)|^2 dQ + C_{d;T} \tau e^{-2\tau\delta} \left[E_{u_{tt}}(0) + E_{u_{tt}}(T) \right], \end{aligned} \tag{4.13}$$

for all τ sufficiently large.

Step 4. We sum up (4.5), (4.7) and (4.13) to obtain

$$\begin{aligned} & \widetilde{C}_{1,\tau} \int\limits_Q e^{2\tau\varphi} \left[|\nabla u|^2 + |\nabla u_t|^2 + |\nabla u_{tt}|^2 \right] dQ \\ & + \widetilde{C}_{2,\tau} \int\limits_Q e^{2\tau\varphi} \left[|u|^2 + |u_t|^2 + |u_{tt}|^2 \right] dQ \\ \leq\ & 4 \int\limits_Q e^{2\tau\varphi} \left[|fR|^2 + |fR_t|^2 + |fR_{tt}|^2 \right] dQ + \\ & C_{d;T} \tau e^{-2\tau\delta} \left[E_u(0) + E_u(T) + E_{u_t}(0) + E_{u_t}(T) + E_{u_{tt}}(0) + E_{u_{tt}}(T) \right], \end{aligned} \tag{4.14}$$

for all τ sufficiently large.

Step 5. We simplify the integral term on the right-hand side of estimate (4.14).

Proposition 4.1. *For the integral term on the right-hand side of* (4.14) , *we have*

$$\int\limits_Q e^{2\tau\varphi} \left[|fR|^2 + |fR_t|^2 + |fR_{tt}|^2 \right] dQ \leq C_R \int\limits_Q e^{2\tau\varphi} \left[|f|^2 \right] dQ, \tag{4.15}$$

$$\begin{aligned} \int\limits_Q e^{2\tau\varphi} |f|^2 dQ \ \leq\ & \left\{ \left(\frac{T}{r^2} \right) (2cT\tau + 1) \right\} \int\limits_\Omega \int\limits_0^{\frac{T}{2}} e^{2\tau\varphi(x,s)} |u_t(x,s)|^2 ds d\Omega \\ & + \left(\frac{T}{r^2} \right) \int\limits_\Omega \int\limits_0^{\frac{T}{2}} e^{2\tau\varphi(x,s)} |u_{tt}(x,s)|^2 ds d\Omega + \frac{T}{r^2} \int\limits_\Omega |u_t(x,0)|^2 d\Omega. \end{aligned} \tag{4.16}$$

Proof of Proposition 4.1

From (2.2) on $R, R_t, R_{tt} \in L^\infty(Q)$, we have

$$\int\limits_Q e^{2\tau\varphi} \left[|fR|^2 + |fR_t|^2 + |fR_{tt}|^2 \right] dQ \leq C_R \int\limits_Q e^{2\tau\varphi} \left[|f|^2 \right] dQ, \tag{4.17}$$

with C_R a constant depending on the $L^\infty(Q)$-norms of R_i, R_{it}, R_{itt} and (4.15) is verified.

To prove (4.16), use use the method suggested in ([9], Theorem 8.2.2, p. 231). We take $t = \frac{T}{2}$ in u-equation $(1.4a)$, invoke the initial condition $(1.4b)$, we find

$$iu_t\left(x,\frac{T}{2}\right)=f\left(x\right)R\left(x,\frac{T}{2}\right). \tag{4.18}$$

From (2.4) , we have $\left|R\left(x.\frac{T}{2}\right)\right|\geq r>0$, then

$$\left|f\left(x\right)\right|\leq\frac{1}{r}\left|u_t\left(x,\frac{T}{2}\right)\right|. \tag{4.19}$$

By (4.19), (3.3) and (3.1), we have

$$\begin{aligned}
&\int\limits_Q e^{2\tau\varphi}\left|f\right|^2dQ=\int\limits_0^T\int\limits_\Omega e^{2\tau\varphi(x,t)}\left|f\left(x\right)\right|^2d\Omega dt\\
&\leq\frac{1}{r^2}\int\limits_0^T\int\limits_\Omega e^{2\tau\varphi(x,t)}\left|u_t\left(x,\frac{T}{2}\right)\right|^2d\Omega dt\leq\frac{T}{r^2}\int\limits_\Omega e^{2\tau\varphi\left(x,\frac{T}{2}\right)}\left|u_t\left(x,\frac{T}{2}\right)\right|^2d\Omega\\
&=\frac{T}{r^2}\left(\int\limits_\Omega\int\limits_0^{\frac{T}{2}}\frac{d}{ds}\left(e^{2\tau\varphi(x,s)}\left|u_t\left(x,s\right)\right|^2\right)dsd\Omega+\int\limits_\Omega e^{2\tau\varphi(x,0)}\left|u_t\left(x,0\right)\right|^2d\Omega\right)\\
&=\frac{T}{r^2}(4c\tau\int\limits_\Omega\int\limits_0^{\frac{T}{2}}\left(\frac{T}{2}-s\right)e^{2\tau\varphi(x,s)}\left|u_t\left(x,s\right)\right|^2dsd\Omega+\\
&2\int\limits_\Omega\int\limits_0^{\frac{T}{2}}e^{2\tau\varphi(x,s)}\left|u_t\left(x,s\right)\right|\left|u_{tt}\left(x,s\right)\right|dsd\Omega+\int\limits_\Omega e^{2\tau\varphi(x,0)}\left|u_t\left(x,0\right)\right|^2d\Omega)
\end{aligned}$$

Using the Schwarz inequality and bearing in mind that $\left(\frac{T}{2}-s\right)\leq\frac{T}{2}$, we get

$$\begin{aligned}
&\int\limits_Q e^{2\tau\varphi}\left|f\right|^2dQ\leq\frac{T}{r^2}((2cT\tau)\int\limits_\Omega\int\limits_0^{\frac{T}{2}}e^{2\tau\varphi}\left|u_t\right|^2dtd\Omega+\\
&\int\limits_\Omega\int\limits_0^{\frac{T}{2}}e^{2\tau\varphi}\left|u_t\right|^2+\left|u_{tt}\right|^2dtd\Omega+\int\limits_\Omega e^{2\tau\varphi(x,0)}\left|u_t\left(x,0\right)\right|^2d\Omega)\\
&\leq\frac{T}{r^2}\left(2cT\tau+1\right)\int\limits_\Omega\int\limits_0^{\frac{T}{2}}e^{2\tau\varphi(x,s)}\left|u_t\left(x,s\right)\right|^2dsd\Omega\\
&+\frac{T}{r^2}\int\limits_\Omega\int\limits_0^{\frac{T}{2}}e^{2\tau\varphi(x,s)}\left|u_{tt}\left(x,s\right)\right|^2dsd\Omega+\frac{T}{r^2}\int\limits_\Omega e^{2\tau\varphi(x,0)}\left|u_t\left(x,0\right)\right|^2d\Omega,
\end{aligned} \tag{4.20}$$

by using in the last step $\varphi\left(x,0\right)\leq-\delta$ by (3.3), so that $e^{2\tau\varphi(x,0)}\leq1$. Then (4.16) follows from (4.20) .

Step 6. We insert the estimates (4.15) , (4.16) into (4.14) , we get the desired estimate for the u-problem $(1.4a)$-$(1.4c)$ in the following proposition.

Proposition 4.2. *In addition to the hypotheses of Theorem 2.1, assume that $f \in H^3(\Omega)$, as in (4.11). Then, the following estimates are true for the u-problem $(1.4a)$-$(1.4c)$, for all $\tau > 0$ large*

$$\begin{aligned} &\widetilde{C}_{1,\tau} \int\limits_Q e^{2\tau\varphi} \left[|\nabla u|^2 + |\nabla u_t|^2 + |\nabla u_{tt}|^2 \right] dQ + K_{2,\tau} \int\limits_Q e^{2\tau\varphi} \left[|u|^2 + |u_t|^2 + |u_{tt}|^2 \right] dQ \\ \leq\ & C_R \left(\frac{T}{r^2} \right) \int\limits_\Omega e^{2\tau\varphi(x,0)} |u_t(x,0)|^2 \, d\Omega + C_{d;T} \tau e^{-2\tau\delta} \\ & [E_u(0) + E_u(T) + E_{u_t}(0) + E_{u_t}(T) + E_{u_{tt}}(0) + E_{u_{tt}}(T)] , \end{aligned}$$

where

$$K_{2,\tau} = \widetilde{C}_{2,\tau} - C_R \left(\frac{T}{r^2} \right) (2cT\tau + 1) - C_R \left(\frac{T}{r^2} \right), \tag{4.21}$$

From (4.21), we deduce

$$\begin{aligned} & K_{2,\tau} \int\limits_Q e^{2\tau\varphi} \left[|u|^2 + |u_t|^2 + |u_{tt}|^2 \right] dQ \leq C_R \left(\frac{T}{r^2} \right) \int\limits_\Omega e^{2\tau\varphi(x,0)} |u_t(x,0)|^2 \, d\Omega \\ + \ & C_{d;T} \tau e^{-2\tau\delta} [E_u(0) + E_u(T) + E_{u_t}(0) + E_{u_t}(T) + E_{u_{tt}}(0) + E_{u_{tt}}(T)] \\ \leq\ & Const_{u,data}, \end{aligned} \tag{4.22}$$

where $Const_{u,data}$ is a constant depending on the data R, T, u, r, δ, d, but not on τ.

Step 7. We note again that from (3.4), (4.22) implies

$$\begin{aligned} & K_{2,\tau} e^{2\tau\sigma} \int\limits_{t_0}^{t_1} \int\limits_\Omega \left[|u|^2 + |u_t|^2 + |u_{tt}|^2 \right] dQ \leq K_{2,\tau} \int\limits_{t_0}^{t_1} \int\limits_\Omega e^{2\tau\varphi} \left[|u|^2 + |u_t|^2 + |u_{tt}|^2 \right] dQ \\ \leq\ & K_{2,\tau} \int\limits_0^T \int\limits_\Omega e^{2\tau\varphi} \left[|u|^2 + |u_t|^2 + |u_{tt}|^2 \right] dQ \\ \leq\ & Const_{u,data}, \end{aligned} \tag{4.23}$$

For $\tau \to +\infty$ in (4.23), $K_{2.\tau}$ is of order τ^3, then (4.23) implies

$$u(x,t) = 0 \qquad (x,t) \in \Omega \times [t_0, t_1] .$$

In particular for $t_0 < \frac{T}{2} < t_1$

$$iu_t \left(x, \frac{T}{2} \right) = f(x) R \left(x, \frac{T}{2} \right) = 0, \quad x \in \Omega. \tag{4.24}$$

after recalling (4.18). (4.24) together with (2.4), yields

$$f(x) = 0, \quad x \in \Omega, \tag{4.25}$$

Step 8. By assumptions (2.2), (2.5) on u-problem $(1.4a)$-$(1.4c)$, the map

$$\begin{array}{rcl} W^{1,\infty}(\Omega) & \to & C\left([0,T], H^1(\Omega)\right), \\ f & \to & u \end{array}$$

is continuous. Hence, by trace theory, the map

$$\begin{array}{rcl} W^{1,\infty}(\Omega) & \to & C\left([0,T], H^{\frac{1}{2}}(\Gamma)\right). \\ \mathcal{T} & : & f \to \mathcal{T}f = u|_{\Sigma} \end{array} \tag{4.26}$$

is also continuous. Since $H^3(\Omega)$ is dense in $W^{1,\infty}(\Omega)$, the conclusion of (4.25) that $\mathcal{T}f = 0$ for $f \in H^3(\Omega)$, can be extended via (4.26) to $\mathcal{T}f = 0$ for all $f \in W^{1,\infty}(\Omega)$ and $R_{x_j}\left(., \frac{T}{2}\right)$, $R_{x_j x_k}\left(., \frac{T}{2}\right)$, $R_{x_j x_k x_l}\left(., \frac{T}{2}\right) \in L^\infty(\Omega)$, $1 \leq j, k, l \leq n$. The proof of Theorem 2.1 is thus complete.

5 Observability inequality

In this section, we recall an observability inequality (see ([12], Theorem 8.4)) for the problem (1.2a)-(1.2c) with homogeneous boundary condition, namely the problem.

$$iy_t(x,t) + \Delta y(x,t) = q(x)y(x,t), \qquad \text{in } Q, \tag{5.1a}$$

$$y(., \frac{T}{2}) = y_0(x), \qquad \text{in } \Omega, \tag{5.1b}$$

$$\frac{\partial y(x,t)}{\partial \nu} = 0, \qquad \text{on } \Sigma. \tag{5.1c}$$

with

$$y_0 \in H^1(\Omega), \ q \in L^\infty(\Omega). \tag{5.2}$$

Then, its solution satisfies

$$y \in C\left([0.T], H^1(\Omega)\right).$$

Theorem 5.1. . *Suppose that the Assumption (**H**) is satisfied. Let y be the finite energy solution of equation $(5.1a)$ satisfying the Neumann B.C. $(5.1c)$ and $y_0 \in H^1(\Omega)$. Then the following observability inequality holds true: there is a constant $C = C(\Omega, T, q) > 0$ such that*

$$\|y_0\|^2_{H^1(\Omega)} \leq C_T \int_0^T \int_{\Gamma_1} \left[|y|^2 + |y_t|^2\right] d\Gamma_1 dt, \tag{5.3}$$

6 Proof of Theorem 2.2

Step 1. Consider the u_t-problem $(4.6a)$-$(4.6c)$, with data (2.2) and (2.4) i.e

$$i\left(u_t\right)_t(x,t)+\Delta\left(u_t\right)(x,t)=q(x)\,u_t(x,t)+f(x)\,R_t(x,t)\,,\quad \text{in } Q, \tag{6.1a}$$

$$u_t\left(.,\frac{T}{2}\right)=-i\,f(.)\,R\left(.,\frac{T}{2}\right)\in H^1(\Omega)\,,\quad \text{in } \Omega, \tag{6.1b}$$

$$\frac{\partial u_t(x,t)}{\partial\nu}=0,\quad \text{on } \Sigma. \tag{6.1c}$$

Set

$$u_t=\bar{u}_t+\tilde{u}_t, \tag{6.2}$$

where

$$i\left(\bar{u}_t\right)_t(x,t)+\Delta\left(\bar{u}_t\right)(x,t)=q(x)\,\bar{u}_t(x,t)\,,\quad \text{in } Q, \tag{6.3a}$$

$$\bar{u}_t\left(.,\frac{T}{2}\right)=-i\,f(.)\,R\left(.,\frac{T}{2}\right),\quad \text{in } \Omega, \tag{6.3b}$$

$$\frac{\partial \bar{u}_t(x,t)}{\partial\nu}=0,\quad \text{on } \Sigma. \tag{6.3c}$$

and

$$i\left(\tilde{u}_t\right)_t(x,t)+\Delta\left(\tilde{u}_t\right)(x,t)=q(x)\,\tilde{u}_t(x,t)+f(x)\,R_t(x,t)\,,\quad \text{in } Q, \tag{6.4a}$$

$$\tilde{u}_t\left(.,\frac{T}{2}\right)=0,\quad \text{in } \Omega, \tag{6.4b}$$

$$\frac{\partial \tilde{u}_t(x,t)}{\partial\nu}=0,\quad \text{on } \Sigma. \tag{6.4c}$$

Step 2. In view of (5.2), we apply the continuous observability inequality (5.3) to the $\bar{u}_t$-problem $(6.3a)$-$(6.3c)$, so that

$$\left\|-i\,f(.)\,R\left(.,\frac{T}{2}\right)\right\|^2_{H^1(\Omega)}\leq C^2_{T,q}\int_0^T\int_{\Gamma_1}\left[\bar{u}_t^2+\bar{u}_{tt}^2\right]d\Gamma_1 dt,$$

Since $\left|R\left(x.\frac{T}{2}\right)\right|\geq r>0$, for $x\in\overline{\Omega}$, (see (2.4)), then using (6.2), we obtain

$$\begin{aligned}\|f\|_{H^1(\Omega)} &\leq C_{T,q,r}\left\{\|\bar{u}_t\|_{L^2(\Sigma_1)}+\|\bar{u}_{tt}\|_{L^2(\Sigma_1)}\right\}\\ &\leq C_{T,q,r}\left\{\|u_t-\tilde{u}_t\|_{L^2(\Sigma_1\times[0,T])}+\|u_{tt}-\tilde{u}_{tt}\|_{L^2(\Gamma_1\times[0,T])}\right\}\\ &\leq C_{T,q,r}\left\{\|u_t\|_{L^2(\Gamma_1\times[0,T])}+\|u_{tt}\|_{L^2(\Sigma_1)}+\|\tilde{u}_t\|_{L^2(\Sigma_1)}+\|\tilde{u}_{tt}\|_{L^2(\Sigma_1\times[0,T])}\right\},\end{aligned} \tag{6.5}$$

with $C_{T,q,r}=\frac{C_{T,q}}{r}.$,

Step 3. We need to drop the lower order term $\left\{\|\tilde{u}_t\|_{L^2(\Gamma_1\times[0,T])}+\|\tilde{u}_{tt}\|_{L^2(\Sigma_1\times[0,T])}\right\}$ in (6.5). We do this by a compactness-uniqueness argument. We need the following lemma.

Lemma 6.1. *Consider the* $\tilde{u}_t$*-problem* $(6.4a)$-$(6.4c)$ *with data*

$$q \in W^{1.\infty}(\Omega), \quad f \in W^{1.\infty}(\Omega), \ R, R_t, R_{tt} \in C\left([0.T], H^1(\Omega)\right). \tag{6.6}$$

Define the following operators

$$\begin{cases} K : W^{1.\infty}(\Omega) \to L^2(\Sigma_1), K\{f\} = \tilde{u}_t|_{\Sigma_1} \\ L : W^{1.\infty}(\Omega) \to L^2(\Sigma_1 \times [0,T]), L\{f\} = \tilde{u}_{tt}|_{\Sigma_1} \end{cases}$$

Then

$$K \text{ and } L \text{ are compact.} \tag{6.7}$$

Proof of Lemma 6.1 ***Step i.*** From (6.6), we have the following regularity for the $\tilde{u}_t$-problem

$$\tilde{u}_t \in C\left([0.T], H^1(\Omega)\right). \tag{6.8}$$

Differentiating the $\tilde{u}_t$-problem $(6.4a)$-$(6.4c)$ in time, we find

$$i(\tilde{u}_{tt})_t(x,t) + \Delta(\tilde{u}_{tt})(x,t) = q(x)\tilde{u}_{tt}(x,t) + f(x)R_{tt}(x,t), \quad \text{in } Q, \tag{6.9a}$$

$$\tilde{u}_{tt}\left(., \frac{T}{2}\right) = -if(x)R_t\left(x, \frac{T}{2}\right), \quad \text{in } \Omega, \tag{6.9b}$$

$$\frac{\partial \tilde{u}_{tt}(x,t)}{\partial \nu} = 0, \quad \text{on } \Sigma. \tag{6.9c}$$

Because of (6.6), we have

$$\tilde{u}_{tt} \in C\left([0.T], H^1(\Omega)\right). \tag{6.10}$$

Step ii. Problem $(6.4a)$-$(6.4c)$ under assumptions (6.6), (6.8) for $\tilde{u}_t$, yields the map

$$\begin{array}{rcl} W^{1,\infty}(\Omega) & \to & C\left([0,T], H^1(\Omega)\right), \\ f & \to & \tilde{u}_t \end{array}$$

is continuous. Hence, by trace theory, the map

$$\begin{array}{rcl} W^{1,\infty}(\Omega) & \to & C\left([0,T], H^{\frac{1}{2}}(\Gamma)\right). \\ \mathcal{T}_1 : f & \to & \mathcal{T}_1 f = \tilde{u}_t|_{\Sigma} \end{array} \tag{6.11}$$

is also continuous. Problem $(6.9a)$-$(6.9c)$ under assumptions (6.6), (6.10) for $\tilde{u}_{tt}$, yields the map

$$\begin{array}{rcl} W^{1,\infty}(\Omega) & \to & C\left([0,T], H^1(\Omega)\right), \\ f & \to & \tilde{u}_{tt} \end{array}$$

is continuous. By trace theory, the map

$$\begin{array}{rcl} \mathcal{T}_2 : W^{1,\infty}(\Omega) & \to & C\left([0,T], H^{\frac{1}{2}}(\Gamma)\right). \\ f & \to & \mathcal{T}_2 f = \tilde{u}_{tt}|_{\Sigma} \end{array} \tag{6.12}$$

is continuous.

Step iii. Since $R_t \in L^{\infty}(Q)$, we deduce from (6.11) that

$f \in W^{1.\infty}(\Omega) \to \tilde{u}_t(f)|_{\Sigma} \in C\left([0,T], H^{\frac{1}{2}}(\Gamma)\right)$ continuously, i.e

$$\|\tilde{u}_t(f)\|_{C\left([0,T],H^{\frac{1}{2}}(\Gamma)\right)} \leq C_R \|f\|_{H^1(\Omega)} \tag{6.13}$$

Similarly, from (6.12), where R_t , $R_{tt} \in L^{\infty}(Q)$, we have

$f \in W^{1.\infty}(\Omega) \to \tilde{u}_{tt}(f)|_{\Sigma} \in C\left([0,T], H^{\frac{1}{2}}(\Gamma)\right)$ continuously, i.e

$$\|\tilde{u}_{tt}(f)\|_{C\left([0,T],H^{\frac{1}{2}}(\Gamma)\right)} \leq C_R\left(\|f\|_{H^1(\Omega)}\right) \tag{6.14}$$

Step iv. We deduce from (6.13) and (6.14), that K and L are compact operators because the embedding $H^{\frac{1}{2}}(\Gamma) \to L^2(\Gamma)$ is compact. This completes the proof of Lemma 6.1.

Step 4.

Assume that the inequality in (2.7) does not hold. Then, there exist a sequence $\{f_n\}_{n\geq 1}$, $f_n \in W^{1,\infty}(\Omega) \subset H^1(\Omega)$, such that

$$\|f_n\|_{H^1(\Omega)} = 1 \quad n \geq 1, \tag{6.15a}$$

$$\lim_{n\to+\infty} \|u_t(f_n)\|_{L^2(\Sigma_1)} + \|u_{tt}(f_n)\|_{L^2(\Sigma_1)} = 0. \tag{6.15b}$$

From $(6.15a)$, there exist subsequence, still denoted by f_n , such that

$$f_n \text{ converges weakly in } L^2(\Omega) \text{ to some } f_0 \in L^2(\Omega), \tag{6.16}$$

by (6.7) we have

$$\lim_{m,n\to+\infty} \|Kf_n - Kf_m\|_{L^2(\Sigma_1)} = \lim_{m,n\to+\infty} \|Lf_n - Lf_m\|_{L^2(\Sigma_1)} = 0. \tag{6.17}$$

Also, from (6.5), and the linearity of the $map\ f \to u(f)$, we find

$$\begin{array}{rcl} \|f_n - f_m\|_{H^1(\Omega)} & \leq & C_{T,q,r}\left\{\|u_t(f_n) - u_t(f_m)\|_{L^2(\Sigma_1)} + \|u_{tt}(f_n) - u_{tt}(f_m)\|_{L^2(\Sigma_1)}\right\} + \\ & & C_{T,q,r}\left\{\|Kf_n - Kf_m\|_{L^2(\Sigma_1)} + \|Lf_n - Lf_m\|_{L^2(\Sigma_1)}\right\} \\ & \leq & C\left\{\|u_t(f_n)\|_{L^2(\Sigma_1)} + \|u_{tt}(f_n)\|_{L^2(\Sigma_1)} + \|u_t(f_m)\|_{L^2(\Sigma_1)} + \|u_{tt}(f_m)\|_{L^2(\Sigma_1)}\right\} + \\ & & C_{T,q,r}\left\{\|Kf_n - Kf_m\|_{L^2(\Sigma_1)} + \|Lf_n - Lf_m\|_{L^2(\Sigma_1)}\right\}, \end{array}$$

then, by $(6.15b)$ and (6.17), we find that

$$\lim_{n,m\to+\infty} \|f_n - f_m\|_{H^1(\Omega)} = 0.$$

By uniqueness of the limit, and (6.16), we obtain

$$\lim_{n\to+\infty} \|f_n - f_0\|_{H^1(\Omega)} = 0. \tag{6.18}$$

Thus, in view of $(6.15a)$, (6.18) implies

$$\|f_0\|_{H^1(\Omega)} = 1 \tag{6.19}$$

We return to the u-problem $(1.4a)$-$(1.4c)$. From hypothesis (2.2) and (2.5) , we have the continuity of the map

$$\begin{array}{rcl} W^{1,\infty}(\Omega) & \to & C\left([0,T], H^1(\Omega)\right), \\ f & \to & u \end{array}$$

Hence, by trace theory, the map

$$\begin{array}{rcl} W^{1,\infty}(\Omega) & \to & C\left([0,T], H^{\frac{1}{2}}(\Gamma)\right), \\ \mathcal{T}:\ f & \to & \mathcal{T}f = u\,|_{\Sigma} \end{array}$$

is also continuous, i.e.

$$\|u(f)\|_{C\left([0,T],\, H^{\frac{1}{2}}(\Gamma)\right)} \leq C_R\left(\|f\|_{H^1(\Omega)}\right),$$

because, $H^{\frac{1}{2}}(\Gamma)$ embeds in $L^2(\Gamma)$, then

$$\|u(f)\|_{C([0,T],\, L^2(\Gamma))} \leq C_R\left(\|f\|_{H^1(\Omega)}\right). \tag{6.20}$$

As the $map\ f \to u(f)|_{\Sigma}$ is linear, it then follows in particular from (6.20), since $f_n,\ f_0 \in W^{1.\infty}(\Omega)$, that

$$\|u(f_n)\,|_{\Sigma_1} - u(f_0)\,|_{\Sigma_1}\|_{C([0,T],\, L^2(\Gamma_1))} \leq C_R\left(\|f_n - f_0\|_{H^1(\Omega)}\right). \tag{6.21}$$

Inserting (6.18) into (6.21), we find

$$\lim_{n\to+\infty} \|u(f_n)\,|_{\Sigma_1} - u(f_0)\,|_{\Sigma_1}\|_{C([0,T],L^2(\Gamma_1))} = 0.$$

Similarly, for u_t-problem $(6.1a)$-$(6.1c)$, we have by hypothesis (2.2) and (2.5) ,

$$\lim_{n\to+\infty} \|u_t(f_n)\,|_{\Sigma_1} - u_t(f_0)\,|_{\Sigma_1}\|_{C([0,T],L^2(\Gamma_1))} = 0. \tag{6.22}$$

Then, by virtue of $(6.15b)$, combined with (6.22), we obtain for $t \in [0,T]$ that

$$u_t(f_0)\,|_{\Sigma_1} = 0. \tag{6.23}$$

For the u-problem $(1.4a)$-$(1.4c)$, with $f = f_n \in W^{1.\infty}(\Omega)$ and $q \in W^{1.\infty}(\Omega)$, $R \in C\left([0.T], H^1(\Omega)\right)$, we have

$$u(f_n)\left(x, \frac{T}{2}\right) = 0,\ x \in \overline{\Omega},$$

and therefore

$$u(f_n)\left(x, \frac{T}{2}\right) = 0,\ x \in \Gamma_1, \tag{6.24}$$

in the sense of trace in $H^{\frac{1}{2}}(\Gamma_1)$, then (6.24) combined with (6.21) yields a-fortiori

$$u(f_0)\left(x, \frac{T}{2}\right) = 0,\ x \in \Gamma_1,$$

and by virtue of (6.23), the desired conclusion

$$u(f_0)(x,t) = 0, \quad (x,t) \in \Sigma_1.$$

Here, $u(f_0)$ solves problem $(1.4a)$-$(1.4c)$ with $f = f_0$.

Finally, we apply Theorem 2.1 to conclude that

$$f_0(x) = 0, \quad x \in \ \Omega.$$

and this contradicts (6.19). Thus the proof of Theorem 2.2 is complete.

7 Proof of Theorem 2.3

From (1.3), we have

$$\begin{aligned} f(x) &= q(x) - p(x), \\ R(x,t) &= z(p)(x,t), \\ u(x,t) &= z(q)(x,t) - z(p)(x,t). \end{aligned} \tag{7.1}$$

Since $R\left(x, \frac{T}{2}\right) = z(p)\left(x, \frac{T}{2}\right) = z_0(x)$, $x \in \Omega$ (see $(1.2b)$), the conditions (2.9) and (2.10) in Theorem 2.3 imply (2.2) for R, R_t, R_{tt}, $R_{x_j}\left(., \frac{T}{2}\right)$, $R_{x_j x_k}\left(., \frac{T}{2}\right)$, $R_{x_j x_k x_l}\left(., \frac{T}{2}\right)$ in Theorem 2.1. Also, the condition (2.11) in Theorem 2.3 implies (2.4) in Theorem 2.1. Moreover, assumption (2.8) for q, p in Theorem 2.3 implies assumption (2.2) for f in Theorem 2.1.
Now, (2.12) together with (7.1) gives

$$u(f)(x,t) = 0, \quad (x,t) \in \Sigma_1.$$

Thus from the uniqueness property in Theorem 2.1 for the u-problem we conclude that

$$f(x) = 0, \quad x \in \Omega,$$

i.e,

$$q(x) = p(x), \quad x \in \Omega.$$

8 Proof of Theorem 2.4

We end this paper by the proof of Theorem 2.4.

From (2.8), (2.9) and (7.1), we have $f \in W^{1.\infty}(\Omega)$ and $R \in C\left([0.T], H^1(\Omega)\right)$. Thus, we can apply Theorem 2.2, to the function $u\{u = z(q) - z(p)\}$ in (7.1), solving problem $(1.4a)$-$(1.4c)$, to obtain the desired stability estimate (2.13), i.e

$$\|q - p\|_{H^1(\Omega)} \leq C\left\{\|z_t(q)(x,t) - z_t(p)(x,t)\|_{L^2[\Sigma_1]} + \|z_{tt}(q) - z_{tt}(p)\|_{L^2[\Sigma_1]}\right\}.$$

References

[1] C. Bardos, G. Lebeau, J. Rauch. Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary. *SIAM J. Control Optim.* 30 (1992), 1024–1165

[2] L. Baudouin and J.-P. Puel. Uniqueness and stability in an inverse problem for the Schrödinger equation. *Inverse Probl.* 18 (2002), 1537.

[3] M. Bellassoued and M. Choulli. Logarithmic stability in the dynamical inverse problem for the Schrödinger equation by arbitrary boundary observation. *J. Math. Pures Appl.* 91 (2009), 233–255.

[4] M. Bellassoued and M. Choulli. Stability estimate for an inverse problem for the magnetic Schrödinger equation from the Dirichlet-to-Neumann map. *J. Funct. Anal.* 258 (2010), 161–195.

[5] M. Bellassoued and M. Yamamoto. Carleman Estimates and Applications to Inverse Problems for Hyperbolic Systems. Springer, Tokyo, 2017.

[6] A.L. Bukhgeĭm. Introduction to the theory of inverse problems. Vol. 19. Walter de Gruyter, 2000.

[7] M. Cristofol and E. Soccorsi. Stability estimate in an inverse problem for non-autonomous magnetic Schrödinger equations. *Applicable Analysis* 90 (2011), 1499-1520.

[8] L. Deng. An inverse problem for the Schrödinger equation with variable coefficients and lower order terms. *J. Math. Anal. Appl.* 427 (2015), 930–940.

[9] V. Isakov. Inverse problems for partial differential equations. Springer, New York, 2006.

[10] X. Huang, Y. Kian, E. Soccorsi and M. Yamamoto. Carleman estimate for the Schrödinger equation and application to magnetic inverse problems. *J. Math. Anal. Appl.* 474 (2019) 116-142.

[11] I. Lasiecka, R. Triggiani and X. Zhang. Nonconservative wave equations with unobserved Neumann BC: Global uniqueness and observability in one shot. *Contemp. Math.* 268 (2000), 227–326.

[12] Lasiecka, R. Triggiani and X. Zhang. Global uniqueness, observability and stabilization of nonconservative Schrödinger equations via pointwise Carleman estimates. Part I: $H^1(\Omega)$-estimates. *J. Inverse Ill-posed Probl.* 12 (2004), 43–123.

[13] S. Liu and R. Triggiani. Global uniqueness in determining electric potentials for a system of strongly coupled Schrödinger equations with magnetic potential terms. *J. Inverse Ill-Posed Probl.* 19 (2011), 223-254.

[14] A. Mercado, Osses and L. Rosier. Inverse problems for the Schrödinger equation via Carleman inequalities with degenerate weights. *Inverse Probl.* 24 (2008), 015017.

[15] V.G. Maz'ya and T.O. Shaposhnikova. Theory of multipliers in spaces of differentiable functions. Monographs and Studies in Mathematics, Vol. 23, Pitman, Boston, 1985.

[16] R.Triggiani. Carleman estimates and exact boundary controllabilty for a system of coupled non-conservative Schödinger Equations. *Rend. Istit. Mat. Univ. Trieste* 28 (1996), 453–504.

[17] R. Triggiani and P. Yao. Inverse/observability estimates for Schrödinger equations with variable coefficients. *Control Cybern.* 28 (1999), 627–664.

[18] R. Triggiani and Z. Zhang. Global uniqueness and stability in determining the electric potential coefficient of an inverse problem for Schrödinger equations on Riemannian manifolds. *J. Inverse Ill-Posed Probl.* 23 (2015), 587–609.

[19] G. Yuan and M. Yamamoto. Carleman estimates for the Schrödinger equation and applications to an inverse problem and an observability inequality. *Chin. Ann. Math. Ser. B* 31 (2010), 555–578.

